Real option analysis in a replicating portfolio perspective

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Abstract

In the last decades, a vast body of literature has arisen on real option analysis (ROA). The use of different approaches and the often implicit adoption of major assumptions may cause confusion on what ROA precisely entails, or in which situations it may be applied.

We assess the field of real option analysis by explicitly linking ROA to the basic principles of option pricing theory and the replicating portfolio concept. From this perspective, we explain how real options adjust to the varying risk profiles of a project, a feature not available in other valuation methods. We also clarify how non-market risks can be dealt with in ROA. We show that a combination of option pricing and decision tree analysis enables us to treat a broad range of investment problems, in a manner that is consistent with pricing theory.

Keywords: Real option analysis, private risk, risk-neutral valuation, replicating portfolio.

1 Introduction

Black & Scholes [1973] introduced an option pricing model allowing market participants to value financial options by applying risk-neutral valuation under a set of restrictive assumptions. This model and subsequent work had a large impact on financial markets; traders started to rely more on mathematical valuation and the implications of market prices. Not long after the introduction of the Black-Scholes model, Myers [1977] recognized its potential to describe real-world investment opportunities under uncertainty.
as well. By considering the value of a project as the underlying asset, the required investment as the strike price, and the opportunity to defer a decision as the right to invest, one could apply option pricing techniques to real investment opportunities. This approach was baptized ‘real’ option pricing as it applied option theory to real-world projects instead of financial assets.

As authors from different research areas rallied to develop real option analysis (ROA) further, a vast body of literature exists by now, containing many variants as well as theoretical assessments (see e.g., Dixit & Pindyck [1994], Trigeorgis [1996], Copeland & Antikarov [2001]). However, ROA is still waiting for a major breakthrough in corporate decision making. We believe one of the reasons for this may be that there is little consensus on what ROA stands for precisely, making it unclear to a practitioner which version should be applied to an investment problem at hand.

The state of the field is that many different approaches to ROA coexist, often without the underlying assumptions and their implications being pointed out explicitly. Decision makers may sense that the assumptions required for financial option valuation are too rigid to apply in the real world, and do not see their concerns addressed in literature (Smith & McCardle [1999]). Also they may not understand in what aspects ROA could yield an improvement over the advanced decision tools they are using already.

When practitioners try to apply ROA on an investment problem using a standard option valuation model, e.g., Black-Scholes, vital assumptions are rarely satisfied for real-world projects. Most importantly, option theory presumes that all risks are liquidly traded on the financial market, and can therefore be hedged, which does not hold for most projects. In general, investment problems are much too complex to be modeled as a standard option, hence the option model must be tailor-made, with standard assumptions no longer applicable. So, application of ROA requires a set of assumptions not as restrictive as for financial options, while retaining the merits of structuring investment problems as real options.

To contribute to solving these impediments for practitioners, we first aim to provide a better insight in what ROA stands for, under which assumptions it can be applied, and how it solves inconsistencies existing in other decision tools. To do this, we return to the basics of option pricing theory, namely the concepts of risk-neutral valuation and replicating portfolios. From this perspective, we compare ROA with the net present valuation (NPV) techniques dominating state-of-the-art practice. NPV techniques have a fundamental theoretical flaw by assuming a constant risk profile for projects incorporating managerial flexibility, and we show that ROA solves this shortcoming.

Next, we point out which approaches can be used for ROA, and what their implications are, based on Borison [2005]. We limit our assessment of differences between these methods to the treatment of non-tradable (‘private’) risk, and expand on the ROA approach of Smith & Nau [1995] to value projects comprising both market and private risk consistent with theory.
1.1 More on real option analysis

Real option analysis is a methodology to value real-world projects by modeling decisions in an option pricing framework. Its application is based on the theory used to value options on financial assets (Luenberger [1998]). In finance, a standard option is the right, but not the obligation, to buy (call option) or sell (put option) an asset at a predefined price, called a strike (price). This allows the holder of the option to defer the investment decision up to a certain date, waiting for new market information (i.e., the asset price) to arrive. A rational holder of an option will only exercise the option if the asset price exceeds the predefined strike price at the decision point.

If the option is not exercised before maturity, the investor loses the cost of the option itself. The so-called ‘classic’ ROA uses an approach highly similar to that of financial options. When the underlying risk of a project behaves as if it is traded, we can apply option pricing theory on real investment decisions. Two conditions required to apply option theory are that the uncertainty associated with the project is market risk (the value-influencing factors are liquidly traded) and that the decision maker has the managerial flexibility to make investment decisions based on new information. In the view we deploy here, pure option theory should only be applied to the part of a project’s risk that is actually traded on the market.

We now illustrate the analogy between financial options and real options by the Black-Scholes option pricing model, which is an application of risk-neutral pricing under strict assumptions (Black & Scholes [1973]). Merton [1998] warned against the application of option theory to real world problems. He stressed to consider the limitations of the model, and keep in mind what purpose it serves. The main limitations and assumptions of classic real option pricing will be assessed in detail in the remainder, along with alternatives that are less restrictive.

The Black-Scholes formula can be used to obtain the value of a European option, i.e., one that can be exercised only at maturity. The major assumptions for the Black-Scholes model and the resulting formula, are the following.

- No arbitrage opportunities exist.
- Cash can be borrowed and lent at a constant risk-free interest rate.
- Buying and short-selling of the underlying asset is unrestricted.
- No transaction costs exist.
- The underlying asset’s price follows a lognormal distribution.
- The underlying asset does not pay dividends.
Under these assumptions we can create a hedged position, so that the value of the portfolio does not depend on the price of the underlying asset. We do this by constructing a portfolio consisting of the option, the underlying and cash (including negative amounts due to short-selling), so that price changes of the asset are offset by the other instruments. It is then possible to apply risk-neutral valuation.

Translated to real options, a call option is the possibility to undertake a project; a put option is the possibility to abandon it, or rather to abstain from it. In real options, the term ‘asset’ should be viewed in a broad sense. It is the value of the project, should it be taken up. We present modifications of the original Black-Scholes formula for the value of call and put options at time $t$, including the effect of continuous dividend payments as well, as they form an integral part of many options. The formulas are the following:

\[
\begin{align*}
\text{call} & = S_t e^{-\delta(T-t)} N(d_1) - X e^{-r_f(T-t)} N(d_2), \\
\text{put} & = X e^{-r_f(T-t)} N(-d_2) - S_t e^{-\delta(T-t)} N(-d_1),
\end{align*}
\]

- $N(\cdot)$: the CDF of a standard-normal distribution,
- $d_1 = \frac{\ln \left( \frac{S_t}{X} \right) + \left( r_f - \delta + 0.5 \sigma^2 \right)(T-t)}{\sigma \sqrt{T-t}}$, $d_2 = \frac{\ln \left( \frac{S_t}{X} \right) + \left( r_f - \delta - 0.5 \sigma^2 \right)(T-t)}{\sigma \sqrt{T-t}}$.

The meanings of the other symbols in terms of financial and real options are provided in Table 1 (Leslie & Michaels [1997]).

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Financial options</th>
<th>Real options</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X$</td>
<td>Strike price</td>
<td>Present value (PV) of required expenditures to exercise the option</td>
</tr>
<tr>
<td>$S_t$</td>
<td>Stock price</td>
<td>PV of expected net cash flows at $t$</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>Volatility of $S_t$</td>
<td>Volatility of $S_t$</td>
</tr>
<tr>
<td>$t$</td>
<td>Current period</td>
<td>Current period</td>
</tr>
<tr>
<td>$T$</td>
<td>Time to expiry</td>
<td>Time that decision is deferred</td>
</tr>
<tr>
<td>$r_f$</td>
<td>Risk-free interest rate</td>
<td>Risk-free interest rate</td>
</tr>
<tr>
<td>$\delta$</td>
<td>Fixed cash dividends</td>
<td>Costs to preserve the option</td>
</tr>
</tbody>
</table>

Table 1: Symbols of the Black-Scholes model in financial and real options.

Nielsen [1993] provides a detailed explanation of the logic behind the Black-Scholes model. We restrict ourselves to a brief rationale for the call option; the one for put options is comparable. For the call option, $N(d_2)$ is the risk-adjusted probability that the option will be exercised, such that the strike price must be paid. $N(d_1)$ can be viewed as the factor by which the expected payoff exceeds the current stock price. As exercise occurs at maturity, payoff and strike price are discounted for dividends and interest respectively. The difference between both terms is the option’s value.
In real options $S_t$ represents the present value at time $t$ of the expected net cash flows, should the option be exercised. The strike price $X$ describes the present value of the expenditures required to exercise the option (Carlsson & Fullér [2003]). These costs are only incurred if the option is actually exercised, such as the costs to acquire an asset (call option) or to abandon a project (put option).

The volatility $\sigma$ is defined as the square root of the variance of the project returns based on the free cash flows. Returns are assumed to follow a Geometric Brownian Motion (i.e., normally distributed and unrelated over time, standard deviation remains constant). The option value increases with volatility, as an option holder profits from favorable movements of the value of the underlying, while downside risk is limited to losing the option value.

In line with option pricing theory, cash flows in ROA are discounted using the risk-free interest rate $r_f$. Finally, for real options, ‘dividends’ $\delta$ represent the costs to preserve the option, or the money draining away during the lifetime of the option (Leslie & Michaels [1997]).

Clearly, the rigid structure of the Black-Scholes model does not suit many real-world investment problems well. We discuss common points of critique on the assumptions posed, and explain how these may be overcome by adopting a less restrictive approach. In the final section of this paper we discuss potential discrepancies between ROA valuation and practice.

A simple European call- or put option can be exercised at the maturity date only, generally making these options unfit to capture the flexibilities embedded in a project. The project might comprise exercise and abandonment decisions at different time points, multiple investment opportunities, strike prices variable over time, time-varying volatility, etc. (Trigeorgis [1993a], Mun [2002]). Consequently, often no analytical solutions can be found. Instead, numerical approaches such as a binomial tree or simulation should be applied (Cortazar [2000], Wood [2007], Fuji et al. [2011]). These methods approximate the option value by dividing its partial differentials in many steps, and allow much more flexibility than analytical methods to value complex options. Therefore, even strong deviations from standard option models need not be considered an obstacle when applying ROA.

The risk-free rate $r_f$ is the (theoretical) return required when an investment has no possibility of default, providing a compensation for the time the invested capital is tied up only. We treat the subject of discounting in detail in Section 3. We apply risk-neutral probabilities to calculate the expected risk-neutral cash flows before discounting them. However, as follows from the assumptions, a hedged position can only be formed for assets which are liquidly traded on the market. Private risk cannot be hedged and as such should not be discounted by $r_f$; the exposure to riskiness calls for a higher discount rate. We address this significant problem in Section 4, and describe a hybrid between option valuation and decision tree analysis applicable to both market and private risks in a manner consistent with theory. In Section
we discuss the practical implementation of risk-neutral pricing, making use of futures’ contracts to estimate risk-neutral drifts.

Examples of dividends in ROA are payments to preserve production rights and money lost through competition. In practice, it might be difficult to forecast and estimate the leakage of cash over the length of the option. Also, losses are generally not constant over time (Trigeorgis [1996]). Some real option practitioners therefore act as if no dividend payments exist, that is, $\delta = 0$ (Davis [1998]). However, for liquidly traded risks it is possible to estimate dividends based on information embedded in futures prices, as we show in Section 6. Furthermore, it should be noted that flexible valuation techniques such as simulation are very well capable of incorporating even complex dividend patterns.

A final difference between financial options and real options we wish to point out is that competitors may have a significant impact on the value of a real option. As opposed to financial options, strategic decision-making (e.g., acting as a leader or a follower) could therefore influence the value of real options as well. Chevalier-Roignant & Trigeorgis [2011], Grenadier [2000], and Huisman [2001] describe how real option analysis and game theory can be combined to address such problems. We do not further treat this subject in this paper, but one should be aware of the influence of strategic decision-making in project valuation.

For the remainder of the paper we take a broad view on ROA, in line with authors such as Dixit & Pindyck [1994] and Dias [2012a]. We do not consider ROA as a pricing technique such as the Black-Scholes model, but rather as a methodology based on risk-neutral valuation. We support the view of Smith & Nau [1995], who modify the concept of risk-neutral valuation to make it applicable to projects containing both market and private risk. We expand on this approach later.

## 2 Comparing ROA and discount-based approaches

The most commonly used valuation methods are based on discounted cash flow (DCF) principles (Drury [2008]). Using such methods, the expected cash flows during the lifetime of the project are estimated and subsequently discounted over time. The discount rate applied should incorporate both the time value of money and compensation for uncertainty of future cash flows (Robichek & Myers [1966]). This rate has a profound impact on the NPV of a long-term project. In this section we address the theoretical background of discount rates and point out the theoretical flaw in DCF methods and its implications for risk adjustment. In the next section, we show how risk-neutral valuation as used in ROA resolves these issues.

It is nearly impossible to obtain a discount rate able to reflect accurately all the risks a project is subject to (Mun [2002]). To mention some, a
project’s value may be influenced by inflation, the size of the company, credit risk, country risk, shareholder decisions, etc. Many random events can occur during the lifetime of a project, making it very hard to derive a proper discount rate analytically. Therefore, we generally resort to discount rates which try to capture how the capital providers perceive the risk they are subject to when investing in the project.

In practice, the most commonly used discount rate is the Weighted Average Cost of Capital (WACC). This is the average cost of capital for the company or the project. In its basic form the WACC assumes that a company is funded with one source of equity and one source of debt, both demanding a single constant return. In reality, companies may raise money from multiple sources requiring different expected returns (e.g., preferred stocks, warrants, etc.); more expansive versions of the WACC could then be applied. Interest costs are deducted from corporate profits, hence the inclusion of corporate tax in the equation. This boils down the following formula:

$$WACC = \left( \frac{E}{E+D} \right) \cdot r_e + \left( \frac{D}{E+D} \right) \cdot r_d \cdot (1 - C)$$

- $E, D$ : the market value of equity respectively debt,
- $r_e, r_d$ : the cost of equity respectively debt,
- $C$ : the corporate tax rate.

We advocate to calculate the WACC for the project itself, regardless of the way it is funded. First, suppose that a project is funded separately, and also pays off as a standalone project. The project must at least earn its WACC to satisfy owners, stock holders and creditors. If the project fails to do so, by assumption rational investors would not be willing to invest in it. Hence, a project discounted at the WACC should at least have an NPV of 0 in order to be attractive to capital providers. The project may have a different risk profile than the company as a whole, meaning that investors would require a different return than the WACC of the company (Mun [2002], Smith [2005]). Alternatively, a project might not be funded separately, but instead have a budget allocated from the company’s means. If this is the case, the project will alter the overall risk profile of the company, because it changes the investment portfolio.\(^1\) By estimating the correlation of the project with market risk, the discount rate can be adjusted better to a specific project (Constantinides [1978], Magni [2007]). Hence, regardless of the manner of funding, using a discount rate based on the project itself yields more accurate and insightful results.

\(^{1}\)Recalculating the firm’s $\beta$ and WACC is rather straightforward and will not yield large differences provided the project’s requirements in assets is small relative to the firm’s.
To calculate the cost of equity, models such as the Capital Asset Pricing Model (CAPM) may be applied. The CAPM states that the expected return of an asset is equal to the risk-free rate plus a market risk premium depending on the relationship between the volatility of the asset’s return and that of the market return (Sharpe [1964], Merton [1973b]). The underlying reasoning of the model is that investors only care about the systemic risk (related to the movements of the market as a whole) of the asset, as all other risks can be diversified away. Diversification means that private risks are offset by holding many uncorrelated assets in a portfolio. The expected return under the CAPM is denoted as

\[ E(r_e) = r_f + \beta (E(r_m) - r_f) \]

with

- \( r_m, r_f \): the market return respectively the risk-free interest rate,
- \( \beta = \frac{\text{cov}(r_e, r_m)}{\text{var}(r_m)} \): the ratio of the covariance of the asset return and market return, and the variance of the market return.

So, \( \beta \) can be viewed as capturing the volatility of the asset relative to the volatility of the market. In other words, it is a measure for part of the asset’s riskiness that cannot be removed through diversification. The risk premium for the asset is given by the term \( \beta (E(r_m) - r_f) \). It follows that an appropriate discount rate for a project depends on the market return, the risk-free rate, and the beta of the project (Ang & Liu [2004]). Though treated as constants, these factors are all variable over time in reality, implying that the discount rate should be time-varying as well. The use of a constant discount rate might be rationalized to some extent by assuming that the portfolio investment opportunity and the systemic risk exposure (i.e., \( \beta \)) remain constant over time (Merton [1973b], Fama & Schwert [1997]).

By incorporating stochastic forecasting models on the aforementioned factors, we could obtain a more realistic \( r_e \) (Geltner & Mei [1995], Schultemerich [2010]). Generally, the theoretical parameters \( r_f \) and \( r_m \) are estimated with bond yields and market indices. The bond should approximate an investment which never defaults, with the same maturity and in the same currency as the investment to exclude currency risk. The market index chosen should represent the portfolio of the investment as well as possible; yet it should be kept in mind that this portfolio should be well-diversified to apply the CAPM.

To finish our assessment of the discount rate, it is essential to note that the risk premium is calculated for the asset (by analogy, the project value); the opportunity to invest in the project (i.e., the option) is subject to a different risk profile, varying with the decisions made. So, the discount rate for a project with embedded flexibilities should be adjusted accordingly. ROA applies risk-neutral valuation instead; by already accounting for risk when estimating the cash flows, \( r_f \) can always be used as the discount rate.
We briefly discuss two discount-based methods, Net Present Value (NPV) analysis and Decision Tree Analysis (DTA). Traditional DCF analysis assumes that future cash flows are deterministic, as soon as the investment decision is made. To reflect both the time value and the riskiness of the project, a constant discount rate is applied to future cash flows. This results in the Net Present Value (NPV), the net worth of the project at time of initial investment. Usually the WACC of the firm is used as the discount rate, and sum of discounted cash flows is the NPV of the project. A positive NPV may be interpreted as a signal to accept the project.

Similar to ROA, Decision Tree Analysis (DTA) incorporates uncertainties and intermediate decision-making in valuation. The nodes are connected in a graph, which paths indicate the change of the project value over time. A distinction is made between decision nodes and uncertainty nodes. Project options are defined as decision nodes to allow managerial flexibility, while uncertainty nodes reflect chance events with certain probabilities assigned. So, decisions may depend on the outcome of chance events. Similar to DCF, future cash flows are discounted by a single discount rate, so DTA can be seen as an enhanced version of DCF (Piesse et al. [2004]). Instead of evaluating a single aggregated scenario, each path is viewed as a possible scenario. As stated before, the discount rate reflects both time value and riskiness.

Traditional DCF assumes that decisions are irreversible, with new information getting available at a later time not altering the cash flows or intermediate decisions made. This is often not realistic, as management has the opportunity to reallocate capital based on the performance and prospects of the project. Active risk management may allow both increasing upside potential and limiting downside potential, leading to a higher expected project value. These effects are ignored in traditional DCF (Prasanna Venkatesan [2005]), making this method unfit for projects with embedded flexibilities.

Real options address several aspects ignored in DCF (Triantis & Borison [2001], Van de Putte [2005]). A real option values flexibility as it includes the possibility to alter the course of the project at the decision points in order to maximize profit or minimize losses given the information available at that time (Copeland & Keenan [1988], Mun [2002], Brandão et al. [2005]).

Compared to ROA, DTA falls short when it comes to risk-adjustment. In a decision tree, chance events and decisions are represented by nodes. A flaw in the DTA approach is that the decisions made over time alter the risk profile of the project, which conflicts with the application of a single risk-adjusted discount rate used to calculate the present value (Brandão et al. [2005]). Investors expect compensation in line with the degree of risk they are exposed to; if the company decides to change course, this will affect expectations. In ROA, this adjustment takes place via risk-neutral valuation, such that an opportunity is valued in line with its risk profile. We clarify this adjustment in the next sections, hence the present example will be continued there for the ROA part.
Example 1 We now illustrate the valuation techniques of NPV and DTA. Say that we have the right to exploit an oil field of unknown size, i.e., it may be ‘small’ (10m barrels), ‘medium’ (20m barrels) or ‘large’ (40m barrels). The associated probabilities are $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{4}$, respectively. At the costs of 50 million dollar (m$), this field may be explored to determine how much oil is present in year 0. If the field is to be exploited, an investment of 800 m$ in year 1 (regardless of field size) is necessary. All cash flows are, for the sake of simplicity, assumed to be end of the year cash flows.

In addition, future oil prices are unknown; with probabilities $\frac{1}{2}$ prices may be ‘low’ (80$ per barrel) or ‘high’ (120$ per barrel) during exploitation. Depending on technological advance, a new technology may be available at the start of the project to reduce extraction costs. The variable extraction costs may be 60$ or 40$ per barrel with a probability of 80% respectively 20%. The field is to be exploited in three years; 20% in year 1, 50% in year 2, and 30% in year 3. Finally, the WACC is 10%.

Figure 1: Structure of the investment scenario for DTA. Dark squares indicate chance events, light squares indicate decision moments, the open circles indicate outcomes. The discounted profits depend on the field size (Large, Medium, Small), the price of oil (High, Low), the technology (New, Old) and the decision whether to exploit (E) the field or not (NE).

With traditional net present valuation, we take the expected values for the field size (20m barrels), oil price (100$ per barrel) and variable costs (56$ per barrel). In the subsequent three years, we then get expected annual cash flows of $4 \cdot 44 - 800 = -624$ m$, 10 \cdot 44 = 440$ m$ and $6 \cdot 44 = 264$ m$. Hence, these
Cash flows result \((WACC = 0.1)\) in an NPV of \(-\frac{624}{1.11} + \frac{440}{1.12} + \frac{264}{1.13} = -5.2893\) m$. Based on this criterion the project should not be taken up.

Next, we apply DTA to the same investment problem; see Figure 1 for the corresponding decision tree, where a crucial assumption is that the decision to test has already been taken in advance and was positive.\(^2\) For DTA, we now calculate the NPV for each branch in the tree, and optimize our decision for each branch. The actual numbers and calculations may be found in the Appendix. Thus, it is possible to determine the discounted project value when accounting for the option to cancel the project after exploration. In our example, the value of the project under DTA and the testing-first assumption increases to approximately 177 m$. This significant improvement of the value found under DTA and NPV stems from the added value of flexibility. Under DTA the project would stand a good chance of being taken up.

3 Risk-neutral valuation

An essential concept in option pricing is risk-neutral valuation to obtain the value of derivatives (Appedu et al. [2012]). Recall that the Black-Scholes model is an application of risk-neutral valuation under strict assumptions. We focus on the main principles of risk-neutral valuation, providing an intuitive insight why it is used. For the mathematical properties of risk-neutral valuation, we refer to Luenberger [1998] or Bingham & Kiesel [2004].

To calculate the present value of an asset, one could take the expected return of an asset, and then discount it based on the preferences of the investor. However, it is difficult to estimate the future growth rate of an asset’s value. Risk-neutral valuation provides a methodology which does not require estimating this rate (Miller & Park [2002]). The application of risk-neutral valuation requires two major assumptions. First, the market must be complete, meaning that every good can be exchanged by any participant in the market without transaction costs (Merton [1973a], Constantinides [1978]). Every agent has perfect market information, so no trader has an advantage through knowledge. Also short-selling and borrowing are unrestricted and can be done at the risk-free rate. Second, arbitrage opportunities are absent; there are no imbalances in the market which allow for the possibility of a risk-free profit at zero cost. When these assumptions hold, a derivative can be replicated by holding a linearly weighted combination of financial instruments (Gisiger [2010]). As arbitrage opportunities cannot exist, this linear combination must have the exact same value as the derivative. If this were not the case, an investor could buy the cheaper of the two and sell the more

\(^2\)DTA would be analogous but considerably more involved as now also an optimal timing issue arises for when to do the test, or whether to do the test at all. Without going into much detail, we found that such an analysis yields a value of 198 m$, a considerable improvement over the number found under the assumption of testing in advance.

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expensive one, thus making a risk-free profit without a cost (Tilley [1992]). Risk-neutral valuation provides the unique arbitrage-free price of the derivative based on this principle. It does so by using the artificial concept of risk-neutral probabilities.

We provide an example based on Gisiger [2010] to explain this concept. Suppose that the economy can be in one of $n$ states at time $t$, with a specific state denoted as $j \in I = \{1, ..., n\}$. For every state a unique so-called Arrow security is available, which pays off a positive amount $x_j$ to the holder of the security when the asset reaches state $j$ and zero otherwise. The real probability that the asset state will shift from an arbitrary state $i$ to state $j$ is denoted by $p_{ij}$, with $\sum_{j \in I} p_{ij} = 1$. We do not assume interest yet. Each security has a price representing the value the market places on this state. This means that the state price need not to be equal to its rationally expected payoff $p_{ij} \cdot x_j$; the market incorporates risk preferences. A security paying off in a certain state could be perceived as a more valuable addition to one’s portfolio (for example because it pays off in a declining market), therefore being priced higher than its rationally expected payoff. The discrete payoff structure described here is illustrated in Figure 2.

![Figure 2: Example of a discrete payoff structure in an economy with $n$ Arrow securities (based on Gisiger [2010]).](image)

Now we introduce a derivative, which returns the payoff of the security matching state $j$. This derivative can be considered as a portfolio of all Arrow securities, priced by using risk-neutral valuation, denoting its value as $\theta_j$. Contrary to what its name may indicate, risk-neutral valuation does not assume investors to be indifferent to risk. Risk-neutral probabilities can be viewed as the sum of state prices (i.e., incorporating risk preferences) compounded to 1; denote these probabilities by $a_{ij}$. Multiplying each risk-neutral probability with the corresponding payoff $x$ in state $j$ provides the
value of the derivative. Note that the real probabilities $p_{ij}$ are not required for this, as their information is incorporated in the security prices. For example, a security yielding a high payoff with large probabilities will most likely have a high price as well, though such a relation need not be linear.

So far, we assumed that an investor is indifferent between receiving money at time 0 or at a later time. However, money has a time value due to time preference of people (they prefer money now over money at a later point in time), leading to the existence of interest. If we would not discount future cash flows, an investor could short-sell the complete set of securities and use the received sum to purchase a risk-free bond, earning the risk-free rate as the price of the securities remain constant. As such, the investor could make a risk-free profit without cost, which contradicts the no-arbitrage assumption. Thus, future payoffs should be discounted at the risk-free rate $r_f$ to obtain the arbitrage-free price of today. The discounted state prices at time 0 then sum up to $\frac{1}{1+r_f}$ instead of 1. The introduction of time value leads to the following equation:

$$\theta_i = \frac{1}{1+r_f} \sum_{j=1}^{n} a_{ij} x_j.$$  

Hence, under risk-neutral valuation the expected prices grow at the risk-free rate. This is a powerful concept, as we are no longer required to estimate the actual growth rate. The drift of the asset value is effectively removed, instead replacing it with the risk-free rate (Kat [1998]).

In a complete market, risk-neutral valuation and valuation under perfect delta hedging\(^3\) (see Section 5) provide the same derivative price. A perfectly hedged portfolio is riskless and as such must provide a risk-free return. Delta hedging and risk-neutral valuation are therefore mathematically equivalent. This helps understanding why the rather artificial risk-neutral valuation principle also applies to the real world when hedging is possible.

If the assumption of a complete market does not hold, the risk-neutral probabilities are not unique. As the derivative in that case cannot be fully replicated by holding securities, no single arbitrage-free price can be obtained for the derivative (Gisiger [2010]). Instead, the value of the derivative will lie between some lower and upper bounds. When calculating the risk-neutral value of an option, uncertainties which cannot be hedged are therefore theoretically not viable for risk-neutral valuation (Dixit & Pindyck [1994], Smith & Nau [1995]). Also other assumptions of the complete market often do not hold in practice. Such issues are sometimes addressed by assuming that the market is approximately complete. In real option settings, the incompleteness of the market may well be too substantial for such

\(^3\)Delta hedging is the practice to reduce exposure to movements in the underlying asset’s price by taking a reverse position in that asset. Under perfect delta hedging, the investor is indifferent to changes in the asset price.
an assumption to hold. Though some authors provide rationales to apply option pricing on an incomplete market, it is theoretically more correct to apply risk-neutral valuation only on risks traded on the market. Section 5 goes in more detail about handling non-hedgeable risks.

Though the assumptions of risk-neutral valuation may sound strong, they are no more restrictive than those adopted for discount-based approaches. In fact, the assumptions for the CAPM and risk-neutral valuation are the same (Birge & Zhang [1998], Cudica [2012]). Thus, accepting DCF methods based on CAPM principles means that the assumptions for risk-neutral valuation should be accepted as well. Next, we show how the possibility to hedge risk allows using risk-neutral valuation in practice.

4 Replicating portfolio concept in ROA

Risk-neutral pricing presumes that a perfect hedge can be constructed for the portfolio held. If this assumption holds, it is possible to construct a replicating portfolio for the project. In that case, holding a portfolio consisting of financial instruments should provide the exact same payoff as the project itself at all times and in all states. We can then also construct a perfect hedging portfolio by mirroring the replicating portfolio (short-selling may be required for this), allowing to apply risk-neutral valuation.

To retain the equivalence between the real project and the replicating portfolio, (continuous) adjustment of the portfolio might be required. We can do this under the assumption that no transaction costs exist in a complete market (Tilley [1992]). In reality, transaction costs are of course present in trading. Therefore, some argue that the rigid complete market assumption significantly affects the validity of the theory (e.g., Mayshar [1981], Haug & Taleb [2011]). Constantinides [1986] justifies the assumption of no transaction costs by stating that the existence of transaction costs does not significantly alter the asset proportions held compared to the theoretical proportions. At the very least, we should keep in mind that we assume the absence of transaction costs when applying option pricing (Merton [1987]).

Another deviation from the complete market observed in practice is the presence of arbitrage opportunities arising from market imperfections. Such opportunities tend to be quickly corrected by the market itself (Tham [2001]), this phenomenon in fact justifies the assumption that a single correct market price exists.

In reality a complete replicating portfolio is rarely found for a project, as only part of the factors influencing its value is traded on the market. One can distinguish market risk and private risk. Market risk can be replicated by financial instruments; it is assumed that individual companies have no influence on it. Market information is revealed over time, thereby solving uncertainty. An example of such risk is the one caused by changing com-
modity prices. It can be hedged by taking a position in these assets.

Private risk comprises all sources of uncertainty that cannot be replicated by financial instruments (Amram & Kulatilaka [2002], Piesse et al. [2004]). Merton [1998] provides a formal definition of private risk, stating that private risk can be measured as the tracking error of the portfolio representing the underlying asset. Mathematically the tracking error can be defined by $\frac{dS_t}{St} - \frac{dP_t}{Pt}$ where $S_t$ is the project value (the underlying) and $P_t$ is the value of the replicating portfolio, both at time $t$. As such, the difference between the value of the replicating portfolio and the value of the underlying asset is considered private risk. Borison [2005] distinguishes five real option approaches, which differ regarding their perspective on dealing with both types of risk. We focus only on their theoretical fundamentals, ignoring differences such as the techniques applied. In our view the integrated approach is the most correct application of ROA, and is preferable over the other methods.

Classic ROA is based on the assumption that the project can be replicated by a portfolio of market-driven instruments that is exactly equivalent (Brennan & Schwartz [1985], Amram & Kulatilaka [1999]). As stated before, such projects rarely exist. Two rationales are used to justify incorporating a certain degree of private risk in classic ROA. It may be presumed that private risk is only minor after the option has been exercised, and will not have a great impact on the payoff (i.e., the market is approximately complete). The tracking error then increases with the amount of private risk. The alternative rationale is to include such uncertainties in the valuation process, but assume that they can be hedged as well. It might be possible to diversify away private risk by trading it with comparable risks, even though these are not liquidly traded on the market (Mattar & Cheah [2006]).

We argue that these rationales fall short for projects containing a significant amount of private risk, and do not recommend using the classic approach in these cases. In an attempt to solve this shortcoming, the revised classic approach proposes to use decision tree analysis when private risk is dominating and option pricing when market risk is dominating. This approach provides only a crude approximation to the project value, and is unable to solve the described theoretical issues.

Some assume that a replicating portfolio can also be derived by subjectively estimating the market value of the project (e.g., Copeland & Antikarov [2001], Amram & Kulatilaka [2002], Brealey et al. [2008]). They justify the subjectively derived asset value by adopting a shareholder view. Their valuation assesses how much a project contributes to the value of the firm, thereby considering the project itself as if it were a traded asset (Borison [2005]). They value the project with traditional DCF (hence without incorporating flexibility) to obtain a subjective estimate of the market value of the project. Some authors deem this value to be the best unbiased estimate, coining the assumption Market Asset Disclaimer or MAD. Although the un-
derlying of a real option is generally not liquidly traded, one may choose to treat it as if it were a financial asset. The rationale is that we seek the arbitrage-free value of the project, as this is comparable to the added value of the project to the market value of the company (Benaroch & Kaufmann [1999]). Wrongly valuing the project would eventually result in arbitrage opportunities which are corrected by the market.

Although the subjective approaches\(^4\) are not as restrictive as the classic one, market completeness remains crucial. Note that using both notions is conflicting by nature; if the market were indeed complete, we would not need subjective estimates but market data to obtain the correct market price. The subjective approach is therefore internally inconsistent, making it hard to justify using this form of ROA.

Finally, the integrated approach considers the market to be partially complete when private risk is incorporated in the project (Smith & Nau [1995], Smith [2005]. Cox et al. [1985] provide a description of a market model which allows applying integrated ROA in a theoretically consistent manner. They adopt the viewpoint of a rational and well-diversified shareholder as described in the CAPM framework.

This shareholder approach is in line with maximizing the market value of the company, which we consider to be a rational objective for real option valuation. Shareholders are assumed to agree with the subjective assessment of management of private risk. Under the assumption that sources of private risk are uncorrelated with the market, their real probability distributions estimated by management are consistent with the risk-neutral approach for well-diversified shareholders. This follows from having $\beta = 0$ in the CAPM, so that shareholders require no additional return on private risk. The risk-neutral distribution is then equivalent to the real distribution. Though not requiring a premium on private risk, reducing private risk leads to better investment decisions, as such increasing value to investors. Therefore private risk should definitely not be viewed as unimportant; yet from a portfolio point of view, the private risks of many assets tend to be (partly) uncorrelated, resulting in a reduced risk of the total portfolio, relative to its expected return. An issue often not assessed in literature is that risk may be correlated with the market, but that no derivative exists for it (Kaufman & Mattar [2002]). For this type of risk, the probability distribution lies between the real distribution and the risk-neutral distribution. We do this by subtracting the risk premium from the drift of the private risk for the part correlated to the market. Within the described market model, the integrated method results in a single theoretically correct option price.

The risk-neutral integrated approach is consistent with a shareholders perspective, and should maximize value for this group. However, projects are not necessarily funded by rational, well-diversified shareholders who are\(^4\)Borison [2003] distinguishes subjective from MAD approaches.
only exposed to systemic risk. Instead, a project may be funded by one or more investors who invest a significant portion of their capital, making them unable to diversify away private risk. In that case the risk-free discount rate would no longer apply to non-systemic risk. The risk preferences of the investors then become of importance, as they have to make an individual assessment of the trade-off between risk and expected return.

The recognition that private risk may not always be diversified away is important to accept the use of real options as a decision tool. It allows removing the assumption that private risk requires no premium if applicable, so that we may apply ROA on a much broader range of investment problems.

Smith & Nau [1995] and Luenberger [1998] propose to perform so-called ‘buying price analysis’ in the case of non-diversifiable private risk, making use of an personal exponential utility function to obtain the unique certainty equivalent of cash flows. We describe this utility function as

\[
U(S_T) = e^{-\gamma(1+r_f)^T}S_T
\]

where \(U(\cdot)\) describes the utility function, \(S_T\) is the stochastic project value at maturity \(T\), \(\bar{S}_T\) is the risk-neutral value of the project value, and \(\gamma\) is a risk aversion coefficient larger than 0. We find the risk-neutral project value by setting equal \(U(\bar{S}_T) = E[U(S_T)]\). This exponential utility function implies a constant absolute risk aversion. It follows that the slope of our utility function decreases when \(S_T\) increases, i.e., our marginal added value diminishes. This is in line with a risk-averse perspective of the investor.

Utility is adjusted for time, this is because the value \(S_T\) would be worth more when received at an earlier time. We may discount the obtained risk-neutral values at the risk-free rate, as they are corrected for risk preferences. This way, we can obtain the real option value by using risk-neutral valuation.

5 Cash flow risk adjustment by futures contracts

The essential characteristic of ROA is that it adjusts the discount rate to the varying risk profiles of the project (Triantis & Borison [2001], Mun [2002], Arnold & Crack [2004]). The expected future cash flows are adjusted for their risk, obtaining their risk-neutral equivalent instead. In such a way a risk-neutral distribution is created, allowing for risk-neutral valuation by discounting the risk-adjusted cash flows at the risk-free interest rate. As an option is a leveraged instrument, it has a more risky profile than the underlying asset (Cudica [2012], Dias [2012b]).

If we were to work with real probabilities, the discount rates would have to be consistent with the varying risk profile of the option to obtain the same value as with risk-neutral valuation (Birge & Zhang [1998]). Risk-neutral valuation is generally much easier to implement. Later on we explain how to use the information embedded in futures contracts when performing ROA.
The cash flows of a project can partially be replicated by one or more market assets for which we are required to estimate the risk-neutral growth rate; recall that private risks are estimated subjectively in the integrated approach. We illustrate the concept of risk-adjustment with the CAPM, showing how the expected growth rate of an asset is composed. Say that we have an asset with return $r_a$ and volatility $\sigma$, and $r_f$ is constant. Recall that the expected return on an asset is given by $r_f + \beta(E(r_m) - r_f)$.

For a correctly priced asset, the future cash flows generated by the asset, discounted at the calculated discount rate, should result in the spot price. If this were not the case, the no-arbitrage assumption would be contradicted. It follows that if we presume the spot price of an asset to be correct, the CAPM provides its expected growth rate, here denoted as $\mu$. The CAPM risk premium may also be expressed as $\lambda \sigma$, such that we get $\mu = r_f + \lambda \sigma$ (Smith [2005], Samis et al. [2007]). The market price of risk $\lambda$ is defined by the Sharpe ratio, which essentially measures the excess return received for the volatility the investor is subject to (Constantinides [1978], Sharpe [1994], Saénz-Diez & Gimeno [2008]):

$$\lambda = \frac{E(r_a - r_f)}{\sigma}.$$

For risk-neutral valuation, we are required to estimate future cash flows based on risk-neutral growth rates of the market assets. To obtain the risk-neutral growth of an asset, we should remove the risk premium from the expected real growth rate of this asset (Tilley [1992], Trigeorgis [1993b]). The risk premium is often assumed to be constant over time, in the discussion we revisit this assumption. After calculating the risk-neutral project cash flows, we can discount them at the riskless interest rate, thereby obtaining the present value of the project (Schwartz & Trigeorgis [2001]). We show how we can observe implied risk premiums for forthcoming cash flows based on futures contracts. For common stocks, under the risk-neutral measure, the expected price simply grows at the risk-free rate, leaving out the need to estimate the real-world drift (Cox & Ross [1976]). However, this method is generally not applicable to commodities or stocks paying dividends. As commodity prices are often relevant in real option settings (e.g., raw materials), we describe a general and practical solution to estimate their risk-neutral drifts based on futures contracts.

Users who physically hold a commodity may be able to profit from temporary shortages. This so-called gross convenience yield fluctuates over time, and is based on an inverse relation with inventory levels (Gibson & Schwartz [1990]). Furthermore, when physically holding a commodity, storage costs decrease the return value. Possible costs when holding a commodity are the costs for the storage facility, maintenance, insurance, etc. Deducting the storage costs from the convenience yield provides a cash flow comparable to
a dividend payment, sometimes referred to as the net convenience yield:

\[ \delta = \text{gross convenience yield} - \text{storage costs} \]

We need to account for this dividend-like payment (usually but not necessarily positive) when estimating the drift of commodities. We illustrate this procedure with a set of equations (Trigeorgis [1996], Dias [2012b]). The total expected growth rate for an investor holding the commodity is given by \( \mu = \alpha + \delta \), with \( \alpha \) describing the real drift for commodity price itself (i.e., when only virtually holding the commodity). We previously established that the growth rate of an asset can also be expressed as \( \mu = r_f + \lambda \sigma \). By setting equal these equations, it follows that \( \alpha - \lambda \sigma = r_f - \delta \). We know that the risk-neutral drift (denoted as \( \hat{\alpha} \)) of an asset is equal to its real drift minus its market-risk premium. Thus, the risk-neutral drift of a dividend-paying asset is given by

\[ \hat{\alpha} = \alpha - \lambda \sigma = r_f - \delta. \]

Historical observations may be used to forecast these parameters, but do not necessarily incorporate insights in future developments. Also there is subjectivity in constructing the forecasting models. Estimating the risk-neutral drift based on historical data may therefore not always be in line with the expectations of the market.

A more convenient method to determine \( \hat{\alpha} \) is to assess futures contracts on the commodity, because they implicitly contain information about the risk-neutral drift (Trigeorgis [1996], Luenberger [1998], Casassus [2004]). A futures contract (or simply ‘futures’) is an agreement between two parties to trade an underlying asset at a specified maturity date for a specified price. Futures are standardized contracts traded on the exchange, and often used as a hedging instrument. Settlement of the contract takes place at maturity either physically or financially, the contracts are often traded many times during their lifetime. To ensure that neither party has an advantage when making the initial agreement, there are no up-front costs to enter into a futures contract except for the transaction costs. In a liquid market, the futures price will therefore be adjusted so that the present value of all expected cash flows is equal to 0, otherwise inducing arbitrage opportunities.

First we will consider this mechanism while ignoring dividends. To prevent arbitrage, the futures price should be equal to the expected spot price at maturity (Mandler [2003]). If this were not the case, a risk-free profit could be made by taking a position in the futures contract and an inverse position in the underlying. If the futures price exceeds the current spot price plus the risk-free return until maturity, the investor is cheaper off by buying the underlying now, missing out only on the interest rate had the money been invested in a risk-free bond instead.

A similar rationale applies when the future price is less than the current spot price growing with the risk-free rate. It follows that the futures price
discounted at the risk-free rate must equal the current spot price. Hence, the futures price implies the risk-neutral drift of the asset until maturity, so that we may simple deduce this drift from readily available futures contracts.

For commodities, the relationship between spot price and futures price is often more complex due to the convenience yield (Trigeorgis [1996], Dincer et al. [2005]). As the commodity is not physically held when holding a futures contract, the ‘dividend’ component should be subtracted from the growth rate $r_f$, provided $\alpha = r_f - \delta$. When futures contracts on commodities in plentiful supply are liquidly traded, their real prices are therefore equivalent to the risk-neutral expectation of the spot prices at time $T$. Hence, we can infer the risk-neutral growth rate $\alpha$ when prices of futures contracts with different maturity dates $T$ are available.

Say that the prices of two futures contracts ($F_1$ and $F_2$) are known, with maturity dates $T_1$ and $T_2$ respectively (with $T_1 < T_2$). Expressed as a function of the spot price growing with the risk-neutral drift until maturity, assuming continuous compounding, the values of these contracts are then given by:

$$F_k = Se^{\alpha T_k} \text{ for } k = 1, 2.$$

All values except for $\alpha$ are known, so between $T_1$ and $T_2$,

$$\alpha = \frac{\ln F_2 - \ln F_1}{T_2 - T_1}.$$

The spot price can be considered as a special case of a futures contract, i.e., one at maturity (future and spot prices converge to the same level at the maturity date to avoid arbitrage). So, $F_1$ may be substituted with $S_t$ as well. The calculated drift depends on the futures contracts used in the equation. When many futures contracts with different maturities are available, a futures curve can be constructed which represents the risk-neutral price development over time. The corresponding curve of the expected real spot price lies above the futures curve by a risk premium $\lambda \sigma$.

We have illustrated all steps necessary to perform ROA, we now apply it on the example analyzed earlier under NPV and DTA.

**Example 2** In Example 1, we implicitly assumed that the oil price is currently 100$ per barrel, has a volatility of 20% and a real drift of 0%. Suppose that the correlation of the oil price with the market is 0.5, we then get $\alpha = \alpha - \lambda \sigma = 0 - 0.5 \cdot 0.2 = -0.1$.

Presenting another computation method, suppose we know that the total growth rate $\mu$ for a commodity holder is 15%. Combined with the real drift we can deduce that $\delta = 0.15$, resulting in $\hat{\alpha} = r_f - \delta = 0.05 - 0.15 = -0.1$.

As these methods require estimating the parameter values in the future, it might prove cumbersome to obtain an accurate value for $\alpha$ using these equations. It is more convenient and more consistent with market expectations
to estimate the risk-neutral drift based on liquidly traded future contracts. Suppose that a futures’ contract maturing in one year exists for a barrel of oil, having a value of 90$. Setting $F_1 = S_t$, we obtain

$$
\hat{\alpha} = \frac{\ln F_2 - \ln F_1}{T_2 - T_1} = \frac{\ln 90 - \ln 100}{1 - 0} \approx -0.1.
$$

This number slightly deviates from the other outcomes because we assume continuous compounding in our formula, which we do not in our example.

After obtaining the risk-neutral drift we can calculate the risk-neutral equivalents for the oil price, using the actual volatility of the price. We get

$$
100 + (100 \cdot -0.1) + 0.5 \cdot (100 \cdot 0.2) = 90 + 20 = 110,
$$

$$
100 + (100 \cdot -0.1) + 0.5 \cdot (100 \cdot -0.2) = 90 - 20 = 70,
$$

as the risk-neutral oil prices for the example. Having effectively removed the market risk component, we may discount these prices at the risk-free rate for the remainder of the analysis.

As we now use arbitrage pricing, the expected oil prices grow by the risk free rate $r_f$ instead of their actual drift. We furthermore assume that technological advance is independent from the market. We apply ROA with these new settings. For this, we use the decision tree structure as shown in Figure 1, modify the oil prices to the risk-neutral equivalents 110$ and 70$, and use the risk-free discount rate of 5%. Remaining calculations may be found again in the Appendix. We find a value of the project of 133 m$. The difference between DTA and ROA is caused by each path in the decision structure altering the risk profile for which DTA does not adapt.

6 Discussion

In comparison to traditional DCF, real option analysis has some distinct advantages. Most importantly, it values flexibility, allowing to respond to new information dynamically. However, more advanced decision tools such as DTA are able to deal with stochastic processes and decision optimization just as well as real options do. It would therefore be incorrect to state that real options bring a decisive advantage with respect to embedding flexibility in general. From an academic point of view, we still prefer ROA over DTA, since the latter is theoretically flawed due its application of a constant discount rate on projects with varying risk profiles. The beta in the CAPM is based on the covariance of the project returns with those of the market. As this covariance differs for each decision path, it is inconsistent to apply a single discount rate. In fact, the only fundamental aspect in which ROA differs from advanced applications of DTA is the risk adjustment towards
the different risk profiles, which is done by applying risk-neutral valuation. ROA is therefore more consistent with pricing theory.

The risk-neutral approach used for market risks in ROA has some favorable elements. It allows estimating the risk-neutral asset drift based on futures contracts, while the discount rate can be based on government bond yields. This objective approach incorporating market information is theoretically superior to subjective estimation of real drifts and discount rates, which can be strongly influenced by personal beliefs and preferences and may deviate from how investors would value the project. The application of ROA should lead to investment decisions which are more in line with the expectations of investors, and consequently to decisions which help to maximize the value of the company.

The integrated ROA is precise and theoretically solid, yet requires each source of risk to be evaluated individually. For a correct implementation, the decision maker should take great care in identifying and modeling risk factors that impact the project value. Such a detailed analysis is not always possible or required; in these cases the use of cruder methods may be justified, be it another form of ROA or DTA. Real options are best suited for projects with large market uncertainties and the managerial flexibility to respond to them (Van de Putte [2005], Kodukula & Papudesu [2006]). When it comes to decision making, real options are particularly useful when the NPV of the project without flexibility is close to 0, so that decisions taken are more likely to have a significant impact on the project value. For decisions which are obviously good or bad beforehand, flexibility provides little additional value. It is important that the decision maker actually has the opportunity to respond in a flexible way to new information becoming available. If this is not possible, an approach such as DCF may have a better fit.

The assumption of a constant risk-free rate and market risk premium is flawed, especially when considering projects with a long time horizon. Treating these factors as stochastic variables could increase realism in project valuation. More research on such stochastic models and their implications is required. In particular, codependencies between the risk-free rate and the risk premium may have profound implications for ROA. Analysis of historical futures contracts in conjunction with changes in the historical risk-free rate might provide fruitful insights on this matter.

A single project may contain more than flexibility; an option to defer an investment, an option to switch, an option to expand, etc. Having multiple options on a project could be considered as a portfolio of options. We have shown that the individual assessment of individual options is well possible. However, the value of the portfolio is generally non-additive due to interdependencies between the options. This means that the total value of flexibility is different from the sum of individual option values; they can be sub- as well as superadditive (Trigeorgis [1993b], Trigeorgis [1996]). Combining multiple interacting options in a single framework may be highly complex or require
long computation times. Gamba [2002] provides some structure for dealing with complex capital budgeting problems, mapping them as a sequence of simple real options, mutually exclusive options and independent options. This approach allows decomposing a complex option into a set of simple ones that can be solved independently.

The risk profile of both the project at hand and the portfolio itself undergo continuous change. Managerial decisions, changes in asset values, fluctuations in the risk-free rate etc., are events requiring the discount rate to be modified. Clearly, it is not possible to do this all the time (semi-continuously). However, in order to obtain insightful results, the risk profile of the full portfolio should indeed be recalculated rather frequently. Making decisions based on an outdated perception of the portfolio may very well undermine the potential accuracy benefits of ROA.

For some practitioners, the frequent violation of option theory assumptions might make it difficult to defend the use of ROA for their investment problems. Though the calculation of the WACC required for traditional methods is partially based on the same principles, these principles become more prominent when applying risk-neutral valuation. These issues may to some extent explain why real option valuation has not been adopted on a large scale in practice so far. The integral method does not require the artificial construct of a complete market, thereby taking away some key objections against ROA.

We stress that it is not necessary to resort to restrictive option pricing techniques such as the Black-Scholes model, or binomial trees. From a methodological point of view there is no constraint on the techniques used for ROA, allowing to incorporate complex processes in the same manner as any other advanced decision tool. We based our explanations partially on the CAPM and its assumptions, for the sake of simplicity. It should be noted that more sophisticated asset price models have been developed, providing a better fit with reality.

Finally, the incorporation of utility functions permits decision makers to apply ROA as well on investment problems requiring a significant proportion of capital, strongly expanding the range of problems on which real options can be applied. Hence, ROA may be used for many practical investment problems. The insights offered by ROA touch upon the very core of the rationale behind project valuation, giving real option analysis the potential to increase the fit between project management and investment decisions.

7 Appendix

Computations for Example 1 Below, all relevant numbers for the DTA of Example 1 are provided: \( S \) is the size of the oil field, Margin denotes the contribution margin per unit, \( PV \) is the present value of the exploitation for
the end of the year amounts presented. For the computation of the present value, the yearly amounts are divided by \((1 + WACC)^t\) where \(t = 1, 2, 3\) refers to the corresponding year. Recall that in the first year 20% of the total size is exploited, then 50% and in the final year 30%.

<table>
<thead>
<tr>
<th>S Margin</th>
<th>PV</th>
<th>Year 1</th>
<th>Year 2</th>
<th>Year 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>40</td>
<td>-399 2 \cdot 40 - 800 = -720</td>
<td>5 \cdot 40 = 200</td>
<td>3 \cdot 40 = 120</td>
</tr>
<tr>
<td>20</td>
<td>20</td>
<td>-563 2 \cdot 20 - 800 = -760</td>
<td>5 \cdot 20 = 100</td>
<td>3 \cdot 20 = 60</td>
</tr>
<tr>
<td>10</td>
<td>80</td>
<td>-71 2 \cdot 80 - 800 = -640</td>
<td>5 \cdot 80 = 400</td>
<td>3 \cdot 80 = 240</td>
</tr>
<tr>
<td>10</td>
<td>60</td>
<td>-235 2 \cdot 60 - 800 = -680</td>
<td>5 \cdot 60 = 300</td>
<td>3 \cdot 60 = 180</td>
</tr>
<tr>
<td>20</td>
<td>40</td>
<td>-71 4 \cdot 40 - 800 = -640</td>
<td>10 \cdot 40 = 400</td>
<td>6 \cdot 40 = 240</td>
</tr>
<tr>
<td>20</td>
<td>20</td>
<td>-399 4 \cdot 20 - 800 = -720</td>
<td>10 \cdot 20 = 200</td>
<td>6 \cdot 20 = 120</td>
</tr>
<tr>
<td>20</td>
<td>80</td>
<td>585 4 \cdot 80 - 800 = -480</td>
<td>10 \cdot 80 = 800</td>
<td>6 \cdot 80 = 480</td>
</tr>
<tr>
<td>20</td>
<td>60</td>
<td>257 4 \cdot 60 - 800 = -560</td>
<td>10 \cdot 60 = 600</td>
<td>6 \cdot 60 = 360</td>
</tr>
<tr>
<td>40</td>
<td>40</td>
<td>585 8 \cdot 40 - 800 = -480</td>
<td>20 \cdot 40 = 800</td>
<td>12 \cdot 40 = 480</td>
</tr>
<tr>
<td>40</td>
<td>20</td>
<td>-71 8 \cdot 20 - 800 = -640</td>
<td>20 \cdot 20 = 400</td>
<td>12 \cdot 20 = 240</td>
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<tr>
<td>40</td>
<td>80</td>
<td>1898 8 \cdot 80 - 800 = -160</td>
<td>20 \cdot 80 = 1600</td>
<td>12 \cdot 80 = 960</td>
</tr>
<tr>
<td>40</td>
<td>60</td>
<td>1242 8 \cdot 60 - 800 = -320</td>
<td>20 \cdot 60 = 1200</td>
<td>12 \cdot 60 = 720</td>
</tr>
</tbody>
</table>

The computation of the project’s value is straightforward if the decision to perform the test is taken first and then the prices of oil and the associated unit costs become known. In case a negative PV arises in the table above, the decision is not to exploit the resource, and the PV of the associated case is set equal to zero. Note that the probabilities of the twelve outcomes are

\[
\begin{bmatrix}
0.05 & 0.2 & 0.05 & 0.2 & 0.025 & 0.1 & 0.025 & 0.1 & 0.025 & 0.1 & 0.025 & 0.1
\end{bmatrix},
\]

(1)

where e.g., the first probability is the likelihood that the first case occurs, i.e., the field happens to be small with probability 0.5, the price is 80 with probability 0.5 and cost are low (40) with probability 0.2, hence the probability of these three events occurring simultaneously is \(0.5 \times 0.5 \times 0.2\).

Five subcases yield a positive present value, hence the value of the exploitation phase of the project is the inner product of (1) and

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 585 & 257 & 585 & 0 & 1898 & 1242
\end{bmatrix}^T,
\]

which amounts to 226.6. For the project a test with costs 50 was performed right at the start, hence these costs should be subtracted from the latter found value. So, using DTA to determine the value of the project, we find a value of approximately 177.

The computations for Example 2 For the computation of the present value, the yearly amounts are divided by \((1 + rf)^t\) where \(t = 1, 2, 3\) refers to the corresponding year.
Here, we calculated the column of present values for a risk-free rate of 5%. Five cases yield positive cash flows and computations with the same probabilities given by (1) and the new vector of PVs

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 502 & 141 & 322 & 0 & 1767 & 1044
\end{bmatrix}^T
\]

we find the value of this project to be equal to approximately \(183 - 50 = 133\), deducting test costs from the expected present value.

## 8 References


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