A \( \sigma_3 \) TYPE CONDITION
FOR HEAVY CYCLES IN WEIGHTED GRAPHS

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Abstract

A weighted graph is a graph in which each edge \( e \) is assigned a non-negative number \( w(e) \), called the weight of \( e \). The weight of a cycle is the sum of the weights of its edges. The weighted degree \( d^w(v) \) of a vertex \( v \) is the sum of the weights of the edges incident with \( v \). In this paper, we prove the following result: Suppose \( G \) is a 2-connected weighted graph which satisfies the following conditions:

1. The weighted degree sum of any three independent vertices is at least \( m \);
2. \( w(xz) = w(yz) \) for every vertex \( z \in N(x) \cap N(y) \) with \( d(x, y) = 2 \);
3. In every triangle \( T \) of \( G \), either all edges of \( T \) have different weights or all edges of \( T \) have the same weight.

Then \( G \) contains either a Hamilton cycle or a cycle of weight at least \( 2m/3 \).

This generalizes a theorem of Fournier and Fraisse on the existence of long cycles in \( k \)-connected unweighted graphs in the case \( k = 2 \). Our proof of the above result also suggests a new proof to the theorem of Fournier and Fraisse in the case \( k = 2 \).

Keywords: weighted graph, (long, heavy, Hamilton) cycle, weighted degree, (weighted) degree sum.

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1. Terminology and Notation

We use Bondy and Murty [4] for terminology and notation not defined here and consider finite simple graphs only.

Let \( G = (V, E) \) be a simple graph. \( G \) is called a weighted graph if each edge \( e \) is assigned a non-negative number \( w(e) \), called the weight of \( e \). For any subgraph \( H \) of \( G \), \( V(H) \) and \( E(H) \) denote the sets of vertices and edges of \( H \), respectively. The weight of \( H \) is defined by

\[
    w(H) = \sum_{e \in E(H)} w(e).
\]

An optimal cycle is one with maximum weight. For each vertex \( v \in V \), \( N_H(v) \) denotes the set, and \( d_H(v) \) the number, of vertices in \( H \) that are adjacent to \( v \). We define the weighted degree of \( v \) in \( H \) by

\[
    d^w_H(v) = \sum_{h \in N_H(v)} w(vh).
\]

When no confusion occurs, we will denote \( N_G(v), d_G(v) \) and \( d^w_G(v) \) by \( N(v), d(v) \) and \( d^w(v) \), respectively. An \( (x, y) \)-path is a path connecting the two vertices \( x \) and \( y \). The distance between two vertices \( x \) and \( y \), denoted by \( d(x, y) \), is the length of a shortest \( (x, y) \)-path. If \( u \) and \( v \) are two vertices on a path \( P \), \( P[u, v] \) denotes the segment of \( P \) from \( u \) to \( v \). The number of vertices in a maximum independent set of \( G \) is denoted by \( \alpha(G) \). For a positive integer \( k \leq \alpha(G) \) we denote by \( \sigma_k(G) \) the minimum value of the degree sum of any \( k \) independent vertices, and by \( \sigma^w_k(G) \) the minimum value of the weighted degree sum of any \( k \) independent vertices. Instead of \( \sigma_1(G) \) and \( \sigma^w_1(G) \), we use the notations \( \delta(G) \) and \( \delta^w(G) \), respectively.

2. Results

There have been many results on the existence of long cycles in graphs. The following three theorems are well-known.

**Theorem A** (Dirac [5]). Let \( G \) be a 2-connected graph such that \( \delta(G) \geq r \). Then \( G \) contains either a Hamilton cycle or a cycle of length at least \( 2r \).

**Theorem B** (Pósa [7]). Let \( G \) be a 2-connected graph such that \( \sigma_2(G) \geq s \). Then \( G \) contains either a Hamilton cycle or a cycle of length at least \( s \).
Theorem C (Fournier and Fraisse [6]). Let $G$ be a $k$-connected graph where \(2 \leq k < \alpha(G)\), such that $\sigma_{k+1}(G) \geq m$. Then $G$ contains either a Hamilton cycle or a cycle of length at least \(2m/(k+1)\).

It is easy to see that Theorem B generalizes Theorem A, and Theorem C in turn generalizes Theorem B.

An unweighted graph can be regarded as a weighted graph in which each edge $e$ is assigned weight $w(e) = 1$. Thus, in an unweighted graph, $d_w(v) = d(v)$ for every vertex $v$, and the weight of a cycle is simply the length of the cycle.

Theorem A and Theorem B were generalized to weighted graphs by the following two theorems, respectively.

Theorem 1 (Bondy and Fan [3]). Let $G$ be a 2-connected weighted graph such that $\delta^w(G) \geq r$. Then either $G$ contains a cycle of weight at least $2r$ or every optimal cycle is a Hamilton cycle.

Theorem 2 (Bondy et al. [2]). Let $G$ be a 2-connected weighted graph such that $\sigma^w_{k}(G) \geq s$. Then $G$ contains either a Hamilton cycle or a cycle of weight at least $s$.

A natural question is whether Theorem C also admits an analogous generalization for weighted graphs. This leads to the following problem.

Problem 1. Let $G$ be a $k$-connected weighted graph where \(2 \leq k < \alpha(G)\), such that $\sigma_{k+1}^w(G) \geq m$. Is it true that $G$ contains either a Hamilton cycle or a cycle of weight at least $2m/(k+1)$?

If the answer to the question of this problem is positive, then the result would be best possible and it would generalize Theorem C and Theorem 2.

It seems very difficult to settle this problem, even for the case $k = 2$. In the next section, we prove that the answer to the case $k = 2$ of Problem 1 is positive if we add some extra conditions. These extra conditions were motivated by a recent generalization of a theorem of Fan to weighted graphs (cf. [8]). Our result is an analogue and also a generalization of Theorem C to weighted graphs in the case $k = 2$.

Theorem 3. Let $G$ be a 2-connected weighted graph which satisfies the following conditions:

1. The weighted degree sum of any three independent vertices is at least $m$;
2. \( w(xz) = w(yz) \) for every vertex \( z \in N(x) \cap N(y) \) with \( d(x, y) = 2 \);

3. In every triangle \( T \) of \( G \), either all edges of \( T \) have different weights or all edges of \( T \) have the same weight.

Then \( G \) contains either a Hamilton cycle or a cycle of weight at least \( 2m/3 \).

3. Proof of Theorem 3

Let \( G \) be a 2-connected weighted graph satisfying the conditions of Theorem 3. Suppose that \( G \) does not contain a Hamilton cycle. Then it suffices to prove that \( G \) contains a cycle of weight at least \( 2m/3 \).

Choose a path \( P = v_1v_2 \cdots v_p \) in \( G \) such that

(a) \( P \) is as long as possible;

(b) \( w(P) \) is as large as possible, subject to (a);

(c) \( d^w(v_1) + d^w(v_p) \) is as large as possible, subject to (a) and (b).

From the choice of \( P \), we can immediately see that \( N(v_1) \cup N(v_p) \subseteq V(P) \).

**Claim 1.** There exists no cycle of length \( p \).

**Proof.** Suppose there exists a cycle \( C \) of length \( p \). Since \( G \) contains no Hamilton cycle and \( G \) is connected, we can find a vertex \( u \in V(G) \setminus V(C) \) and a path \( Q \) from \( u \) to a vertex \( v \in V(C) \), such that \( Q \) is internally disjoint from \( C \). The subgraph \( C \cup Q \) of \( G \) contains a path longer than \( P \), contradicting the choice of \( P \) in (a).

**Claim 2.** \( v_1v_p \notin E(G) \).

**Proof.** If \( v_1v_p \in E(G) \), then we can find a cycle \( C = v_1v_2 \cdots v_pv_1 \) of length \( p \), contradicting Claim 1.

**Claim 3.** If \( v_i \in N(v_1) \), then \( v_{i-1} \notin N(v_p) \).

**Proof.** Suppose \( v_i \in N(v_1) \) and \( v_{i-1} \in N(v_p) \). Then we can form a cycle \( C = v_1v_iv_{i+1} \cdots v_pv_{i-1}v_{i-2} \cdots v_1 \) with length \( p \), again contradicting Claim 1.

**Claim 4.** If \( v_i \in N(v_1) \), then \( w(v_i v_{i+1}) \geq w(v_1 v_i) \). If \( v_j \in N(v_p) \), then \( w(v_j v_{j+1}) \geq w(v_j v_p) \).
**Proof.** If \( v_i \in N(v_1) \), the path \( P' = v_{i-1}v_{i-2} \cdots v_1v_i \cdots v_p \) has the same length as \( P \). So, because of (b), we must have \( w(P) \geq w(P') \), hence \( w(v_{i-1}v_i) \geq w(v_1v_i) \). The second assertion can be proved similarly. ■

Since \( G \) is 2-connected, by Lemma 1 of [1], there is a sequence of internally disjoint paths \( P_1, P_2, \ldots, P_m \) such that

1. \( P_k \) has end vertices \( x_k \) and \( y_k \), and \( V(P_k) \cap V(P) = \{x_k, y_k\} \) for \( k = 1, 2, \ldots, m; \)
2. \( v_1 = x_1 < x_2 < y_1 \leq x_3 < y_2 \leq x_4 < \cdots < y_{m-2} \leq x_m < y_{m-1} < y_m = v_p \), where the inequalities denote the order of the vertices on \( P \).

By Claim 2, we have \( m \geq 2 \). It is not difficult to see that we can choose these paths such that

3. if \( v_i \in N(v_1) \), then \( v_i \in P[v_2, x_2] \cup P[y_3, x_3] \) for \( m \geq 3 \), or \( v_i \in P[v_2, x_2] \cup P[y_1, v_{p-1}] \) for \( m = 2; \)
4. if \( v_j \in N(v_p) \), then \( v_j \in P[y_{m-2}, x_m] \cup P[y_{m-1}, v_{p-1}] \) for \( m \geq 3 \), or \( v_j \in P[v_2, x_2] \cup P[y_1, v_{p-1}] \) for \( m = 2. \)

Now denote by \( C_k \) the cycle \( P_k \cup P[x_k, y_k] \) for \( k = 1, 2, \ldots, m \), and let \( C \) be the cycle whose edge set is the symmetric difference of the edge sets of these cycles \( C_k \). By (3), (4) and Claim 3 we have for all \( v_i \in N(v_1) \setminus \{y_1\} \) and \( v_j \in N(v_p) \setminus \{x_m\} \) that \( v_{i-1}v_i, v_jv_{j+1} \in E(C) \) and \( v_{i-1}v_i \neq v_jv_{j+1} \). Also note that since \( N(v_1) \cup N(v_p) \subseteq V(P) \), we must have \( P_1 = v_1y_1 \) and \( P_m = x_mv_p \). Using Claim 4, this shows that

\[
    w(C) \geq \sum_{v_i \in N(v_1) \setminus \{y_1\}} w(v_{i-1}v_i) + \sum_{v_j \in N(v_p) \setminus \{x_m\}} w(v_jv_{j+1})
    + w(v_1y_1) + w(x_my_p)
\]

\[
    \geq \sum_{v_i \in N(v_1)} w(v_1v_i) + \sum_{v_j \in N(v_p)} w(v_jv_p)
\]

\[
    = d^w(v_1) + d^w(v_p).
\]

Without loss of generality, we can assume that \( d^w(v_1) \leq w(C)/2 \).

Since \( G \) is 2-connected, \( v_2 \) is adjacent to at least one vertex on \( P \) other than \( v_2 \). Choose \( v_k \in N(v_1) \cap V(P) \) such that \( k \) is as large as possible. By Claim 2 it is clear that \( 3 \leq k \leq p - 1 \).

Now we consider two cases.
Case 1. There exists a vertex \( v_i \in V(P) \) such that \( v_1v_i \in E(G) \) but \( v_1v_{i-1} \notin E(G) \) for some \( i \) with \( 3 \leq i \leq k \).

By Claim 3 we know that \( v_{i-1}v_p \notin E(G) \), so the three vertices \( v_1, v_{i-1} \) and \( v_p \) are independent. From Condition 2 of the theorem and the fact \( d(v_1,v_{i-1}) = 2 \) we know that \( v_{i-1}v_{i-2} \cdots v_1v_1 \cdots v_p \) is another longest path with the same weight as \( P \). By the choice of \( P \) in (c), we have \( d^w(v_{i-1}) \leq d^w(v_1) \leq w(C)/2 \). With \( d^w(v_1)+d^w(v_p) \leq w(C) \), we have \( d^w(v_1)+d^w(v_{i-1})+d^w(v_p) \leq 3w(C)/2 \). It follows from Condition 1 of the theorem that the weight of the cycle \( C \) is at least \( 2m/3 \).

Case 2. \( v_1v_i \in E(G) \) for all \( i \) with \( 3 \leq i \leq k \).

Case 2.1. \( w(v_1v_{i-1}) = w(v_1v_i) = w(v_{i-1}v_i) = w^* \) for all \( i \) with \( 3 \leq i \leq k \). For every \( i \) with \( 2 \leq i < k-1 \), \( v_i \) cannot be adjacent to any vertex outside \( P \). Otherwise, there will be a path of length \( p \), contradicting the choice of \( P \) in (a). Since \( G \) is 2-connected, there must be an edge \( v_jv_s \in E(G) \) with \( j < k < s \). Choose \( v_jv_s \in E(G) \) such that \( j < k < s \) and \( s \) is as large as possible. From Claim 3 we have \( s < p \).

Case 2.1.1. \( s \geq k+2 \).

By the choice of \( v_k \) we know that \( v_1v_{s-1} \notin E(G) \). If \( v_{s-1}v_p \in E(G) \), then we can form a cycle \( v_1v_{j+1} \cdots v_{s-1}v_p \cdots v_sv_j \cdots v_1 \) of length \( p \), contradicting Claim 1. So, the three vertices \( v_1, v_{s-1} \) and \( v_p \) are independent. By the choice of \( v_k \), we have \( d(v_1,v_s) = 2 \). If \( v_jv_{s-1} \in E(G) \), then \( d(v_1,v_{s-1}) = 2 \). Then it follows from Condition 2 of the theorem that \( w(v_jv_{s-1}) = w(v_1v_j) = w^* \), and from Condition 3 of the theorem we get \( w(v_{s-1}v_s) = w^* \). If \( v_jv_{s-1} \notin E(G) \), then \( d(v_jv_{s-1}) = 2 \). This implies that \( w(v_{s-1}v_s) = w(v_jv_s) = w^* \). Thus, in both cases the path \( v_{s-1}v_s \cdots v_{j+1}v_1 \cdots v_jv_s \cdots v_p \) is another longest path with the same weight as \( P \). By the choice of \( P \) in (c), we know that \( d^w(v_{s-1}) \leq d^w(v_1) \leq w(C)/2 \). With \( d^w(v_1)+d^w(v_p) \leq w(C) \), we have \( d^w(v_1)+d^w(v_{s-1})+d^w(v_p) \leq 3w(C)/2 \). It follows from Condition 1 of the theorem that the weight of the cycle \( C \) is at least \( 2m/3 \).

Case 2.1.2. \( s = k+1 \).

By Claim 3 we may assume that \( k+1 < p \). From the 2-connectedness of \( G \) and the choice of \( v_s \), there must be an edge \( v_kv_t \in E(G) \) such that \( t \geq k+2 \). By the choice of \( v_k \), we know that \( v_1v_{t-1} \notin E(G) \). On the other hand, if \( v_{t-1}v_p \in E(G) \), then we can form a cycle \( v_1v_{j+1} \cdots v_kv_{t} \cdots v_pv_{t-1} \cdots v_{k+1} \cdots v_1 \).
v_j \cdots v_1 \text{ of length } p, \text{ contradicting Claim 1. So, the three vertices } v_1, v_{t-1} \text{ and } v_p \text{ are independent.}

If \( v_kv_{t-1} \in E(G) \), then from Condition 2 of the theorem we have \( w(v_kv_{t-1}) = w(v_kv_k) = w(v_1v_k) = w^* \), and from Condition 3 of the theorem, the edge \( v_{t-1}v_t \) has weight \( w^* \). If \( v_kv_{t-1} \notin E(G) \), then from Condition 2 of the theorem we also get \( w(v_{t-1}v_t) = w^* \). Thus, in both cases the path \( v_{t-1}v_{t-2} \cdots v_{k+1}v_k \cdots v_1 \) is another longest path with the same weight as \( P \). By the choice of \( P \) in (c), \( d(v_{t-1}) \leq d^w(v_1) \leq w(C)/2 \). With \( d^w(v_1) + d^w(v_p) \leq w(C) \), we have \( d^w(v_1) + d^w(v_{t-1}) + d^w(v_p) \leq 3w(C)/2 \). It follows from Condition 1 of the theorem that the weight of the cycle \( C \) is at least \( 2m/3 \).

This completes the proof of Case 2.1.

Case 2.2. There is some vertex \( v_i \) with \( 3 \leq i \leq k \) such that \( w(v_1v_i) \), \( w(v_iv_i) \) and \( w(v_{i-1}v_i) \) are all different. In this case, choose vertex \( v_j \) such that \( w(v_1v_j) \), \( w(v_jv_j) \) and \( w(v_jv_{j+1}) \) are all different, and \( j \) is as large as possible. Denote the weight of \( v_1v_j \), \( v_jv_{j-1} \) and \( v_{j-1}v_1 \) by \( w_1 \), \( w_2 \) and \( w_3 \), respectively. It follows from Condition 3 (or Condition 2 if \( j = k \)) that \( w(v_{j-1}v_j) = w_j \neq w_1 = w(v_jv_{j+1}) \), and from Condition 2 of the theorem that \( v_{j-1}v_{j+1} \in E(G) \). If \( j < k \), then the weight of the edge \( v_{j-1}v_{j+1} \) is different from the weight \( w_1 \) of the edge \( v_{j+1}v_{j+2} \) since there is a triangle \( v_{j-1}v_{j-1}v_{j+1}v_1 \) and \( w(v_1v_{j-1}) = w_3 \neq w_1 = w(v_1v_{j+1}) \). With the same argument, we can prove that \( v_{j-1}v_i \in E(G) \) for all \( i \) with \( j \leq i \leq k+1 \). By the choice of \( v_k \), we have that \( w(v_{j-1}v_{k+1}) = w_3 \).

Suppose first that \( v_kv_{k+2} \in E(G) \). Then \( d(v_1, v_{k+2}) = 2 \). This shows that \( w(v_kv_{k+2}) = w(v_1v_k) = w_1 \). From \( w(v_kv_{k+1}) = w(v_kv_{k+2}) = w_1 \) and Condition 3 of the theorem we know that \( w(v_{k+1}v_{k+2}) = w_1 \). Therefore, there must be an edge \( v_{j-1}v_{k+2} \in E(G) \) since the two edges \( v_{j-1}v_{k+1} \) and \( v_{k+1}v_{k+2} \) have different weights. Again, by the fact \( d(v_1, v_{k+2}) = 2 \), we obtain that \( w(v_{j-1}v_{k+2}) = w(v_{j-1}v_j) = w_3 \). This leads to a triangle \( v_{j-1}v_{k+1}v_{k+2}v_{j-1} \) in which \( w(v_{j-1}v_{k+1}) = w(v_{j-1}v_{k+2}) = w_3 \) and \( w(v_{k+1}v_{k+2}) = w_1 \), contradicting Condition 3 of the theorem. Hence \( v_kv_{k+2} \notin E(G) \). Thus \( d(v_k, v_{k+2}) = 2 \). This implies that \( w(v_{k+1}v_{k+2}) = w(v_kv_{k+1}) = w_1 \). Therefore, there must be an edge \( v_{j-1}v_{k+2} \in E(G) \) and \( w(v_{j-1}v_{k+2}) = w_3 \). This also leads to a triangle \( v_{j-1}v_{k+1}v_{k+2}v_{j-1} \) which is impossible by Condition 3 of the theorem.

The proof of the theorem is complete. ■
4. Remarks

The proof of Theorem C in [6] is very complicated. It is clear that our proof of Theorem 3 provides a simpler proof for Theorem C in the case $k = 2$. We do not know whether the extra conditions in Theorem 3 are necessary. The results in [8] indicate that for some generalizations of long cycle results to weighted graphs one cannot avoid such additional conditions. We do not believe that there is an analogous generalization of Theorem C for the case $k \neq 2$.

References


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