DEFLECTION OF AN ECCENTRIC TOOTH OF A COMB DRIVE IN AN ELECTROSTATIC FIELD

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ABSTRACT

The elastic deflection of a comb drive tooth in an electrostatic field is considered. The tooth can be symmetrically located between two rigid teeth of the matching comb, in which case the problem reduces to a pure bifurcation problem for which the critical voltage can be determined. Alternatively, due to an approximate straight-line mechanism, the tooth can have a uniform initial lateral displacement and a smooth curve of equilibria is found which has a limit point, after which pull-in occurs.

An assumed deflection shape and a series expansion of the electrostatic capacity yield the deflection curves for the case with a uniform initial lateral displacement. This shows that pull-in occurs at a voltage that is reduced by a factor that is about proportional to the two-third power of the relative lateral initial displacement.

The theoretical results have been experimentally tested. The results show a qualitative agreement, but the experimental deflections are larger and the pull-in voltages are lower. These differences can be explained from neglected fringe fields and deviations from the nominal shape.

INTRODUCTION

Electrostatic comb drives, first demonstrated by Tang et al. [1], are common as actuation elements in microelectromechanical systems. They consist of two arrays of interlocking teeth or fingers, each connected to a base. By maintaining a voltage difference between them, a longitudinal force is generated that is almost independent of the actuator displacement over a fairly large range of motion. The lateral force is ideally zero, but as the electric field gives rise to a negative lateral stiffness, comb drives can suffer from instability if the support stiffness is not sufficiently high, which can result in a sudden pull-in [2]. Normally, the instability occurs globally, where a comb is pulled in as a whole, but also locally, individual teeth can be pulled in, while the base remains in place. The case of a tooth centrally located between two rigid teeth has been considered by Elata and Leus [3], who analytically derived the critical voltage. Some experiments on buckling were performed later [4], which confirmed the theory.

The combs are usually guided by an elastic straight line mechanism. In some designs, the approximate nature of the guidance causes the teeth to have a lateral or rotational offset. For instance, in a tilted folded flexure with a length of 1000 μm and a tilt angle of five degrees, the lateral displacement is about 20 nm and the rotation about 0.7 mrad for a longitudinal displacement of 100 μm [5]. The precise values depend strongly on the specific guidance used and the configuration. In this paper, only the influence of a uniform lateral offset is considered, where it is assumed that the rotation is small due to a symmetry in the configuration. The influence of a rotation can be investigated in a similar way as presented for the lateral offset. Moreover, the analysis is

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The static equilibrium is characterized by a stationary value of the potential energy, which has contributions from the elastic deflection of the tooth, the electric field and the voltage source.

For the considered small deflections, the elastic energy per unit length is \( \frac{EI(u'')}{2} \), where \( u'' \) denotes a derivative with respect to the material coordinate \( s \) and \( EI = Eht^3/12 \) is the flexural rigidity of the tooth. Over the part of the tooth between the two adjacent teeth of the matching comb, \( al \leq s \leq l \), the electric field energy per unit of length is \( q^2/(2C(u)) \), where \( q \) is the charge per unit of length of the tooth and \( C(u) \) is the capacity per unit of length, approximated by the parallel-plate formula as

\[
C(u) = \varepsilon h \left( \frac{1}{d-u} + \frac{1}{d+u} \right) = \frac{2\varepsilon hd}{d^2-u^2}.
\]

Here, \( \varepsilon \) is the permittivity of the air or vacuum in the gap, \( \varepsilon \approx 8.86 \text{pF/m} \). The contribution of the out-of-plane fringe fields at the top and the bottom of the tooth are neglected, which is admissible if the height of the tooth \( h \) is many times larger than the thickness \( t \) and the gap width \( d \). The fringe fields increase the capacity and change the dependence on the displacement, which has been studied in [6, 7].

The energy of the voltage source is \(-V\) times the charge, which has to be considered because of the charge changes as the tooth deflects. The total potential energy \( P \) can be expressed as

\[
P = \int_0^l \left[ \frac{1}{2} EI(u'')^2 + \frac{1}{2} \frac{q^2}{C(u)} - Vq \right] ds.
\]

Because the in-plane fringe fields at the tip and at the part \( 0 \leq s \leq al \) are neglected, the charge as well as the capacity is zero for this part, so the two electrical terms have no contribution to the integral over this part of the tooth. Taking variations with respect to the charge per unit of length \( q \) yields

\[
q = C(u)V,
\]

which agrees with the definition of a capacity and justifies the definition (1). On the other hand, taking variations of \( P \) with respect to \( u \) yields

\[
\delta P = \int_0^l \left[ \frac{EIu''^2}{2C^2} \frac{dC}{du} \right. \delta u + \left. \frac{q^2}{2C^2} \frac{dC}{du} \right] ds - \int_0^l \left[ \frac{EIu'''^2}{2C^3} \frac{dC}{du} \right. \delta u + \left. [EIu''^2 \delta u' - EIu''' \delta u']_0 \right] ds,
\]

where we have made use of partial integration. As \( \delta P = 0 \) at an equilibrium and the variations are independent,

\[
EIu''' - \frac{q^2}{2C^2} \frac{dC}{du} = 0.
\]
The kinematic boundary conditions at \( s = 0 \) are

\[
\begin{align*}
  u(0) &= u_0, \quad u'(0) = 0, \\
  E l u''(0) &= 0, \quad E l u'''(0) = 0.
\end{align*}
\]

where \( u_0 \) is the lateral initial offset. The dynamic boundary conditions at \( s = l \) are

\[
\begin{align*}
  E l u''(l) &= 0, \quad E l u'''(l) = 0.
\end{align*}
\]

The energy functional can be modified by eliminating the charge by the relation (3) as

\[
P^* = \int_0^l \left[ \frac{1}{2} E l (u'')^2 - \frac{1}{2} C(u) V^2 \right] ds.
\]

As the potential \( V \) is a parameter, this is now a purely mechanical energy functional that contains the elastic energy and the negative energy of the attracting distributed electrostatic forces. Taking variations with respect to \( u \) yields the mechanical equations, equivalent to Eq. (5),

\[
E l u''' - \frac{V^2}{2} \frac{dC}{du} = 0,
\]

or written out,

\[
\frac{E l l^3}{12} - l u''' - \frac{2V^2}{2} \frac{dC}{du} = 0,
\]

where it is understood that the term with \( V^2 \) vanishes for \( 0 \leq s < \alpha l \).

**ANALYTIC BIFURCATION PROBLEM**

The bifurcation analysis for the case that the tooth is centrally located in the gap can be performed analytically, as has been shown by Elata and Leus [3]. The difference here is that we assume plane stress instead of plane strain, as in elementary beam theory, which gives a more conservative estimate of the critical voltage. Eq. (10) can be linearized for small \( u \) as

\[
\frac{E l l^3}{12} u''' - \frac{2V^2}{2} \frac{dC}{du} = 0,
\]

or

\[
l^4 u''' - \frac{V^2}{V_0^2} u = 0,
\]

where

\[
V_0 = \sqrt{\frac{E l l^3}{24 l^2}}.
\]

The boundary conditions are

\[
\begin{align*}
  u(0) = 0, \quad u'(0) = 0, \quad E l u''(0) = 0, \quad E l u'''(0) = 0.
\]

If \( \alpha = 0 \), the general solution that satisfies the boundary conditions at \( s = 0 \) is

\[
u = A (\cos \lambda s - \cosh \lambda s) + B (\sin \lambda s - \sinh \lambda s)
\]

with the undetermined constants \( A \), \( B \); \( \lambda \) is defined as

\[
\lambda = \frac{1}{l} \sqrt{\frac{V}{V_0}}.
\]

With Eq. (15), the boundary conditions (14) at \( s = l \) become

\[
\begin{align*}
  E l \lambda^2 [(\cos \lambda l - \cos \lambda l) A + (\sin \lambda l - \sin \lambda l) B] &= 0, \\
  E l \lambda^3 [(\sin \lambda l - \sinh \lambda l) A + (\cos \lambda l - \cos \lambda l) B] &= 0.
\end{align*}
\]

In order to have a non-trivial solution, the determinant of the matrix of coefficients of the linear equations has to be zero. The root \( \lambda = 0 \) still leads to a trivial solution, so only \( \lambda > 0 \) need be considered. After dividing by the non-zero constant \( E l^2 \lambda^3 \), this determinant is

\[
\left| \begin{array}{c}
  - \cos \lambda l - \cos \lambda l \\
  \sin \lambda l - \sin \lambda l
\end{array} \right| = 2(1 + \cos \lambda l \cos \lambda l).
\]

The smallest positive solution for \( \lambda \) is approximately \( \lambda = 1.8751/l \), so the critical voltage is

\[
V_{cr} = (1.8751)^2 V_0 = \frac{3.5160}{l^2} \sqrt{\frac{E l l^3}{24 l^2}}.
\]

The buckling mode shape has \( A = -0.5 \) and \( B = 0.36705 \), which is the same as the mode shape of a vibrating cantilever beam, and is shown in Fig. 2 as the exact solution.

For \( \alpha > 0 \), the parts with \( 0 \leq s < \alpha l \) and \( \alpha l \leq s < l \) have to be considered separately and the solutions have to be continuous with continuous derivatives up to the third order. Details can be found in [3]. For \( \alpha = 0.1, 0.2 \) and 0.5, the numerical value in (19) becomes 3.51606, 3.51720 and 3.60761, respectively, so for small values of \( \alpha \), the critical voltage does not change so much and the value for \( \alpha = 0 \) is a conservative estimate.
Approximate Solution

An approximate solution can be found by assuming some deflection modes, substituting these modes with undetermined participation factors into the functional of Eq. (8) and finding non-trivial stationary values. We take a single assumed mode that satisfies the kinematic and dynamic boundary conditions, and is proportional to the deflection caused by a uniformly distributed lateral load,

\[ u = u_0 + (u_1 - u_0) \left( \frac{1}{3} \xi^4 - \frac{4}{3} \xi^3 + 2 \xi^2 \right), \]  

(20)

where \( \xi = s/l, \) \( 0 \leq \xi \leq 1, \) is the dimensionless material coordinate along the tooth, \( u_0 \) is the base displacement, which is zero for the bifurcation problem, and \( u_1 - u_0 \) is the undetermined participation factor for the deflection mode, the deflection at the tip of the tooth, \( u_t, \) minus the base displacement. The mode shape for \( u_0 = 0 \) is shown in Fig. 2 as the approximation.

Expansion of the capacity \( C(u) \) of Eq. (1) in powers of \( u \) leads to

\[ C(u) = \frac{2 \rho h}{d} \left[ 1 + \frac{u^2}{d^2} + \frac{u^4}{d^4} + \cdots \right]. \]  

(21)

For the bifurcation problem, only terms up to quadratic ones in the energy functional (8) need be included, so we can use the truncated functional

\[ P_2^* = \int_0^l \left[ \frac{1}{2} EI(u'')^2 - \frac{\rho h}{d^3} u^2 V^2 \right] ds, \]  

(22)

where the constant term has been omitted. Substitution of Eq. (20) with \( u_0 = 0 \) into the functional (22) and evaluating the integrals for \( \alpha = 0 \) results in

\[ P_2^* = \frac{2}{15} \frac{E h t^3}{d^3} u_1^2 - \frac{104}{405} \frac{\rho h}{d^3} u_1^2 V^2, \]  

(23)

which ceases to be positive definite if the second derivative is zero, which occurs at

\[ V_{cr} = \sqrt{\frac{162}{13} \frac{V_0}{t^2} \sqrt{\frac{E t^3 d^3}{24 \epsilon}}}. \]  

(24)

This value differs only 0.4 % from the analytic value, so the assumed mode can be considered to be a good approximation for any small deflection. For \( \alpha = 0.1, 0.2 \) and 0.5, the numerical values in (24) become 3.53014, 3.53150 and 3.63174, respectively, so even for \( \alpha = 0.5, \) the difference from the analytic solution is smaller than 0.7 %.

Deflection Problem

For the case in which an initial offset is present, the problem changes from a bifurcation problem to a deflection problem. For small offsets, the problem can be seen as a perturbed bifurcation problem and an asymptotic analysis can be made according to the post-buckling theory by Koiter [8, 9]. In the present analysis, we stick to the approximation of the deflection by the single mode (20) and a direct analysis can be made. We expand the functional (8) further to quartic terms, \( P^* \approx P_2^* + P_4^* \), with

\[ P_4^* = -\int_0^l \left[ \frac{\rho h}{d^3} u^4 V^2 \right] ds. \]  

(25)

Substituting the approximation (20) in the expansion, evaluating the integrals and again omitting some constant terms gives the result

\[ P^* \approx P_2^* + P_4^* = \frac{2}{15} \frac{E h t^3}{d^3} (u_1 - u_0)^2 \]

\[ -\frac{\rho h}{d^3} \left[ \frac{4}{5} u_0 (u_1 - u_0) + \frac{104}{405} (u_1 - u_0)^2 \right] \]

\[ -\frac{\rho h}{d^3} \left[ \frac{8}{15} u_0 (u_1 - u_0) + \frac{208}{135} \left( u_1 - u_0 \right)^2 \right] \]

\[ + \frac{9344}{12285} u_0^3 (u_1 - u_0)^3 + \frac{347488}{2297295} (u_1 - u_0)^4 \]  

(26)
With the dimensionless quantities

\[ \bar{u} = \frac{u}{d}, \quad \bar{u}_0 = \frac{u_0}{d}, \quad \bar{V} = \frac{V}{V_{cr}} = \sqrt{\frac{52e}{27Et^3d^3}}, \quad P^* = \frac{15P^3}{4Eht^2 d^2}, \]

the energy expression can be rewritten, again with some constant terms left out, as

\begin{equation}
P^* \approx \frac{1}{2}(\bar{u}_l - \bar{u}_0)^2 - \bar{V}^2 \left[ \frac{81}{52} \bar{u}_0(\bar{u}_l - \bar{u}_0) + \frac{1}{2}(\bar{u}_l - \bar{u}_0)^2 \right.
\end{equation}

\begin{equation}
+ \frac{81}{26} \bar{u}_0^3(\bar{u}_l - \bar{u}_0) + 3\bar{a}_0^2(\bar{u}_l - \bar{u}_0)^2
\end{equation}

\begin{equation}
+ \frac{1752}{1183} \bar{a}_0(\bar{u}_l - \bar{u}_0)^3 + \frac{65154}{221221} (\bar{u}_l - \bar{u}_0)^4 \right].
\end{equation}

Equilibria are found from the equation obtained by putting the derivative with respect to \( \bar{u}_l \) equal to zero. For a limit point, the resultant stiffness is zero, so this point can be calculated by simultaneously solving the equation obtained by putting the second derivative equal to zero.

For moderate values of \( \bar{V} \), we can restrict ourselves to the quadratic terms and we obtain the formula for the deflection

\begin{equation}
\bar{u}_l = \bar{u}_0 \left[ 1 + \frac{81\bar{V}^2}{52(1-\bar{V}^2)} \right], \quad (29)
\end{equation}

For an asymptotic analysis near the limit point, we can even simplify the energy expression (28) further by noting that \( \bar{V} \) differs little from the value of one and \( \bar{u}_l \) is much larger than \( \bar{u}_0 \). This gives the simplified expression

\begin{equation}
P^* \approx \frac{1}{2}(1 - \bar{V}^2)(\bar{u}_l - \bar{u}_0)^2 - \frac{81}{52} \bar{u}_0(\bar{u}_l - \bar{u}_0) - \frac{65154}{221221} (\bar{u}_l - \bar{u}_0)^4.
\end{equation}

Putting \( dP^*/d\bar{u}_l = 0 \) and \( d^2 P^*/d\bar{u}_l^2 = 0 \) yields the asymptotic analytic expressions for the limit point voltage and the corresponding limit point deflection as

\begin{equation}
\bar{V}_{lp,an} \approx 1 - 1.341 \bar{u}_0^{2/3}, \quad \bar{u}_{l,lp,an} \approx \bar{u}_0 + 0.871 \bar{u}_0^{1/3} \approx 0.871 \bar{u}_0^{1/3}.
\end{equation}

The linear term in the critical tip displacement can be omitted, as terms of the same order have been neglected in the calculations. Omitting this term will make the results more accurate, as will appear in the next section.

**SEMINUMERICAL CALCULATIONS**

In order to see how accurate the asymptotic results of Eq. (31) are, some seminumerical calculations are made. The approximation for the deflection as in Eq. (20) is retained and the energy is expanded up to eighth-order terms. With this expansion and approximation, results up to a tip deflection that is half the gap width may be expected to be relatively accurate.

The deflection curves for several values of the initial deflection \( \bar{u}_0 \) are shown in Fig. 3. The solutions in the initial rising part are stable, whereas the solutions beyond the limit point are unstable.

The approximate values of the dimensionless limit-point voltage, \( \bar{V}_{lp,an} \), and the values obtained in a seminumerical way, \( \bar{V}_{lp,num} \), are compared in Fig. 4. It appears that the asymptotic approximation is rather good in comparison with the more detailed approximation up to a dimensionless lateral displacement.
FIGURE 5. SEM picture of a test specimen with a lateral offset of 0.5 µm after pull-in

Experimental samples were produced by etching structures from a silicon on insulator (SOI) wafer, which consists of a device layer of monocrystalline silicon on an insulating layer of silicon oxide, which in turn rests on a substrate layer of silicon. The height of the device layer, and hence the height of the structures, was $h = 50$ µm, whereas the insulating layer was 1 µm thick. Each structure consisted of a single tooth fixed to a sturdy base located at a fixed position in a slit etched out of a solid block. The teeth had a length $l = 120$ µm and a width of $t = 3$ µm. The slit had a width of 9 µm, so the nominal gap width was $d = 3$ µm. The overlap was chosen as large as practically possible, 110 µm, which resulted in $\alpha = 10/120 = 0.0833$. The position of the tooth in the slit could be central without offset, or with a uniform lateral offset $u_0 = 0.25$ µm or $u_0 = 0.50$ µm. The direction of the tooth was in the $<110>$ direction of the crystal, so $E = 169$ GPa. As an example, a test specimen with a lateral offset of 0.5 µm is shown in Fig. 5, where pull-in has occurred and the tooth sticks to the wall.

Deflection curves were measured by applying a voltage difference between the tooth and the slit and observing the deflection by stroboscopic light microscopy with a microsystem analyser (Polytec MSA-400). Software for planar motion analysis was used to extract the deflections with a resolution of 15 nm rms. The pull-in voltage was determined by slowly increasing the applied voltage until pull-in took place. The pull-in voltage was determined for the point where a current through the device started to flow owing to short-circuiting caused by contact between the tooth and the wall.

Results
The observed deflection curves, where the displacement is scaled with the gap width and the voltage with the theoretical pull-in voltage of $V_{cr} = 186$ V, are shown in Fig. 6. The measurement produced loops for an increasing voltage up to its maximum and then a decreasing voltage back to zero. The spread of the lines is mainly a result of the limited resolution of the optical measurement system. It is seen that the measured deflections are larger than the theoretical deflections.

Figure 7 shows the observed pull-in voltages, which are lower than the voltages predicted by the theory.

Discussion
The observed difference between the measured values and the theoretical values of the deflections and the pull-in voltages can originate from different sources. A concern is the difference between the real electrical field and the assumed electrical field. At the edges of the tooth, additional field contributions are present that are not included in the formula for the capacity (1). Another effect is that the observed flexural rigidity is smaller than the flexural rigidity calculated from the nominal outer dimen-
FIGURE 7. Measured dimensionless pull-in voltages shown by the Union Jack crosses. Also shown are the numerically determined limit voltages as in Fig. 4, and this curve reduced by a factor 0.85.

sions. Some preliminary calculations with a three-dimensional finite-element multiphysics model where measured dimensions were used and the effects of the fringe fields was taken into account showed a better agreement with the experimental observations. Especially the electric field between the tooth and the grounded substrate of the wafer gives a major contribution for eccentric teeth, because the attractive force tends to twist the deflected tooth.

To take into account the contributions of the fringe fields and the difference in the shape of the tooth, the theoretical critical voltage obtained from the simple model is reduced by a factor 0.85. In Figs. 6 and 7, lines are shown for this reduced critical voltage. These lines give a better agreement with the measurement results. The low pull-in voltage for the centrally located tooth can be explained from the sensitivity for small imperfections, for instance small lateral initial displacements. A reduction of the thickness near the centre of the cross-section has been observed. Further investigations are needed to shed more light on this discrepancy between the theory and the experimental results.

CONCLUSIONS
An asymptotic expression for the pull-in voltage for a tooth with a uniform lateral initial displacement has been derived, together with an asymptotic analytic expression for the tip displacement at the onset of pull-in. The decrease of the dimensionless pull-in voltage is proportional to the two-third power of the dimensionless initial lateral displacement, whereas the corresponding dimensionless tip displacement is proportional to the cubic root of the dimensionless initial lateral displacement. The asymptotic expressions have been checked by a more accurate seminumerical approximation, which validates their correctness and shows their applicability if the lateral initial displacement is smaller than one tenth of the nominal gap width.

The flexural rigidity used here is the expression from classical beam theory, which is valid for small deflections, rather than the plane-strain stiffness as was used in [3]. As the flexural rigidity from classical beam theory is smaller, we are on the conservative side.

An experimental validation of some of the presented theoretical results shows that the observed deflections are larger than the theoretical values and the observed pull-in voltage is lower than the theoretically predicted value. More detailed finite-element calculations have shown that these differences can be explained from edge effects in the electrical field and differences from the nominal shape of the tooth. Further investigations are under way for clearing up these observations.

REFERENCES