Hamiltonian discontinuous Galerkin FEM for linear, rotating incompressible Euler equations: inertial waves

S. Nurijanyan, J.J.W. van der Vegt, O. Bokhove*

Department of Applied Mathematics, University of Twente, P.O. Box 217, 7500 AE, Enschede, The Netherlands

Abstract

A discontinuous Galerkin finite element method (DGFEM) has been developed and tested for linear, three-dimensional, rotating incompressible Euler equations. These equations admit complicated wave solutions.

The numerical challenges concern: (i) discretisation of a divergence-free velocity field; (ii) discretisation of geostrophic boundary conditions combined with no-normal flow at solid walls; (iii) discretisation of the conserved, Hamiltonian dynamics of the inertial-waves; and, (iv) large-scale computational demands owing to the three-dimensional nature of inertial-wave dynamics and possibly its narrow zones of chaotic attraction. These issues have been resolved: (i) by employing Dirac’s method of constrained Hamiltonian dynamics to our DGFEM for linear, compressible flows, thus enforcing the incompressibility constraints; (ii) by enforcing no-normal flow at solid walls in a weak form and geostrophic tangential flow — along the wall; (iii) by applying a symplectic time discretisation; and, (iv) by combining PETSc’s linear algebra routines with our high-level software.

We compared our simulations with exact solutions of three-dimensional compressible and incompressible flows, in (non)rotating periodic and partly periodic cuboids (Poincaré waves). Additional verifications concerned semi-analytical eigenmode solutions in rotating cuboids with solid walls.

*Corresponding author.

Email addresses: s.nurijanyan@math.utwente.nl (S. Nurijanyan), j.j.w.vandervegt@math.utwente.nl (J.J.W. van der Vegt), o.bokhove@math.utwente.nl (O. Bokhove)

The three-dimensional, incompressible Navier-Stokes equations exhibit complicated wave behavior in closed, rotating domains of sufficient size. Wave focussing onto chaotic attractors occurs in asymmetric domains as Maas showed \[16, 18\]. The waves that arise for this constant-density fluid are so-called inertial waves (see \[2, 3, 9, 16, 17, 18, 31\] and references therein). Complex wave focusing phenomena can appear in asymmetric domains because Coriolis forces caused by the background rotation of the domain act as a restoring force for wave motion. In addition, boundary conditions of zero flow hold in the direction normal to the wall as well as nonzero geostrophic or consistency relations for the velocity components tangential to the wall. Inertial waves are best studied in isolation, in the absence of viscosity and nonlinearity. We therefore focus on the development and testing of finite element numerical solution techniques for the linear, three-dimensional incompressible Euler equations in rotating (closed) domains, instead of a focussing directly on the Navier-Stokes equations.

It is useful to contrast two types of waves admitted by the linear, incompressible Euler equations: (i) inertial waves in closed rotating domains, and (ii) surface-trapped waves in half-closed domains with the free surface of the liquid acting under gravity. Surface waves arise due to the restoring force of gravity at the interface between a heavier fluid (e.g., sea water) and a lighter fluid or vacuum. Linear surface waves in the absence of Coriolis forces only involve the potential-flow component, while the vortical components of the velocity or the vorticity (the three-dimensional curl of the velocity vector) are zero. In contrast, inertial waves involve nonzero vortical components of the velocity and exhibit multi-scale behavior, especially when wave focusing occurs. These inertial-wave solutions are thus challenging to compute, either analytically or numerically. The linear three-dimensional Euler equations form a Hamiltonian system. The wave dynamics of both wave types thus concern geometric, Hamiltonian dynamics as an initial value problem, in which invariants such as mass, energy and phase-space volume derive from this geometric structure. Furthermore, the Hamiltonian system
is constrained since the total density is constant and the divergence of the velocity field is zero. Preservation of these discrete invariants in the numerical discretisation ensures numerical stability without any loss of wave amplitude due to artificial numerical damping. The compatible numerical discretisation we aim to develop for these linear incompressible Euler equations should thus preferably inherit a discrete analog of this characteristic Hamiltonian geometric structure.

To wit, our goal is to develop and test a Hamiltonian discontinuous Galerkin finite element method (DGFEM) for inertial-wave dynamics of the linear, incompressible, three-dimensional, rotating Euler equations. The features of the inertial waves indicate that the following mathematical and numerical challenges should be met: (i) The constraint of incompressibility of the flow or zero divergence of the velocity needs to be inherited by the discretisation in a weak or strong form. This is a classical issue in computational fluid dynamics, in which the pressure acts as a Lagrange multiplier to ensure time consistency of the secondary constraint of incompressibility (zero divergence). The zero perturbation density acts here as primary constraint. (ii) The discretisation needs to satisfy the geostrophic balance relations along the wall together with the no-normal flow condition imposed either weakly or strongly. Rotation in combination with the no-normal flow requirement at solid walls yields geostrophic balance conditions on the tangential velocity components. (iii) A discrete analog of the geometric Hamiltonian structure needs to be established to ensure conservation properties of the system. In particular, it would guarantee preservation of wave amplitude and phase space volume, such that long-time calculations remain stable and relevant over many wave periods. (iv) Large computations should be efficient and sufficiently fast, which is nontrivial since the inertial-wave problem is inherently three-dimensional. These four challenges and the associated approaches to meet them are discussed in turn.

The linear incompressible Euler equations emerge from the linear compressible Euler equations describing acoustic waves, in the limit of constant perturbation density. The variables in these compressible acoustic equations are conjugate, being the perturbation density and the velocity field. The constant, nonzero background density plus this small perturbation density comprise the total density of the fluid after a linearization procedure of the nonlinear compressible Euler equations around a state of rest. Enforcing zero perturbation density strongly, leading to a constant total density, as constraint also in time leads to the incompressibility condition of zero
divergence as secondary constraint. In [32], we derived a DGFEM for two-
dimensional linear Hamiltonian hyperbolic systems. A Hamiltonian DGFEM
for the compressible Euler equations thus exists as the linear Euler equations
are hyperbolic, and the extension to three dimensions is straightforward.
The generalized Poisson bracket for linear acoustic flow is bilinear and in-
dependent of the conjugate flow variables, the perturbation scalar density
and velocity field. Consequently, this skew-symmetric bracket automati-
cally satisfies the Jacobi identity. Guided by the requirement to preserve
the skew-symmetry of the discrete bracket, the relevant fluxes at element
interfaces were readily derived in [32]. Our first computational step is to
test the algorithm and implementation of this three-dimensional Hamilt-
onian DGFEM. The result is a finite-dimensional unconstrained Hamiltonian
system for acoustics, essentially consisting of a large number of coupled linear
oscillators. Consequently, Dirac’s theory of constrained Hamiltonian systems
[8, 19] can immediately be applied to this discrete Hamiltonian system, by
imposing zero perturbation density as primary constraint. It yields a dis-
crete form of zero divergence as secondary constraint, which is subsequently
imposed using discrete pressures as Lagrange multipliers. This application of
Dirac’s theory in numerical discretisations of fluid flows appears to b e novel.
We also briefly discuss how it differs from classical approaches in finite ele-
ment analysis of incompressible flows. So far, the Hamiltonian discretisations
remained time continuous, resulting in stiff systems of ordinary differential
equations. The use of stable dissipative, time integrators would destroy the
carefully preserved geometric structure of the spatial discretisation designed
for Hamiltonians in classical mechanics. A better alternative which preserves
the Hamiltonian structure is a symplectic or modified mid-point integrator.

The zero normal-velocity condition holds at the solid impenetrable walls.
Preservation of this condition in time for the linear acoustic equations in
a rotating domain leads to a balance between the Coriolis force and the
pressure gradient normal to the wall. The latter gradient is proportional to
the gradient of the perturbation density for the linear acoustic equations.
Consequently, the tangential components of the flow at the wall are bal-
ced by the normal gradient of the perturbation density. It is nontrivial
to satisfy these consistency boundary conditions discretely, see Ambati and
Bokhove [1]. Using geometric properties of the discrete Hamiltonian formu-
lation and weakly imposing the zero normal-velocity boundary conditions
we ensured the satisfaction of the geostrophic balance for the system. This
procedure is performed for the compressible case and, as before, the appli-
cation of Dirac’s theory then by construction immediately yields the proper geostrophic boundary conditions for the DGFEM of the incompressible flow.

The efficient and fast solution of the resulting linear systems is important because the numerical solutions of inertial waves are inherently three-dimensional. Computations thus quickly become large, and costly. The computational demands become even higher when chaotic attractors exist for which locally fine-scale resolution is required and strong gradients emerge. The need to deal with local fine scales and the presence of strong gradients led to our choice for discontinuous Galerkin finite element methods in the first place. Furthermore, DGFEM permits large gradients and $hp$-refinement. The computational linear algebra demands are handled by using PETSc \cite{27,28} in our versatile DGFEM software environment hpGEM \cite{24}.

The outline of the paper is as follows. In Section 2, we review the equations of motion for the linear compressible and incompressible Euler equations and their Hamiltonian formulations. It also includes an exposition of Dirac’s method of constraints for the linear compressible Euler equations, with zero perturbation density as primary constraint \cite{6,25}. In Section 3, we digress briefly on a finite-volume discretisation of incompressible Hamiltonian dynamics from the Hamiltonian structure for a compressible flow, to clearly demonstrate our methodology. We continue with developing the general Hamiltonian DGFEM for incompressible flows. In Section 4, we present a proper time integrator for the presented Hamiltonian dynamics and discuss some of the properties of the resulting time and space discrete numerical schemes. Numerical verifications are given in Section 5, where DGFEM simulations are compared with exact solutions of compressible flow in (non-rotating) closed cuboids, with incompressible flow in rotating triple-periodic domain and a partially closed cuboid with periodicity in one direction, and with semi-analytical series solutions for incompressible flow in closed cuboids. Conclusions are drawn in Section 6.

2. Continuum theory for (in)compressible fluid

2.1. Governing equations

Compressible fluid flow in a domain $D$ is governed by the non-linear compressible Euler equations prescribed with angular velocity $\mathbf{\Omega} = (\Omega_1, \Omega_2, \Omega_3)^T$
in a rotating frame:

$$\frac{\partial \hat{u}}{\partial t} = -2\Omega \times \hat{u} - (\hat{u} \cdot \nabla)\hat{u} - \hat{\rho}^{-1} \nabla \hat{P}(\hat{\rho}) - \nabla (gz), \quad (1a)$$

$$\frac{\partial \hat{\rho}}{\partial t} = -\nabla \cdot (\hat{\rho} \hat{u}), \quad (1b)$$

where $\hat{u} = \hat{u}(x, y, z, t) = (\hat{u}, \hat{v}, \hat{w})^T$ is the three-dimensional velocity field, $\hat{\rho} = \hat{\rho}(x, y, z, t)$ a scalar density field, and $\hat{P} = \hat{P}(\hat{\rho})$ the barotropic pressure. Cartesian coordinates $\mathbf{x} = (x, y, z)$ and time $t$ are used; the three-dimensional differential operator is given by $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)^T$; and, $g$ is the constant gravitational acceleration acting in the negative z-direction. The boundaries of the domain $D$ are denoted with $\partial D = \cup_i \partial D_i$.

We linearise the compressible Euler equations (1) around a rest state $(\mathbf{u}_0 = 0, \rho_0)$ with $\hat{u} = 0 + \epsilon \mathbf{u}$ and $\hat{\rho} = \rho_0 + \epsilon \rho$, where $\mathbf{u}$ and $\rho$ are the perturbation velocity and density fields, respectively. Linearisation results in the linear compressible Euler equations in a rotating domain

$$\frac{\partial \mathbf{u}}{\partial t} = -\nabla \left( \frac{c_0^2}{\rho_0} \rho \right) - 2\Omega \times \mathbf{u}, \quad (2a)$$

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho_0 \mathbf{u}), \quad (2b)$$

where $c_0 = \sqrt{\partial \hat{P}/\partial \rho}|_{\rho = \rho_0}$ is the constant wave speed. Two types of boundary conditions will be discussed: periodic and solid-wall boundary conditions. For fixed, solid-wall boundary conditions the normal component of the velocity field at the boundaries is zero $\mathbf{u} \cdot \hat{n} = 0$, with $\hat{n}$ the outward normal vector at the boundary. If we multiply both sides of the momentum equation (2a), restricted to the domain boundary, with the normal vector $\hat{n}$,

$$\frac{\partial (\mathbf{u} \cdot \hat{n})}{\partial t} = -\nabla \left( \frac{c_0^2}{\rho_0} \rho \right) \cdot \hat{n} - (2\Omega \times \mathbf{u}) \cdot \hat{n}, \quad (3)$$

and apply the no-normal flow condition $\mathbf{u} \cdot \hat{n} = 0$, we obtain a restriction on the density gradient

$$\frac{c_0^2}{\rho_0} \nabla \rho \cdot \hat{n} = -(2\Omega \times \mathbf{u}) \cdot \hat{n}. \quad (4)$$

In the absence of domain rotation, the right side of (4) is zero at the boundary, which indicates that the normal component of the density gradient is
also zero on the boundary. With rotation, in contrast, the normal component of the density gradient is balanced by the rotational velocity field. This balance between the density/pressure gradient force and the Coriolis force is called geostrophic balance. Therefore, the implementation of the boundary conditions becomes more challenging, due to the mandatory satisfaction of geostrophic balance.

The linear incompressible Euler equations arise from in the limit of zero Mach number, \( M_0 = V_0/c_0 \to 0 \), with \( V_0 \) a reference velocity of the fluid

\[
\frac{\partial \mathbf{u}}{\partial t} = -2\Omega \times \mathbf{u} - \nabla P, \quad (5a)
\]
\[
\nabla \cdot \mathbf{u} = 0, \quad \rho = 0, \quad (5b)
\]

where \( P \) is the pressure. Note that the constraint on the perturbation density ensures that the total density is constant for all time.

2.2. Hamiltonian framework

In the following sections we introduce the Hamiltonian framework for linear compressible and incompressible fluid flows, including the connection with the corresponding partial differential equation (PDE) models. A general Hamiltonian system consists of a phase-space and two geometric objects, a scalar energy functional \( \mathcal{H} \) and a Poisson bracket \( \{,\} \) \([4, 21, 26]\). The Hamiltonian dynamics is given by the time evolution of a general state functional \( \mathcal{F} \) via the bracket form

\[
\frac{d\mathcal{F}}{dt} = \{\mathcal{F}, \mathcal{H}\} \quad (6)
\]

for a specific Hamiltonian functional, or energy, \( \mathcal{H} \). This (generalized) Poisson bracket \( \{\mathcal{F}, \mathcal{H}\} \) has to satisfy the following properties:

(i) skew-symmetry: \( \{\mathcal{F}, \mathcal{H}\} = -\{\mathcal{H}, \mathcal{F}\} \),

(ii) linearity in the first component: \( \{\alpha \mathcal{F} + \beta \mathcal{G}, \mathcal{H}\} = \alpha \{\mathcal{F}, \mathcal{H}\} + \beta \{\mathcal{G}, \mathcal{H}\} \),

(iii) Jacobi identity: \( \{\mathcal{F}, \{\mathcal{G}, \mathcal{H}\}\} + \{\mathcal{G}, \{\mathcal{H}, \mathcal{F}\}\} + \{\mathcal{H}, \{\mathcal{F}, \mathcal{G}\}\} = 0 \), and

(iv) Leibniz identity: \( \{\mathcal{F}\mathcal{G}, \mathcal{H}\} = \mathcal{F}\{\mathcal{G}, \mathcal{H}\} + \{\mathcal{F}, \mathcal{H}\}\mathcal{G} \),

where \( \alpha \) and \( \beta \) are constants and \( \mathcal{F}, \mathcal{G}, \mathcal{H} \) arbitrary functionals. Skew-symmetry of the bracket automatically results in the energy conservation property: \( d\mathcal{H}/dt = \{\mathcal{H}, \mathcal{H}\} = 0 \).
2.2.1. Bracket for linearised compressible flow

Hamiltonian dynamics of compressible fluid flow [7, 22] governed by the linear equations (2) in \( D \subset \mathbb{R}^3 \) is given by

\[
\frac{d\mathcal{F}}{dt} = \{\mathcal{F}, \mathcal{H}\} = \int_D \left( \frac{\delta \mathcal{H}}{\delta \rho} \nabla \cdot \frac{\delta \mathcal{F}}{\delta \mathbf{u}} - \frac{\delta \mathcal{F}}{\delta \rho} \nabla \cdot \frac{\delta \mathcal{H}}{\delta \mathbf{u}} - \frac{2\Omega}{\rho_0} \times \frac{\delta \mathcal{H}}{\delta \mathbf{u}} \cdot \frac{\delta \mathcal{F}}{\delta \mathbf{u}} \right) dx, \tag{7}
\]

with Hamiltonian energy functional

\[
\mathcal{H} = \mathcal{H}[\mathbf{u}, \rho] \equiv \int_D \left( \frac{1}{2} \rho_0 \mathbf{u}^2 + \frac{c_0^2}{\rho_0} \rho^2 \right) dx. \tag{8}
\]

The definition of the functional derivative is

\[
\delta \mathcal{H} \equiv \lim_{\epsilon \to 0} \frac{\mathcal{H}[\mathbf{u} + \epsilon \delta \mathbf{u}, \rho + \epsilon \delta \rho] - \mathcal{H}[\mathbf{u}, \rho]}{\epsilon} = \int_D \left[ \frac{\delta \mathcal{H}}{\delta \mathbf{u}} \cdot \delta \mathbf{u} + \frac{\delta \mathcal{H}}{\delta \rho} \delta \rho \right] dx. \tag{9}
\]

Therefore, the functional derivatives of \( \mathcal{H} \) are

\[
\frac{\delta \mathcal{H}}{\delta \mathbf{u}} = \rho_0 \mathbf{u}, \quad \frac{\delta \mathcal{H}}{\delta \rho} = \frac{c_0^2}{\rho_0} \rho. \tag{10}
\]

The Poisson bracket \( \{ , \} \) satisfies all properties: skew-symmetry is easy to spot from the structure of the bracket; the bracket is obviously bilinear, thus the linearity and Leibniz identity are automatically satisfied; and, the Jacobi identity can be checked directly.

To specify Hamiltonian dynamics in the domain \( D \) one has to specify appropriate boundary conditions. Mathematical models based on PDEs usually specify boundary conditions on the relevant variables at the boundary. Similarly, in the Hamiltonian formulation boundary conditions can be imposed by choosing appropriate function spaces for the arbitrary functional \( \mathcal{F} \).

As an example, we will show the equivalence between the Hamiltonian framework [7] and PDE representation [22] of compressible fluid flow in the rotating domain \( D \) bounded by solid walls. The momentum and continuity equations can be obtained if the following functionals are chosen

\[
\mathcal{F}_u \equiv \int_D \mathbf{u}(\mathbf{x}, t) \cdot \Phi(\mathbf{x}) d\mathbf{x} \tag{11a}
\]
\[
\mathcal{F}_\rho \equiv \int_D \rho(\mathbf{x}, t) \phi(\mathbf{x}) d\mathbf{x} \tag{11b}
\]
with $\phi \in Q$ and $\Phi \in Y$ arbitrary test functions, where

$$Q = \{ \phi \in L^2(D) \}$$

$$Y = \{ \Phi \in (L^2(D))^3 \text{ and } \nabla \cdot \Phi \in L^2(D) : \hat{n} \cdot \Phi = 0 \text{ at } \partial D \},$$

and $L^2(D)$ is the space of square integrable functions on $D$. To incorporate slip flow boundary conditions at $\partial D$ we restrict the space for the test functions $\Phi$ at the boundary. Corresponding functional derivatives of (11a) and (11b) are

$$\frac{\delta F_\rho}{\delta \rho} = \phi(x) \text{ and } \frac{\delta F_u}{\delta u} = \Phi(x), \text{ with } \frac{\delta F_u}{\delta u} \cdot \hat{n} = 0 \text{ at } \partial D.$$

Using functionals (11a) and (11b), with corresponding functional derivatives (14) and (10), in the bracket formulation (9) yields the momentum (2a) and continuity (2b) Euler equations for compressible flow, respectively. The restricted test function arising from functional $F_u$ ensures the satisfaction of the boundary conditions at the PDE level.

2.2.2. Construction of a Dirac-bracket for incompressible flow

Dirac’s theory of constrained Hamiltonian systems ([8, 25, 29]) is used to derive the incompressible Euler equations as a limit of the compressible Hamiltonian structure. Basically, Dirac’s theory enforces a constant density constraint via Lagrange multipliers onto the derived compressible Hamiltonian framework ([6, 7]).

Due to linearization, the constant total density constraint $\dot{\rho} = \text{const}$ transforms into the perturbation density constraint

$$\rho(x) = 0.$$

It will act as a primary constraint, to be incorporated into compressible Hamiltonian dynamics ([7]) via a Lagrange multiplier field. In a consistent theory, the constraint must be preserved by the evolution of the system. This leads to three possible outcomes; (i) the consistency requirement yields modulo the constraints to an equation of essentially the form $1 = 0$; (ii) to an equation of the form $0 = 0$; (iii) we obtain an equation which resolves the unknown Lagrange multiplier, or it yields a secondary constraint. Case (i) implies inconsistent equations of motion; they do not posses any solution. Case (ii) is the desired outcome. Case (iii) introduces new secondary constraints, preservation of which must be checked by repeating the procedure.
until either we encounter case (i) or all constraints lead to case (ii). Thus is
the main idea of Dirac’s algorithm.

A Lagrange multiplier $\lambda_\rho(x, t)$ is introduced to enforce the primary con-
straint. This constraint, or any arbitrary functional $F[\rho]$ thereof, must be
preserved in time. Hence, the evolution of this functional must remain naught

$$\frac{dF[\rho]}{dt} = 0 = \{F[\rho], \mathcal{H}\} + \int_D \lambda_\rho(x') \{F[\rho], \rho(x')\} dx'.$$  \hspace{1cm} (16)

From Poisson bracket (7), we deduce that $\{F[\rho], \rho(x')\} = 0$, and therefore,
the Lagrange multiplier remains undetermined. It gives, however, rise to a
secondary constraint

$$0 = \{F[\rho], \mathcal{H}\} = - \int_D \frac{\delta F[\rho]}{\delta \rho} \nabla \cdot (\rho_0 u(x)) \, dx.$$  \hspace{1cm} (17)

Since the functional $F[\rho]$ is arbitrary in (17), it follows that

$$\nabla \cdot u = 0$$

should hold as well. Note that $\delta F[\rho]/\delta \rho$ serves as arbitrary test function and
that the secondary constraint implies that the velocity is divergence-free.
Next, both constraints

$$\rho(x) = 0 \quad \text{and} \quad \Delta(x) = \nabla \cdot u(x) = 0$$

will be enforced as primary constraints together, also in time.

For that reason, we introduce Lagrange multipliers $\lambda_\rho = \lambda_\rho(x, t)$ and
$\lambda_\Delta = \lambda_\Delta(x, t)$. The two consistency requirements are stated in weak form
by using two (different) arbitrary functionals $F[\rho]$ and $F[\Delta]$, as follows

$$\frac{dF[\rho]}{dt} = 0 = \{F[\rho], \mathcal{H}\} + \int_D \lambda_\Delta(x') \{F[\rho], \Delta(x')\} dx'$$  \hspace{1cm} (20a)

$$\frac{dF[\Delta]}{dt} = 0 = \{F[\Delta], \mathcal{H}\} + \int_D \lambda_\rho(x') \{F[\Delta], \rho(x')\} dx'$$

$$+ \int_D \lambda_\Delta(x') \{F[\Delta], \Delta(x')\} dx'$$  \hspace{1cm} (20b)

where we omitted stating the explicit time dependence. An elaborate calcu-
lation of the brackets in (20a) yields

$$0 = \int_D \frac{\delta F}{\delta \rho} (-\nabla \cdot (\rho_0 u) + \nabla^2 \lambda_\Delta) \, dx - \int_{\partial D} \frac{\delta F}{\delta \rho} \hat{n} \cdot \nabla \lambda_\Delta \, dS$$  \hspace{1cm} (21)
with surface element $dS$. By using the secondary constraint in (21), and the arbitrariness of the functional $\mathcal{F}[\rho]$ in the interior and at the boundary, we find that

$$\nabla^2 \lambda_\Delta = 0 \quad \text{with} \quad \hat{n} \cdot \nabla \lambda_\Delta = 0. \quad (22)$$

Its solution is $\lambda_\Delta = \text{cst.}$

To analyse (20b), we first relate the functional derivative of $\mathcal{F}[\Delta]$ with respect to $\Delta$ to the one with respect to $u$, as follows

$$\delta \mathcal{F}[\Delta] = \int_D \delta \mathcal{F}[\Delta] \delta \Delta d\mathbf{x} = -\int_D \nabla \nabla^2 \lambda_\rho \rho \delta \Delta d\mathbf{x}, \quad (23)$$

where we used that $\hat{n} \cdot \delta u = 0$. The last term in (20b) cancels after an integration by parts, by using $\hat{n} \cdot \nabla \lambda_\Delta = 0$ and $\hat{n} \cdot \nabla (\delta \mathcal{F}[\Delta]/\delta \Delta) = 0$. We subsequently find that (20b) becomes

$$0 = \int_D \delta \mathcal{F} \left( \nabla \cdot (2 \Omega \times u) + \nabla^2 \lambda_\rho \right) d\mathbf{x} - \int_{\partial D} \nabla \cdot \delta \mathcal{F} (2 \Omega \times u + \nabla \lambda_\rho) dS. \quad (24)$$

The arbitrariness of $\mathcal{F}[\Delta]$ in (24), in the interior and at the boundary, then implies that

$$\nabla \cdot (2 \Omega \times u) + \nabla^2 \lambda_\rho = 0 \quad \text{on } D \quad \text{with} \quad (2 \Omega \times u + \nabla \lambda_\rho) \cdot \hat{n} = 0 \quad \text{on } \partial D. \quad (25)$$

Details in the above calculations have been relegated to Appendix A.

The bracket formulation for incompressible flow is now given by

$$\frac{d\mathcal{F}[u]}{dt} = \{\mathcal{F}, \mathcal{H}\} + \int_D \lambda_\rho(x') \{\mathcal{F}, \rho(x')\} d\mathbf{x}'. \quad (26)$$

The dynamics is then obtained from (26) combined with (24) for the Lagrange multiplier ($\lambda = \lambda_\rho$)

$$\frac{d\mathcal{F}}{dt} = \{\mathcal{F}, \mathcal{H}\}_{\text{inc}} = \int_D \left[ -\frac{2 \Omega}{\rho_0} \times \frac{\delta \mathcal{H}}{\delta u(x)} \cdot \frac{\delta \mathcal{F}}{\delta u(x)} + \lambda(x) \nabla \cdot \frac{\delta \mathcal{F}}{\delta u(x)} \right] d\mathbf{x}, \quad (27a)$$

$$0 = \int_D \frac{\delta \mathcal{F}}{\delta \Delta} \left( \nabla \cdot (2 \Omega \times u) + \nabla^2 \lambda \right) d\mathbf{x} - \int_{\partial D} (2 \Omega \times u + \nabla \lambda) \cdot \hat{n} \frac{\delta \mathcal{F}}{\delta \Delta} dS, \quad (27b)$$
with constrained energy functional

$$\mathcal{H} = \int_D \frac{1}{2} \rho \mathbf{u}^2 \, dx. \quad (27c)$$

It is obtained after application of the primary constraint $\rho = 0$. The incompressible, linear Euler equations can be derived from (27) by choosing functionals

$$\mathcal{F}_u \equiv \int_D \mathbf{u}(x, t) \cdot \Phi(x) \, dx \quad \text{and} \quad \mathcal{F}_\Delta \equiv \int_D \Delta(x, t) \tilde{\phi}(x) \, dx, \quad (28)$$

where $\Phi(x) \in \mathcal{Y}$ and $\tilde{\phi}(x) \in \mathcal{Q}$ with the additional requirement that $\mathbf{n} \cdot \nabla \tilde{\phi} = 0$. The functionals in (28) lead to the system of equations

$$\frac{\partial \mathbf{u}}{\partial t} = -\nabla \lambda - 2\Omega \times \mathbf{u} \quad \text{and} \quad \nabla^2 \lambda = -\nabla \cdot (2\Omega \times \mathbf{u}), \quad (29)$$

with slip flow $\mathbf{u} \cdot \mathbf{n} = 0$ and geostrophic balance $(2\Omega \times \mathbf{u} + \nabla \lambda) \cdot \mathbf{n} = 0$ at the solid-wall boundary. Notice that the Lagrange multiplier $\lambda = P$ plays the role of the pressure $P$.

## 3. Discrete Hamiltonian formulation

Discretisations of the earlier derived compressible and incompressible continuous Hamiltonian formulations will be derived next. There are two possible choices for a derivation of discrete Hamiltonian dynamics for incompressible fluid flow: direct discretisation of the continuous bracket formulation (27) for incompressible fluid flow, or application of Dirac’s theory on the discretised Hamiltonian formulation of compressible flow. The latter approach is preferable for several reasons: (i) a discretisation of the compressible Hamiltonian formulation is becoming an intermediate check point for the introduced discretisation algorithm; (ii) avoidance of dealing with discontinuities of unknown Lagrange multipliers simplifies the process; and, (iii) the relatively easy incorporation of boundary conditions which are set automatically by Dirac’s theory given the proper boundary conditions for the compressible case. Before proceeding to a discontinuous Galerkin FEM discretisation, we demonstrate key aspects of the algorithm on a finite volume (FV) discretisation of the compressible Hamiltonian formulation with consecutive application of Dirac’s theory on a discrete level. This FV discretisation is equivalent to a DG discretisation with constant basis functions.
3.1. Finite volume discretisation for linear Euler equations
3.1.1. Discrete compressible dynamics

\[
\frac{d}{dt} \begin{pmatrix} \bar{U}_{i,j,k} \\ \bar{V}_{i,j,k} \\ \bar{W}_{i,j,k} \\ \bar{R}_{i,j,k} \end{pmatrix} = - \left( 2\Omega \times \begin{pmatrix} \bar{U}_{i,j,k} \\ \bar{V}_{i,j,k} \\ \bar{W}_{i,j,k} \\ 0 \end{pmatrix} \right) - \begin{pmatrix} \bar{G}^1_{i,j,k} \\ \bar{G}^2_{i,j,k} \\ \bar{G}^3_{i,j,k} \\ \bar{G}^4_{i,j,k} \end{pmatrix}, \tag{30a}
\]

where \( \bar{U}_{i,j,k} = \bar{U}_{i,j,k}(t), \bar{V}_{i,j,k} = \bar{V}_{i,j,k}(t), \bar{W}_{i,j,k} = \bar{W}_{i,j,k}(t) \) and \( \bar{R}_{i,j,k} = \bar{R}_{i,j,k}(t) \) are the time-dependent mean values of \( u, v, w, \rho \) in the \( (i, j, k) \)-th element.
and the flux functions are defined by

\[
\bar{G}_{1}^{i,j,k} = -\frac{(\bar{R}_{i+1,j,k}(1-\theta) + \bar{R}_{i,j,k}\theta) - (\bar{R}_{i,j,k}(1-\theta) + \bar{R}_{i-1,j,k}\theta)}{\Delta x},
\]

\[
\bar{G}_{2}^{i,j,k} = -\frac{(\bar{R}_{i,j+1,k}(1-\theta) + \bar{R}_{i,j,k}\theta) - (\bar{R}_{i,j,k}(1-\theta) + \bar{R}_{i,j-1,k}\theta)}{\Delta y},
\]

\[
\bar{G}_{3}^{i,j,k} = -\frac{(\bar{R}_{i,j,k+1}(1-\theta) + \bar{R}_{i,j,k}\theta) - (\bar{R}_{i,j,k}(1-\theta) + \bar{R}_{i,j,k-1}\theta)}{\Delta z},
\]

\[
\bar{G}_{4}^{i,j,k} = -\frac{(\bar{U}_{i,j,k}(1-\theta) + \bar{U}_{i,j+k}(1-\theta) + \bar{U}_{i,j,k+1}(1-\theta) + \bar{U}_{i,j+1,k}(1-\theta))}{\Delta x} - \frac{(\bar{V}_{i,j,k}(1-\theta) + \bar{V}_{i,j+k}(1-\theta) + \bar{V}_{i,j,k+1}(1-\theta) + \bar{V}_{i,j+1,k}(1-\theta))}{\Delta y} - \frac{(\bar{W}_{i,j,k}(1-\theta) + \bar{W}_{i,j+k}(1-\theta) + \bar{W}_{i,j,k+1}(1-\theta) + \bar{W}_{i,j+1,k}(1-\theta))}{\Delta z},
\]

(30b, 30c, 30d, 30e)

with \(\Delta x\), \(\Delta y\), and \(\Delta z\) the respective mesh sizes, and \(0 \leq \theta \leq 1\).

Energy conservation can be shown by a series of straightforward calculations: multiply equations (30a) by \((\bar{U}_{i,j,k}, \bar{V}_{i,j,k}, \bar{W}_{i,j,k}, \bar{R}_{i,j,k})\), and sum over all elements. Discretisation (30) then leads to energy conservation, i.e.,

\[
\frac{dH}{dt} = 0, \quad \text{with} \quad H = \sum_{(i,j,k)} \frac{1}{2} \left( U_{i,j,k}^2 + V_{i,j,k}^2 + W_{i,j,k}^2 + R_{i,j,k}^2 \right).
\]

(31)

The latter result suggests there is a discrete Hamiltonian formulation for (30). We first calculate the partial derivatives of the Hamiltonian

\[
\frac{\partial H}{\partial U_{i,j,k}} = \bar{U}_{i,j,k}, \quad \frac{\partial H}{\partial V_{i,j,k}} = \bar{V}_{i,j,k}, \quad \frac{\partial H}{\partial W_{i,j,k}} = \bar{W}_{i,j,k}, \quad \frac{\partial H}{\partial R_{i,j,k}} = \bar{R}_{i,j,k}
\]

(32)

and use these in the RHS of (30a). Subsequently, we multiply the four equations in (30a) by \(\partial F/\partial U_{i,j,k}, \partial F/\partial V_{i,j,k}, \partial F/\partial W_{i,j,k}, \text{ and } \partial F/\partial R_{i,j,k}\), respectively, add them up, and sum over all cells. After some algebraic manip-
ulations, it yields the Hamiltonian finite-dimensional dynamics

\[
\frac{dF}{dt} = [F, H]_{c_0} = -\frac{\partial F}{\partial U_k} \cdot 2\Omega \times \frac{\partial H}{\partial U_k} + \frac{\partial H}{\partial U_k} \cdot \left( (1 - \theta) \begin{pmatrix} \frac{\Delta x}{\partial R_{i+1}} & -\frac{\partial F}{\partial R_k} \\ \frac{\Delta y}{\partial R_{j+1}} & -\frac{\partial F}{\partial R_k} \\ \frac{\Delta z}{\partial R_{k+1}} & -\frac{\partial F}{\partial R_k} \end{pmatrix} \right) + \theta \begin{pmatrix} \frac{\Delta x}{\partial R_k} & -\frac{\partial F}{\partial R_k} & -\frac{\partial F}{\partial R_{k-1}} \\ \frac{\Delta y}{\partial R_k} & -\frac{\partial F}{\partial R_k} & -\frac{\partial F}{\partial R_{k-1}} \\ \frac{\Delta z}{\partial R_k} & -\frac{\partial F}{\partial R_k} & -\frac{\partial F}{\partial R_{k-1}} \end{pmatrix},
\]

(33)

where \( U = (\bar{U}, \bar{V}, \bar{W})^T \) and \( k = (i, j, k) \), and we used the shorthand notation \( R_{i+1} = R_{i+1,j,k} \), et cetera. Repeated indices indicate summation, if not stated otherwise. By inspection, (33) is seen to be anti-symmetric, and independent of the variables involved. It therefore satisfies all requirement of a (noncanonical) Hamiltonian system. The \( c_0 \) subscript indicates the cosymplectic form of the Poisson bracket for a FV discretisation of the compressible Euler equations. Since several brackets appear below, we will use such subscripts to distinguish the different brackets used hereafter.

3.1.2. Discrete incompressible dynamics

The discrete compressible Hamiltonian dynamics (33) with (31) is our starting point for Dirac’s theory of constraints. It will be used to enforce the density as a constraint into the compressible Hamiltonian formulation. For simplicity of illustration, we will only consider the case with \( \theta = 0 \) in (33).
With some minor index renumbering, we obtain

\[
\frac{dF}{dt} = [F, H]_{c_0} \equiv -\left(2\Omega \times \frac{\partial H}{\partial U_k}\right) \cdot \frac{\partial F}{\partial U_k}
\]

\[-\left(\frac{\partial H}{\partial R_{i+1,j,k}} - \frac{\partial H}{\partial R_k}\right) \frac{1}{\Delta x} \frac{\partial F}{\partial U_k} - \left(\frac{\partial H}{\partial R_{i,j+1,k}} - \frac{\partial H}{\partial R_k}\right) \frac{1}{\Delta y} \frac{\partial F}{\partial V_k}\]

\[-\left(\frac{\partial H}{\partial R_{i,j,k+1}} - \frac{\partial H}{\partial R_k}\right) \frac{1}{\Delta z} \frac{\partial F}{\partial W_k} - \left(\frac{\partial H}{\partial U_k} - \frac{\partial H}{\partial U_{i-1,j,k}}\right) \frac{1}{\Delta x} \frac{\partial F}{\partial R_k}\]

\[-\left(\frac{\partial H}{\partial V_k} - \frac{\partial H}{\partial V_{i,j-1,k}}\right) \frac{1}{\Delta y} \frac{\partial F}{\partial R_k} - \left(\frac{\partial H}{\partial W_k} - \frac{\partial H}{\partial W_{i,j,k-1}}\right) \frac{1}{\Delta z} \frac{\partial F}{\partial R_k} = (34)\]

For the continuous problem the primary constraint is the zero perturbation density, \(\rho(x, y, z, t) = 0\). For the FV discretisation, it means that the mean value of the density is zero everywhere,

\[\bar{R}_k = 0.\]  

(35)

The primary constraints should hold in time, as a consistency requirement,

\[0 = \frac{d\bar{R}_k}{dt} = [\bar{R}_k, H]_{c_0} + \mu_p [\bar{R}_k, \bar{R}_p]_{c_0},\]  

(36)

with Lagrange multipliers \(\mu_p\) and \(p = (p, q, r)\). From (34), it follows that \([\bar{R}_k, R_p]_{c_0} = 0\). The Lagrange multiplier thus remains undetermined and a secondary constraint arises from (36) as

\[
E_k \equiv -[\bar{R}_k, H]_{c_0} = \frac{\partial H}{\partial U_k} + \frac{\partial H}{\partial U_{i-1,j,k}} \frac{1}{\Delta x} + \frac{\partial H}{\partial V_k} + \frac{\partial H}{\partial V_{i,j-1,k}} \frac{1}{\Delta y} + \frac{\partial H}{\partial W_k} + \frac{\partial H}{\partial W_{i,j,k-1}} \frac{1}{\Delta z}\]

\[
= \frac{(\bar{U}_k - \bar{U}_{i-1,j,k})}{\Delta x} + \frac{(\bar{V}_k - \bar{V}_{i,j-1,k})}{\Delta y} + \frac{(\bar{W}_k - \bar{W}_{i,j,k-1})}{\Delta z} = 0.
\]

(37)

A closer look at the constraint (37) shows that it is a first-order discretization of the divergence-free velocity condition (18). Subsequently, both constraints

16
are enforced together, with mandatory preservation in time

\[
0 = \frac{d\bar{R}_k}{dt} = [\bar{R}_k, H]_{c_0} + \lambda^R_p[\bar{R}_k, E_p]_{c_0}, \tag{38a}
\]

\[
0 = \frac{dE_k}{dt} = [E_k, H]_{c_0} + \lambda^R_p[E_k, \bar{R}_p]_{c_0} + \lambda^U_p[E_k, E_p]_{c_0}. \tag{38b}
\]

Application of primary constraint \((35)\) in \((38a)\) yields \(\lambda^U_p[\bar{R}_k, E_p]_{c_0} = 0\). The latter equation is a FV discretisation of the Laplacian, as follows shortly, and has as solution \(\lambda^U_p = 0\). That result simplifies \((38b)\) to

\[
0 = [E_k, H]_{c_0} + \lambda^R_p[E_k, \bar{R}_p]_{c_0}. \tag{39}
\]

Further valuation of \((39)\) using \((33)\) gives an equation for \(\lambda \equiv \lambda^R\)

\[
\begin{align*}
\lambda_{i+1,j,k} - 2\lambda_k + \lambda_{i-1,j,k} &\quad \Delta x^2 \\
\lambda_{i,j+1,k} - 2\lambda_k + \lambda_{i,j-1,k} &\quad \Delta y^2 \\
\lambda_{i,j,k+1} - 2\lambda_k + \lambda_{i,j,k-1} &\quad \Delta z^2 \\
\end{align*}
\]

\[
\begin{align*}
\frac{\lambda_{i,j,k+1} - \lambda_{i,j,k-1}}{\Delta z} &\quad \frac{\lambda_{i+1,j,k} - \lambda_{i-1,j,k}}{\Delta x} \\
\frac{\lambda_{i,j+1,k} - \lambda_{i,j-1,k}}{\Delta y} &\quad - \frac{2\Omega \times \left( \frac{\partial H}{\partial U_k} - \frac{\partial H}{\partial U_{i-1,j,k}} \right) \Big|_1}{\Delta x} \\
\frac{2\Omega \times \left( \frac{\partial H}{\partial V_k} - \frac{\partial H}{\partial U_{i,j,k-1}} \right) \Big|_2}{\Delta y} - \frac{2\Omega \times \left( \frac{\partial H}{\partial W_k} - \frac{\partial H}{\partial U_{i,j,k-1}} \right) \Big|_3}{\Delta z}.
\end{align*}
\tag{40}
\]

Note that \((40)\) is a discretisation of the Laplacian acting on the Lagrange multiplier, \(\nabla^2\lambda\), on the left-hand-side and the divergence of the rotational effects on the right-hand-side, corresponding to a discrete version of the continuous case \((25)\). Finally, the bracket for the incompressible case becomes

\[
\frac{dF}{dt} = [F, H]_{co}^{inc} \equiv - \left( 2\Omega \times \frac{\partial H}{\partial U_k} \right) \cdot \frac{\partial F}{\partial U_k} - \lambda_{i+1,j,k} - \lambda_k \frac{\partial F}{\partial U_k} \\
\lambda_{i,j,k+1} - \lambda_{i,j,k} \frac{\partial F}{\partial V_k} - \lambda_{i,j,k+1} - \lambda_{i,j,k} \frac{\partial F}{\partial W_k}. \tag{41}
\]

Hence, the constrained dynamics for incompressible fluid flow results in the Dirac-bracket formulation \((41)\) coupled with \((39)\) for the Lagrange multiplier \(\lambda\), and the discrete energy functional

\[
H = \sum_{(i,j,k)} \frac{1}{2} \left( \bar{U}_k^2 + \bar{V}_k^2 + \bar{W}_k^2 \right). \tag{42}
\]
Proofs of energy conservation and preservation of zero divergence in time will be presented later for the more general, DGFEM discretisation of incompressible Hamiltonian dynamics.

3.2. Discontinuous Galerkin FEM discretisation for the linearised Euler equations

In this section, we will introduce a discontinuous Galerkin FEM discretisation for the Hamiltonian structure of linear, compressible and incompressible flows. The FV discretisation of the Hamiltonian system will be used as a guide in the choice of the numerical flux.

3.2.1. Finite element space

Let $\mathcal{I}_h$ denote a tessellation of the domain $D$ with elements $K$. The set of all edges in the tessellation $\mathcal{I}_h$ is $\Gamma$, with $\Gamma_i$ the set of interior edges and $\Gamma_D$ the set of edges at the domain boundary $\partial D$. Additional notation is introduced for the numerical flux, to be introduced shortly. Let $e$ be a face between "left" and "right" elements $K_L$ and $K_R$, respectively, with corresponding outward normals $n_L$ and $n_R$. If $f$ is a continuous function on $K_L$ and $K_R$, but possibly discontinuous across the face $e$, let $f_L = (f|_{K_L})|_e$ and $f_R = (f|_{K_R})|_e$ denote the left and right traces, respectively. Let $P_p(K)$ be the space of polynomials of at most degree $p$ on $K \in \mathcal{I}_h$, with $p \geq 0$. The finite element spaces $Q_h$ and $Y_h$ required are

$$Q_h = \{ q \in L^2(D) : q|_K \in P^p(K), \forall K \in \mathcal{I}_h \},$$

$$Y_h = \{ Y \in (L^2(D))^3 : Y|_K \in (P^p(K))^3, \forall K \in \mathcal{I}_h \text{ and } (n \cdot Y)|_{\partial D} = 0 \}. \tag{43b}$$

Note that space $Y_h$ accommodates the solid-wall boundary condition. The number of degrees of freedom on an element is denoted by $N_K = \dim(P^p(K))$.

The discrete energy on the tessellated domain, cf. (8), thus becomes

$$H = \frac{1}{2} \sum_K \int_K (\mathbf{u}_h^2 + \rho_h^2) \, dK, \tag{44}$$

where $\rho_h \in Q_h$ and $\mathbf{u}_h \in Y_h$. Corresponding variational derivatives are

$$\frac{\delta H}{\delta \mathbf{u}_h} = \mathbf{u}_h \quad \text{and} \quad \frac{\delta H}{\delta \rho_h} = \rho_h. \tag{45}$$
There is some abuse of notation here, because we use functions $F$ and $H$ for functionals. However, if approximations $u_h$ and $\rho_h$ are viewed as finite-dimensional expansions, then function derivatives with respect to the expansion coefficients emerge.

### 3.2.2. Hamiltonian DGFEM discretisation for linearised compressible flow

In this section, we derive a DGFEM discretisation of the Hamiltonian structure for linearised compressible flow \[^7\]. The specific functional $F[u_h] \equiv \int_D u_h \cdot \Phi \, dx$ is chosen to obtain the discretised momentum equations in a Hamiltonian framework, with $\Phi \in \mathcal{Q}_h$ an arbitrary test function. The definition of a functional derivative is

$$\delta F := \lim_{\epsilon \to 0} \frac{F[u_h + \epsilon \delta u_h] - F[u_h]}{\epsilon} = \int_D \Phi \cdot \delta u_h \, dx.$$  \hspace{1cm} (46)

The functional derivative with respect to the velocity thus equals

$$\frac{\delta F}{\delta u_h} = \Phi.$$  \hspace{1cm} (47)

Likewise, a functional $F[\rho_h] \equiv \int_D \rho_h \phi \, dx$ is needed, with $\phi \in \mathcal{Q}_h$ an arbitrary test function. Its functional derivative equals

$$\frac{\delta F}{\delta \rho_h} = \phi.$$  \hspace{1cm} (48)

Our starting point is to simply limit functionals in the Poisson bracket \[^7\] on tessellation $\mathcal{I}_h$ to ones on the approximate finite element space, as follows

$$\frac{dF}{dt} = [F, H] \equiv \sum_K \int_K \left( \delta H \frac{\delta F}{\delta u_h} \nabla_h \cdot \delta u_h - \delta F \frac{\delta H}{\delta u_h} \nabla_h \cdot \delta u_h - 2\Omega \times \frac{\delta H}{\delta u_h} \cdot \frac{\delta F}{\delta u_h} \right) \, dx$$  \hspace{1cm} (49)

with elementwise differential operator $\nabla_h$. After integration by parts of the first two terms on the right-hand-side of \[^49\] and introduction of numerical fluxes, we obtain

$$\frac{dF}{dt} = \sum_K \int_K \left( -\nabla_h \frac{\delta H}{\delta \rho_h} \cdot \frac{\delta F}{\delta u_h} + \frac{\delta H}{\delta \rho_h} \cdot \nabla_h \frac{\delta F}{\delta u_h} - \frac{\delta F}{\delta \rho_h} \cdot \nabla_h \frac{\delta F}{\delta u_h} - 2\Omega \times \frac{\delta H}{\delta u_h} \cdot \frac{\delta F}{\delta u_h} \right) \, dK +$$

$$\sum_K \int_{\partial K} \left( \frac{\delta H}{\delta \rho_h} n \cdot \frac{\delta F}{\delta u_h} - \frac{\delta H}{\delta \rho_h} \cdot n \frac{\delta F}{\delta \rho_h} \right) \, d\Gamma,$$  \hspace{1cm} (50a)
with element boundaries \( \partial K \). Wide hats on the expressions in the boundary integrals indicate numerical fluxes. We chose the following numerical fluxes

\[
\begin{align*}
\hat{\delta F} = (1 - \theta) \frac{\delta F}{\delta u_h} + \theta \frac{\delta F}{\delta u_R} \quad \text{and} \quad \hat{\delta H} = (1 - \theta) \frac{\delta H}{\delta u_h} + \theta \frac{\delta H}{\delta u_R},
\end{align*}
\]

where \( L \) and \( R \) indicate the traces from the left and right elements connected to the faces, and \( 0 \leq \theta \leq 1 \). We emphasize the equivalence of the numerical fluxes used in (50) and in the FV discretisation (33) for the case with constant basis and test functions on each element.

We use numerical fluxes (50) and rewrite the sum over element boundaries into a sum over all faces. It gives us with (44) the following DGFEM discretisation for linear, compressible Hamiltonian dynamics

\[
\frac{dF}{dt} = \sum_K \int_K \left( -\nabla_h \frac{\delta H}{\delta \rho_h} \cdot \frac{\delta F}{\delta u_h} + \frac{\delta H}{\delta u_h} \cdot \nabla_h \frac{\delta F}{\delta \rho_h} - 2 \Omega \times \frac{\delta H}{\delta u_h} \cdot \frac{\delta F}{\delta \rho_h} \right) dK + \sum_{e \in \Gamma_i} \int_e \left( \frac{\delta H}{\delta \rho_h} - \frac{\delta H}{\delta \rho_R} \right) n \cdot \left( (1 - \theta) \frac{\delta F}{\delta u_h} + \theta \frac{\delta F}{\delta u_R} \right) + \left( \frac{\delta F}{\delta \rho_h} - \frac{\delta F}{\delta \rho_L} \right) n \cdot \left( \frac{\delta H}{\delta u_h} \theta + \frac{\delta H}{\delta u_R} (1 - \theta) \right) d\Gamma.
\]

Here \( n = n^L \) is the exterior normal vector connected with element \( K^L \).

Technically speaking, periodic boundary conditions can be specified in ghost cells (denoted with subscript \( R \)), where values of the variables exactly coincide with the face-adjacent cell values (denoted with subscript \( L_p \)) at the other side of the periodic boundary

\[
\frac{\delta H}{\delta U_R} \cdot n = \frac{\delta H}{\delta U_{L_p}} \cdot n \quad \text{and} \quad \frac{\delta H}{\delta \rho_R} = \frac{\delta H}{\delta \rho_{L_p}},
\]

with \( n \) the normal to the boundary face. Geometrically speaking, there are of course only internal cells in a periodic domain. In the case of a three-dimensional cuboid bounded by solid walls, the numerical fluxes on both the test functions and the Hamiltonian derivatives must vanish, cf. our specifications in (43). In terms of ghost cells, it implies that

\[
(1 - \theta) \frac{\delta F}{\delta u_h} + \theta \frac{\delta F}{\delta u_R} = 0 \quad \text{and} \quad (1 - \theta) \frac{\delta H}{\delta u_h} + \theta \frac{\delta H}{\delta u_R} = 0 \quad \text{at} \quad \Gamma_d. \quad (53)
\]
We will use shortly that boundary conditions for incompressible flow are automatically satisfied by using Dirac’s theory, given that those boundary conditions are satisfied for the discrete, compressible Hamiltonian discretisation.

By construction, the bracket \( [51] \) remains skew-symmetric. Unconventional is that the numerical flux is also acting on the test functions \( \delta F / \delta u_h \).

We refer to \([32]\) for a proof that the bracket \( [51] \) can be transformed to a classical, discontinuous Galerkin finite element weak formulation with alternating fluxes, provided \( \theta = 1/2 \) at boundary faces and for constant material parameters. When material parameters are a function of space, then the Hamiltonian formulation with its division between bracket and Hamiltonian becomes crucial. Not only the skew-symmetric or alternating fluxes matter then but also the dual, Hamiltonian projection. The DG discretisation with a polynomial approximation of order zero will exactly coincide with our FV discretisation. Note that for \( \theta = 1/2 \) the well-known image boundary condition emerges from \( [53] \). We emphasize, though, that for \( \theta \neq 1/2 \) our general condition \( [53] \) still applies, but that it seems no longer quite equivalent to the standard, alternating flux formulation.

Variables are expanded on each element \( K \) in terms of local basis functions: \( u_h = \phi_\beta u_\beta \) and \( \rho_h = \phi_\beta \rho_\beta \). Both coefficients and test functions require elemental superscripts, which we silently omit. Greek numerals are used locally on each element \( K \). Variational and function derivatives are then related as follows

\[
\delta F = \sum_K \int_K \frac{\delta F}{\delta u_h} \delta u_h + \frac{\delta F}{\delta \rho_h} \delta \rho_h dK \tag{54a}
\]

\[
= \sum_K \left( \int \frac{\delta F}{\delta u_h} \phi_\beta dK \right) \delta u_\beta + \left( \int \frac{\delta F}{\delta \rho_h} \phi_\beta dK \right) \delta \rho_\beta \tag{54b}
\]

\[
= \sum_K \frac{\partial F}{\partial u_\beta} \delta u_\beta + \frac{\partial F}{\partial \rho_\beta} \delta \rho_\beta. \tag{54c}
\]

Similarly, by starting from \( [54c] \) and using the relation

\[
M_{\alpha\beta} u_\beta = \int_K \phi_\beta u_h dK \tag{55}
\]

with local mass matrix \( M_{\alpha\beta} = M^K_{\alpha\beta} \), one can derive

\[
\frac{\delta F}{\delta u_h} = M^{-1}_{\beta\gamma} \frac{\partial F}{\partial u_\beta} \phi_\gamma \quad \text{and} \quad \frac{\delta F}{\delta \rho_h} = M^{-1}_{\beta\gamma} \frac{\partial F}{\partial \rho_\beta} \phi_\gamma. \tag{56}
\]
By substitution of (56) into (51), we immediately derive the desired form of the finite-dimensional Hamiltonian discretisation

\[
\frac{dF}{dt} = \sum_K \left( \frac{\partial H}{\partial u_\beta} \frac{\partial F}{\partial \rho_\alpha} - \frac{\partial F}{\partial u_\beta} \frac{\partial H}{\partial \rho_\alpha} \right) \cdot E_{\mu\gamma} M^{-1}_{\beta\gamma} M^{-1}_{\mu\alpha} - 2\Omega \times \frac{\partial H}{\partial u_\alpha} \cdot \frac{\partial F}{\partial u_\beta} M^{-1}_{\alpha\beta} \\
+ \sum_{e \in \Gamma_i} (1 - \theta) \left( \frac{\partial H}{\partial \rho^L_\alpha} \frac{\partial F}{\partial u^L_\beta} - \frac{\partial F}{\partial \rho^L_\alpha} \frac{\partial H}{\partial u^L_\beta} \right) \cdot G^{LL}_{\mu\gamma} M^{-L}_{\alpha\mu} M^{-L}_{\beta\gamma} \\
+ \theta \left( \frac{\partial H}{\partial \rho^R_\alpha} \frac{\partial F}{\partial u^R_\beta} - \frac{\partial F}{\partial \rho^R_\alpha} \frac{\partial H}{\partial u^R_\beta} \right) \cdot G^{LR}_{\mu\gamma} M^{-L}_{\alpha\mu} M^{-R}_{\beta\gamma} \\
+ (1 - \theta) \left( \frac{\partial H}{\partial \rho^R_\alpha} \frac{\partial F}{\partial u^L_\beta} - \frac{\partial F}{\partial \rho^R_\alpha} \frac{\partial H}{\partial u^L_\beta} \right) \cdot G^{RL}_{\mu\gamma} M^{-R}_{\alpha\mu} M^{-L}_{\beta\gamma} \\
+ \theta \left( \frac{\partial H}{\partial \rho^L_\alpha} \frac{\partial F}{\partial u^R_\beta} - \frac{\partial F}{\partial \rho^L_\alpha} \frac{\partial H}{\partial u^R_\beta} \right) \cdot G^{RR}_{\mu\gamma} M^{-R}_{\alpha\mu} M^{-R}_{\beta\gamma} 
\]  

(57)

with elemental (vector) matrices \( E_{\mu\gamma} \) and \( G^{LL}_{\mu\gamma} \), et cetera. These read

\[
E_{\mu\gamma} = \int_K \phi_\gamma \nabla_h \phi_\mu dK \quad \text{and} \quad G^{LL}_{\mu\gamma} = \int_e n_\alpha \phi^L_\mu \phi^L_\gamma d\Gamma. \tag{58}
\]

Finally, after substitution of (56), the Hamiltonian (44) becomes

\[
H = \frac{1}{2} \sum_K M_{\alpha\beta} \left( u_\alpha \cdot u_\beta + \rho_\alpha \rho_\beta \right). \tag{59}
\]

A global formulation is useful for the incompressible case. We therefore introduce a reordering into global coefficients \( U_i = U_i(t) = (U, V, W)^T \) and \( R_k(t) \), instead of the elemental ones, in the finite element expansion of \( u_h \) and \( \rho_h \) with the indices running over their respective, global ranges. It turns out that the local matrices \( M_{\alpha\beta} \) and \( E_{\mu\gamma} \) in (57) and (59) readily extend to global matrices \( M_{ij} \) and \( E_{ij} \). These have a block structure in which each elemental matrix fits in separation from the others. The contribution of the numerical fluxes lead to coupling between the elements, which can be incorporated into a global matrix \( G_{ij} \). The latter is straightforwardly defined computationally by a loop over the faces, and we will therefore not provided an explicit expression. The resulting, global Hamiltonian formulation then
becomes
\[
\frac{dF}{dt} = [F, H]_d = \left( \frac{\partial F}{\partial U_i} \frac{\partial F}{\partial R_j} - \frac{\partial F}{\partial U_j} \frac{\partial H}{\partial R_i} \right) \cdot \mathbf{E}_{kl} M^{-1}_{ik} M^{-1}_{jl} 
- 2\Omega \times \frac{\partial H}{\partial U_i} \cdot \frac{\partial F}{\partial U_j} M^{-1}_{ij} + \left( \frac{\partial H}{\partial R_i} \frac{\partial F}{\partial U_j} - \frac{\partial F}{\partial U_i} \frac{\partial H}{\partial R_j} \right) \cdot \mathbf{G}_{kl} M^{-1}_{ik} M^{-1}_{jl}
\]

(60a)

with global Hamiltonian
\[
H = \frac{1}{2} M_{ij} (U_i \cdot U_j + R_i R_j). 
\]

(60b)

The resulting equations of motion arising from (60) are
\[
\dot{U}_i = M^{-1}_{ik} R_l (\mathbf{E}_{kl} - \mathbf{G}_{ik}) - 2\Omega \times U_i 
\]

(61a)

\[
M_{kl} \dot{R}_l = -U_j \cdot (\mathbf{E}_{jk} - \mathbf{G}_{kj}) 
\]

(61b)

with the dot denoting a time derivative.

3.2.3. Hamiltonian DGFEM discretisation for linearised incompressible flow

In close analogy with the continuous case and the FV-case, Dirac’s theory is applied to the Hamiltonian dynamics (60). The density expansion coefficients are all restricted in every local element such that the resulting density in the element is zero. The following primary constraints are imposed on the discrete density field
\[
D_k = M_{kl} R_l. 
\]

(62)

Preservation of the constraints in time leads to the following consistency relation
\[
0 = \dot{D}_k = [D_k, H]_d + \lambda_l [D_k, G_l]_d. 
\]

(63)

Using (62) in the bracket (60a) shows that \([D_k, G_l]_d = 0\). The Lagrange multipliers \(\lambda_l\) in (63) thus remain undetermined, but the consistency requirement gives rise to secondary constraints \(L_k = [D_k, H]_d = 0\). Analogous to the continuous case and FV-case, the secondary constraint is the discrete version of the divergence-free velocity field property (18). To wit
\[
L_k = [D_k, H] = (\mathbf{E}_{ik} - \mathbf{G}_{kl}) \cdot U_l \equiv \text{DIV}_{kl} \cdot U_l, 
\]

(64)
in which we defined the discrete divergence, vector operator $\text{DIV}_{kl}$. We start again with both primary constraints and require these to remain preserved under the Hamiltonian dynamics. We obtain

$$0 = \dot{D}_k = [D_k, H]_d + \mu_t[D_k, L_t]_d, \quad (65a)$$

$$0 = \dot{L}_k = [L_k, H]_d + \lambda_t[L_k, D_t]_d + \mu_t[L_k, L_t]_d. \quad (65b)$$

Application of the primary constraint implies that $L_k = [D_k, H] = 0$ in (65a). Hence,

$$\mu_t[D_k, L_t] = \mu_t\text{DIV}_{lm}M_{mj}^{-1} \cdot \text{DIV}_{kj} = 0. \quad (66)$$

The matrix acting on $\mu_t$ is a discrete Laplacian. It is nonsingular, whence $\mu_k = 0$. Consequently, (65b) reduces to

$$0 = \dot{L}_k = [L_k, H]_d + \lambda_t[L_k, D_t]_d$$

$$= -\text{DIV}_{kj} \cdot 2\Omega \times U_j - \lambda_t\text{DIV}_{km}M_{mj}^{-1} \cdot \text{DIV}_{lj}, \quad (67a)$$

which is the discrete equivalent of the Poisson equation in (29). Finally, the resulting discrete, linear, incompressible Hamiltonian dynamics is given by

$$\frac{dF}{dt} = [F, H]_{inc} \equiv -\frac{\partial F}{\partial U_j} \cdot (2\Omega \times \frac{\partial H}{\partial U_i}M_{ij}^{-1} + \lambda_tM_{jk}^{-1}\text{DIV}_{lk}) \quad (68a)$$

with energy function

$$H = \frac{1}{2}M_{ij}U_i \cdot U_j. \quad (68b)$$

The final system consists of the ordinary differential equations following directly from (68) after using $F = U_j$:

$$\dot{U}_j = -2\Omega \times U_j - \lambda_tM_{jk}^{-1} \cdot \text{DIV}_{lk}, \quad (69)$$

combined with (67):

$$\lambda_t\text{DIV}_{km}M_{mj}^{-1} \cdot \text{DIV}_{lj} = -\text{DIV}_{kj} \cdot 2\Omega \times U_j. \quad (70)$$

4. Time Integrator

We consider a symplectic time integrator for the time discretisation of the linear, compressible (61a) and incompressible (69) Hamiltonian dynamics. Symplectic time integrators form the subclass of geometric integrators which, by definition, are canonical transformations. The modified midpoint time integrator was chosen from amongst other symplectic schemes [14]. It is implicit, which requires more computation, but pays off in dealing with the momentum and continuity equations, in a rotating frame of reference.
4.1. Linear, compressible flow

Applying the modified midpoint scheme to the discrete compressible Hamiltonian dynamics (60a) presented in matrix form in (61), one gets

\[ \frac{U^n_{i+1} - U^n_i}{\Delta t} = M^{-1}_{ik} \left( R^{n+1}_l + R^n_l \right) (E_{kl} - G_{lk}) - \Omega \times (U^{n+1}_i + U^n_i) \quad (71a) \]

\[ \frac{M_{kl} (R^{n+1}_l - E^n_l)}{\Delta t} = - \frac{U^{n+1}_m + U^n_m}{2} \cdot (E_{mk} - G_{km}) \quad (71b) \]

**Proposition 1.** The numerical scheme for linear, compressible fluid flow given by (71) is exactly energy conserving, such that

\[ M_{ij} (U^{n+1}_i \cdot U^{n+1}_j + R^{n+1}_i R^{n+1}_j) = M_{ij} (U^n_i \cdot U^n_j + R^n_i R^n_j). \]

**Proof.** Multiply equations (71) with \( M_{ij} U^{n+1}_j, R^{n+1}_l \) and \( M_{ij} U^n_j, R^n_l \). Thereafter, add them up. After some manipulation, the Hamiltonian on the \((n+1)\)-th time level can be shown to equal the Hamiltonian on the \(n\)-th level. \( \square \)

4.2. Incompressible flow

The midpoint time integrator is also applied to the incompressible discrete Hamiltonian dynamics (68a) in a matrix form (69), giving

\[ \frac{(U^{n+1}_j - U^n_j)}{\Delta t} = -\Omega \times (U^{n+1}_j + U^n_j) - \lambda^{n+1}_l M^{-1}_{jk} \cdot \text{DIV}_{lk} \quad (72a) \]

\[ \lambda^{n+1}_l \text{DIV}_{km} M^{-1}_{mj} \cdot \text{DIV}_{lj} = -\text{DIV}_{kj} \cdot \Omega \times (U^{n+1}_j + U^n_j). \quad (72b) \]

**Proposition 2.** The numerical scheme for linear, incompressible fluid flow given by (72) exactly conserves energy and the discrete zero-divergence property in time, such that \( \text{DIV}_{lm} \cdot U^{n+1}_m = 0 \) given that \( \text{DIV}_{lm} \cdot U^n_m = 0 \) and \( M_{ij} U^{n+1}_j = M_{ij} U^n_j \).

**Proof.** Firstly, we present the proof for the conservation of the discrete zero-divergence property, under the assumption that the present velocity is discretely divergence free, i.e., \( L_t = \text{DIV}_{lm} \cdot U^n_m = 0 \). We apply the discrete divergence operator on both sides of the (72a) and use that the present velocity is divergence free, to obtain

\[ \text{DIV}_{mj} \cdot U^{n+1}_j / \Delta t = -\text{DIV}_{mj} \cdot \Omega \times (U^{n+1}_j + U^n_j) \]

\[ -\text{DIV}_{mj} \cdot \lambda^{n+1}_l M^{-1}_{jk} \cdot \text{DIV}_{lk}. \quad (73) \]
The right hand side of (73) exactly coincides with (72b) and therefore, $\text{DIV} l_m \cdot U_{n+1}^m = 0$.

Secondly, energy conservation means that the discrete Hamiltonian energy functional (60b) stays unchanged in time. Multiplication of (72a) with $M_{ij} U_{i}^{n+1}$ and $M_{ij} U_{i}^{n}$ followed by summation of both equations yields

$$\frac{M_{ij} (U_{i}^{n+1} \cdot U_{i}^{n+1} - U_{i}^{n} \cdot U_{i}^{n})}{\Delta t} = - \lambda_{i}^{n} \text{DIV} l_{i} \cdot (U_{i}^{n+1} + U_{i}^{n}) = 0,$$

(74)

because the terms involving rotational effects cancel and because in the last step we use that present and future time velocities are divergence free, as shown in the first part of this proof. Hence, the difference of the energy at the $(n + 1)$-th and $n$-th level is zero.

4.3. Initial conditions

As proven above, the numerical scheme for linearised incompressible fluid flow discretely conserves energy and is divergence free. The proofs required exact satisfaction of the discrete, divergence-free equation for the initial condition, i.e., $\text{DIV} l_m \cdot U_{0}^m = 0$. This condition is not guaranteed automatically, since the projection of the initial, divergence-free velocity field on the chosen discontinuous Galerkin finite element space only satisfies discrete zero-divergence up to the order of accuracy. We therefore require a preprocessing step on this projected velocity $U$. We are looking for a $U^*$ for which $\text{DIV} l_m \cdot U^*_m = 0$ exactly and $||U^* - U||$ is minimal. Note that the (“vector”) matrix $\text{DIV} l_m$ is not square. Hereafter, we denote the associated, grand matrix by $\text{DIV}$ and the grand vector of velocity unknowns as $U$ (so without indices). Basically, the latter problem transforms into a well-known problem in vector calculus: find a projection of the vector $U$ on the space $A$: the nullspace of discrete divergence matrix operator $\text{DIV}$

$$A = \{ Q \in \mathbb{R}^{3N_{dof}} : \text{DIV} Q = 0 \}.$$  

(75)

with $N_{dof}$ the degrees of freedom per velocity component (assuming them to be equal for simplicity).

From linear algebra [10, 11], we obtain that the closest vector from the $A$-space will be the projection of vector $U$ on the space, which is

$$U^* \equiv \text{proj}_A U = U + U_\perp.$$  

(76)
Applying the $\text{DIV}$–operator on (76), we find
\[
0 = \text{DIV} U^* = \text{DIV} U + \text{DIV} U_\perp, \tag{77}
\]
which results in
\[
\text{DIV} U_\perp = -\text{DIV} U, \tag{78}
\]
The latter equation is solved for $U_\perp$ via a least-square approximation [13] up to machine precision. Hence, the projected velocity is preprocessed using (76), and the resulting velocity field has become exact discrete divergence free, as required.

4.4. Other properties of the algebraic system

A direct DGFEM discretisation of the incompressible Navier-Stokes equations (or the Euler equations as special case) generally requires the inf-sup condition to be satisfied stable [12, 20]. In order to get a stable pressure approximation, two different strategies are often pursued: either a pressure stabilization term is used or the approximation spaces for velocity and pressure are chosen such that an inf-sup compatibility condition is fulfilled. Nonetheless, our numerical discretisation for linear, incompressible flow does not suffer from those drawbacks. The exact preservation of the Hamiltonian dynamics as well as the constraints makes the system unconditionally stable.

The three-dimensionality of the problem results in a large algebraic system, which we represent using the sparse matrix structures available in PETSc [27, 28]. Figure 3 shows the sparsity of a matrix, needed to determine $U_m^{n+1}$ and $\lambda_m^{n+1}$ in the discretisation for incompressible flow (71). In spite of this sparse structure, the resulting matrix for the finest meshes possible, given memory restrictions, has on the order of 100 million degrees of freedom. We use a linear, iterative solver to converge to the desired tolerance. To make the convergence of the iterative solver faster, ILU preconditioners were used with controlled memory usage of the resulting algebraic system.
Figure 3: A structure of square sparse matrix $P$ with more than $10^8$ non-zero elements and $\Omega_1 = \Omega_2 = 0$. $\Lambda$ denotes the vector of unknown Lagrange multipliers.
5. Tests of numerical scheme

The discretisations for linear compressible and incompressible flows were implemented for discontinuous Galerkin methods in the hpGEM C++ software framework \[24\]. The developed applications are consequently highly object-oriented, easy to maintain and extend. Although the tests considered concern cuboids, the implementation can cope with general geometries and meshes. The three-dimensionality of the problem poses, however, significant requirements on speed and memory use. The sparse matrix data structures available in the PETSc library are therefore used. An ILU preconditioner is applied to the linear algebraic system before applying a GMRES linear solver \[27, 28\]. The number of iterations varies for the different test cases and strongly depends on the dimensions of the algebraic system, e.g., the grid size and the amount of basis functions. In the case of quadratic polynomials with a grid of \(64 \times 64 \times 64\) elements, which is the computationally most demanding case, one needs approximately forty iterations to reach the tolerance \(10^{-14}\) for resolving the incompressible flow. Various test cases will be discussed next. Although the main goal is to simulate inertial waves in a rectangular box, four extra test cases will be presented to verify the approaches and techniques used. In the tests, \(\theta = 1/2\) was used in the numerical flux. For various test cases on one mesh also other value \(0 \leq \theta \leq 1\) were used, with similar results.

5.1. Compressible harmonic waves in a periodic domain

Consider linear, compressible fluid flow with zero rotation \(\Omega = (0, 0, 0)\) in a rectangular triple periodic domain \(D = [0, 1]^3\). The following expressions satisfy the linear Euler equations

\[
\begin{align*}
u & = A_1 \cos(2k_x(t - x)), \\
v & = A_2 \cos(2k_y(t - y)), \\
w & = A_3 \cos(2k_z(t - z)), \\
\rho & = A_1 \cos(2k_x(t - x)) + A_2 \cos(2k_y(t - y)) + A_3 \cos 2k_z(t - z). 
\end{align*}
\]

Each component of the velocity vector is a traveling wave in the direction of the corresponding axes. The numerical discretisation is initialised with \(79\) at \(t = 0\), \(k_x = k_y = k_z = 2\pi\) and \(A_1 = A_2 = A_3 = 1\). Numerical and exact solutions were compared during several periods. Figure \[4\] presents the numerical density profile during one time period of the traveling waves.
Discrete energy is conserved up to machine precision even after one hundred
periods. The results of a convergence analysis, presented in Table 1, show
that the numerical scheme is first, second and third order accurate in space
for, respectively constant, linear and quadratic polynomial approximations.

Table 1: Convergence of the error for compressible traveling waves in a three-dimensional
periodic domain. Due to symmetry all velocity components have the same error.

| Grid  | $p = 0$ | $|l^2|$-error | order | $|l^2|$-error | order | $|l^2|$-error | order |
|-------|-----|---------|------|---------|------|---------|------|
| 4x4x4 | u   | 1.011e+0 | –    | 4.3024E-1 | –    | 1.8532E-2 | –    |
|       | ρ   | 1.751e+0 | –    | 7.4520E-1 | –    | 3.2098E-2 | –    |
| 8x8x8 | u   | 8.249E-1 | 0.3  | 1.3087E-1 | 1.7  | 3.0944E-3 | 2.6  |
|       | ρ   | 1.428e+0 | 0.3  | 2.2667E-1 | 1.7  | 5.3598E-3 | 2.6  |
| 16x16x16 | u | 2.405E-1 | 1.8  | 2.9192E-2 | 2.1  | 3.4723E-4 | 3.2  |
|        | ρ  | 4.166E-1 | 1.8  | 5.0554E-2 | 2.1  | 6.0142E-4 | 3.2  |
| 32x32x32 | u | 7.193E-2 | 1.7  | 5.0432E-3 | 2.5  | 2.9366E-5 | 3.6  |
|        | ρ  | 1.245E-1 | 1.7  | 8.7321E-3 | 2.5  | 5.0864E-5 | 3.6  |
| 64x64x64 | u | 2.638E-2 | 1.5  | 1.0813E-3 | 2.2  | 1.8266E-6 | 4.0  |
|        | ρ  | 4.579E-2 | 1.4  | 2.3901E-3 | 1.8  | 3.1638E-6 | 4.0  |

5.2. Compressible waves with slip-flow boundary conditions

Next, linear, acoustic fluid flow is considered in domain $D$, but now slip-
flow boundary conditions are used with zero normal component of the veloc-
ity field at domain boundaries. Boundary conditions are effectively imple-
mented with the help of ghost cells, where the values of velocity and density
fields are specified to satisfy the boundary conditions. One can check that

\[
\begin{align*}
  u &= -A_1 \sin(2k_x x) \cos(\omega t), \\
  v &= -A_2 \sin(2k_y y) \cos(\omega t), \\
  w &= -A_3 \sin(2k_z z) \cos(\omega t), \\
  \rho &= A_1 \sin(2k_x x) \cos(\omega t) + A_2 \sin(2k_y y) \cos(\omega t) + A_3 \sin(2k_z z) \cos(\omega t).
\end{align*}
\]

exactly satisfy the linearised compressible Euler equations with slip-flow
boundary conditions. The numerical scheme is initialised using (80) at $t = 0,
where $\omega = k_x = k_y = k_z = 2\pi$ and $A_1 = A_2 = A_3 = 1$. Figure 5 shows the
numerical density profile during a full time period. Discrete energy again
is conserved up to machine precision, after one hundred wave periods. A
convergence analysis is given in Table 2.
Figure 4: Plots of the density field computed with the discretised compressible Hamiltonian formulation. A 32 x 32 x 32 grid with time step $\Delta t = T/20$, where $T$ is the time period of the harmonic waves, was used in a periodic domain.

Table 2: Convergence of the error for compressible standing waves in a cuboid with solid walls. Due to symmetry all velocity components have the same error.

<table>
<thead>
<tr>
<th>Grid</th>
<th>$p = 0$</th>
<th>$p = 1$</th>
<th>$p = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$l^2$-error</td>
<td>order</td>
<td>$l^2$-error</td>
</tr>
<tr>
<td>4x4x4</td>
<td>u</td>
<td>7.909E-1</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>$\rho$</td>
<td>2.391e+0</td>
<td>–</td>
</tr>
<tr>
<td>8x8x8</td>
<td>u</td>
<td>5.012E-1</td>
<td>0.7</td>
</tr>
<tr>
<td></td>
<td>$\rho$</td>
<td>1.134e+0</td>
<td>0.6</td>
</tr>
<tr>
<td>16x16x16</td>
<td>u</td>
<td>8.794E-2</td>
<td>2.5</td>
</tr>
<tr>
<td></td>
<td>$\rho$</td>
<td>3.877E-1</td>
<td>1.4</td>
</tr>
<tr>
<td>32x32x32</td>
<td>u</td>
<td>4.013E-2</td>
<td>1.1</td>
</tr>
<tr>
<td></td>
<td>$\rho$</td>
<td>1.033E-1</td>
<td>1.9</td>
</tr>
<tr>
<td>64x64x64</td>
<td>u</td>
<td>2.003E-2</td>
<td>1.0</td>
</tr>
<tr>
<td></td>
<td>$\rho$</td>
<td>2.973E-2</td>
<td>1.8</td>
</tr>
</tbody>
</table>
Figure 5: The results are obtained on a 32 x 32 x 32 grid with $\Delta t = T/20$, where the $T$ is the time period of the standing, compressible waves in a closed cuboid.
5.3. Incompressible waves in a periodic domain

The compressible test cases were mainly interesting as a quality assurance step for linearised incompressible fluid flow, which we consider next. An exact solution was found for the linear, incompressible, rotational Euler equations with periodic boundary conditions

\[ u = \frac{1}{2\pi} \left[ \sqrt{3} \cos \left( 2\pi(x + y + z) + \frac{\sqrt{3}t}{3} \right) + 3 \sin \left( 2\pi(x + y + z) + \frac{\sqrt{3}t}{3} \right) \right], \quad (81a) \]

\[ v = \frac{1}{2\pi} \left[ \sqrt{3} \cos \left( 2\pi(x + y + z) + \frac{\sqrt{3}t}{3} \right) - 3 \sin \left( 2\pi(x + y + z) + \frac{\sqrt{3}t}{3} \right) \right], \quad (81b) \]

\[ w = -\frac{1}{\pi} \sqrt{3} \cos \left( 2\pi(x + y + z) + \frac{\sqrt{3}t}{3} \right), \quad (81c) \]

\[ P = \frac{1}{2\pi^2} \cos \left( 2\pi(x + y + z) + \frac{\sqrt{3}t}{3} \right), \quad (81d) \]

where the rotation vector is \( \Omega = (0, 0, 1) \) and \( P \) is the pressure. This exact solution is used for the initialisation in a periodic domain \( D = [0, 1]^3 \). As was already discussed, the Lagrange multiplier \( \lambda = P \) plays the role of the pressure in our incompressible Hamiltonian discretisation. The numerical velocity and pressure fields are compared against the exact solution during several wave periods. Figure 6 gives an example of the numerical solution during one period. Figure 7 shows that the conservation of energy and discrete zero-divergence in time are satisfied up to machine precision. To ensure that the velocity field has zero-divergence, one has to initialise the numerical scheme with an exact discrete divergence-free velocity field. Thus, adjustment of the initial projection of the velocity field is required to satisfy this condition exactly (up to machine precision). The energy change observed at \( t = 0 \) in Figure 7 originated from this projection, is within the order of accuracy of the numerical approximation. Table 3 presents the rate of the convergence.

5.4. Poincaré waves in a channel

Poincaré waves in a channel with incompressible fluid are considered next. The channel is periodic in the \( y \)-direction and closed with walls in the \( x \)- and \( z \)-directions. The angular rotation vector is equal to \( \Omega = (0, 0, 1) \). An exact
Table 3: Convergence of the error for incompressible periodic waves in a cuboid. Due to symmetry all velocity components have the same error.

<table>
<thead>
<tr>
<th>Grid</th>
<th>$p = 0$</th>
<th>$p = 1$</th>
<th>$p = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$l^2$-error</td>
<td>order</td>
<td>$l^2$-error</td>
</tr>
<tr>
<td>4x4x4</td>
<td>2.665E-1</td>
<td>–</td>
<td>1.6340E-1</td>
</tr>
<tr>
<td></td>
<td>2.872E-2</td>
<td>–</td>
<td>1.3876E-2</td>
</tr>
<tr>
<td>8x8x8</td>
<td>1.477E-1</td>
<td>0.9</td>
<td>5.3412E-2</td>
</tr>
<tr>
<td></td>
<td>1.411E-2</td>
<td>1.0</td>
<td>4.6244E-3</td>
</tr>
<tr>
<td>16x16x16</td>
<td>7.587E-2</td>
<td>1.0</td>
<td>1.8100E-2</td>
</tr>
<tr>
<td></td>
<td>7.141E-3</td>
<td>1.0</td>
<td>1.4475E-3</td>
</tr>
<tr>
<td>32x32x32</td>
<td>3.822E-2</td>
<td>1.0</td>
<td>5.7218E-3</td>
</tr>
<tr>
<td></td>
<td>3.737E-3</td>
<td>0.9</td>
<td>4.5473E-4</td>
</tr>
<tr>
<td>64x64x64</td>
<td>1.919E-2</td>
<td>1.0</td>
<td>1.6772E-3</td>
</tr>
<tr>
<td></td>
<td>2.169E-3</td>
<td>0.8</td>
<td>1.3692E-4</td>
</tr>
</tbody>
</table>

(a) Pressure profile at $t=0$
(b) Pressure profile at $t=T/4$
(c) Pressure profile at $t=T/2$
(d) Pressure profile at $t=3T/4$

Figure 6: The results concern an incompressible Hamiltonian discretisation on a 32 x 32 x 32 grid with $\Delta t = T/20$ and period $T$. The implementation concerns a quadratic polynomial approximation on local elements in a periodic domain.
solution for Poincaré waves in $D = [0, 1]^3$ reads
\begin{align}
    u &= -\frac{k\sigma^3}{1 - \sigma^2}\left(1 + \frac{l^2}{(\sigma k)^2}\right)\sin(kx) \sin(l(y - \sigma t)) \cos(2\pi z), \tag{82a} \\
    v &= \left(-l\sigma \cos(kx) + \frac{1}{k}\sin(kx)\right) \cos(l(y - \sigma t)) \cos(2\pi z), \tag{82b} \\
    w &= \sigma \left(\cos(kx) + \frac{l}{\sigma k}\sin(kx)\right) \sin(l(y - \sigma t)) \sin(2\pi z), \tag{82c} \\
    P &= -\sigma^2 \left(\cos(kx) + \frac{l}{\sigma k}\sin(kx)\right) \cos(l(y - \sigma t)) \cos(2\pi z), \tag{82d}
\end{align}

where $k = 2\pi$, $l = 2\pi$, $\sigma = 1/\sqrt{k^2 + l^2 + 1}$. At the solid walls in the $x$- and $z$-directions we need to satisfy the geostrophic balance on the boundaries, due to the rotation of the domain. In Figures 8 and 9 we present a numerical solution (velocity vector and scalar pressure fields) during one period. Figure 10 demonstrates discrete conservation of the energy and zero-divergence velocity properties. Convergence test results are given in Table 4.

5.5. Inertial waves

Finally, we consider linear, incompressible, rotational fluid flow in a rectangular box with solid wall boundary conditions on all sides. Such kind of flow will lead to inertial waves in the interior of the domain. An improved semi-analytical solution of this problem analysed in [17] has been derived in
Table 4: Convergence of error for incompressible Poincaré waves in a channel.

<table>
<thead>
<tr>
<th>Grid</th>
<th>$t^2$-error $p = 0$</th>
<th>order</th>
<th>$t^2$-error $p = 1$</th>
<th>order</th>
<th>$t^2$-error $p = 2$</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>4x4x4</td>
<td>$u$ 7.834e+0</td>
<td>–</td>
<td>1.3599e+0</td>
<td>–</td>
<td>2.9126E-1</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>$v$ 7.492e+0</td>
<td>–</td>
<td>1.4793e+0</td>
<td>–</td>
<td>2.5108E-1</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>$w$ 8.935e+0</td>
<td>–</td>
<td>1.6662e+0</td>
<td>–</td>
<td>2.4933E-1</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>$p$ 5.819e+0</td>
<td>–</td>
<td>4.7072e+0</td>
<td>–</td>
<td>2.1281E-1</td>
<td>–</td>
</tr>
<tr>
<td>8x8x8</td>
<td>$u$ 3.889e+0</td>
<td>1.0</td>
<td>5.7839E-1</td>
<td>1.3</td>
<td>2.8640E-2</td>
<td>3.3</td>
</tr>
<tr>
<td></td>
<td>$v$ 3.802e+0</td>
<td>1.0</td>
<td>6.1504E-1</td>
<td>1.3</td>
<td>2.0699E-2</td>
<td>3.6</td>
</tr>
<tr>
<td></td>
<td>$w$ 4.042e+0</td>
<td>1.1</td>
<td>6.7238E-1</td>
<td>1.3</td>
<td>2.0075E-2</td>
<td>3.6</td>
</tr>
<tr>
<td></td>
<td>$p$ 2.290e+0</td>
<td>1.3</td>
<td>9.1548E-1</td>
<td>2.3</td>
<td>2.9025E-2</td>
<td>2.9</td>
</tr>
<tr>
<td>16x16x16</td>
<td>$u$ 2.192e+0</td>
<td>0.8</td>
<td>1.9858E-1</td>
<td>1.5</td>
<td>3.1792E-3</td>
<td>3.1</td>
</tr>
<tr>
<td></td>
<td>$v$ 2.229e+0</td>
<td>0.8</td>
<td>2.4017E-1</td>
<td>1.4</td>
<td>2.3963E-3</td>
<td>3.1</td>
</tr>
<tr>
<td></td>
<td>$w$ 2.015e+0</td>
<td>1.0</td>
<td>2.4244E-1</td>
<td>1.5</td>
<td>2.2733E-3</td>
<td>3.1</td>
</tr>
<tr>
<td></td>
<td>$p$ 1.179e+0</td>
<td>1.0</td>
<td>3.2671E-1</td>
<td>1.5</td>
<td>3.2460E-3</td>
<td>3.1</td>
</tr>
<tr>
<td>32x32x32</td>
<td>$u$ 1.136e+0</td>
<td>0.9</td>
<td>6.3126E-2</td>
<td>1.6</td>
<td>4.1469E-4</td>
<td>2.9</td>
</tr>
<tr>
<td></td>
<td>$v$ 1.169e+0</td>
<td>0.9</td>
<td>8.4394E-2</td>
<td>1.5</td>
<td>3.2670E-4</td>
<td>2.9</td>
</tr>
<tr>
<td></td>
<td>$w$ 1.065e+0</td>
<td>0.9</td>
<td>8.5931E-2</td>
<td>1.5</td>
<td>3.1967E-4</td>
<td>2.8</td>
</tr>
<tr>
<td></td>
<td>$p$ 5.932E-1</td>
<td>1.0</td>
<td>1.0317E-1</td>
<td>1.7</td>
<td>4.3572E-4</td>
<td>2.9</td>
</tr>
<tr>
<td>64x64x64</td>
<td>$u$ 5.726E-1</td>
<td>1.0</td>
<td>1.8031E-2</td>
<td>1.8</td>
<td>5.5452E-5</td>
<td>2.9</td>
</tr>
<tr>
<td></td>
<td>$v$ 5.899E-1</td>
<td>1.0</td>
<td>2.4461E-2</td>
<td>1.8</td>
<td>4.5485E-5</td>
<td>2.9</td>
</tr>
<tr>
<td></td>
<td>$w$ 5.258E-1</td>
<td>1.0</td>
<td>2.3687E-2</td>
<td>1.9</td>
<td>4.3963E-5</td>
<td>2.9</td>
</tr>
<tr>
<td></td>
<td>$p$ 2.961E-1</td>
<td>1.0</td>
<td>3.1627E-2</td>
<td>1.7</td>
<td>4.3572E-5</td>
<td>2.9</td>
</tr>
</tbody>
</table>
Figure 8: Numerical $u$ and $v$ components are presented for Poincaré waves. Numerical results concern the incompressible Hamiltonian discretisation on a $32 \times 32 \times 32$ grid with $\Delta t = T/20$. We consider a quadratic polynomial approximation in local elements.
Figure 9: $w$ component of velocity and linearized scalar pressure fields are presented during one period of a Poincaré-wave simulation.
It is used as test solution for the implementation of our incompressible Hamiltonian discretisation with slip-flow boundary conditions. Due to the slow convergence of the semi-analytical solutions, the comparison can, however, only be done for restricted mesh sizes.

Since the exact solution is unknown, we use solutions on a sequence of uniform meshes to obtain an estimate for the rate of convergence, which is called Richardson extrapolation \[30\]. We take a uniformly refined sequence of meshes \( h_4 = h_{ref} < h_3 < h_2 < h_1 \), where the mesh-size \( h_i \) (\( i = 1, 2, 3, 4 \)) is doubled for each finer mesh, and calculate the convergence rate \( s \) by numerically solving the following

\[
\frac{h_s^i - h_s^j}{h_s^i - h_s^k} = \frac{||U_{ref} - U_{h_1}|| - ||U_{ref} - U_{h_2}||}{||U_{ref} - U_{h_2}|| - ||U_{ref} - U_{h_3}||}, \tag{83}
\]

in which \( h_s^i \) is \( h_i \) to the power \( s \). The numerical velocity field and the mesh size for the different meshes are given with subscript notation, where \( ref \) is the finest mesh. By taking the sequence of meshes \( 64 \times 64 \times 64, 32 \times 32 \times 32, 16 \times 16 \times 16, 8 \times 8 \times 8 \), we numerically solve (83). The convergence rate in the \( L_\infty \) norm is roughly as expected, \( s \approx 2.89 \), for the implementation with quadratic polynomials.

The extensive tests reported above convince us that the presented numerical scheme is actually more accurate than the slowly converging semi-
analytical solutions. In Figures 11 and 12 we present all components of the numerical velocity vector and pressure fields produced by a simulation of incompressible fluid flow initialised with one of the eigenmodes of the semi-analytical solution. The domain is a rectangular box $D = [0, 2\pi] \times [0, \pi]^2$. Figure 13 shows conservation of energy and discrete zero-divergence.

6. Concluding remarks

We have derived a DGFEM discretisation of Hamiltonian dynamics for linear incompressible fluid flow. The discretisation was obtained by applying Dirac’s constrained Hamiltonian theory on a DGFEM formulation of compressible fluid flows. As an interim result a discretisation of Hamiltonian dynamics of compressible flow was derived, implemented and tested against exact solutions. Dirac’s theory proved to be a suitable tool for reducing the compressible system to the incompressible limit. The resulting system, as a consequence of the exact preservation of the constraints, does not require the inf-sup condition or stabilisation common to direct DGFEM discretisation of incompressible flows.

It was a challenge to derive and implement the boundary conditions for a Hamiltonian structure preserving discretisation in a rotating frame, due to the mandatory satisfaction of geostrophic balance for the flow along fixed walls. Moreover, for exact preservation of energy and zero-divergence, the presented numerical scheme requires the projection of the initial velocity profile to be exactly divergence free on the discrete level. A preprocessing step was thus introduced to correct the projection of the initial velocity, within the order of accuracy.

Several tests of the inertial waves in rotating domains were presented. The simulation of Poincaré inertial waves in a channel assessed the proper implementation of no-normal flow boundary conditions in rotating domains. Next, an inertial-wave simulation in a cuboid with fixed solid walls showed agreement up to $10^{-2}$ with slowly converging semi-analytical solutions available from [17]. Richardson extrapolation with sequencing meshes proved that our numerical solution is more accurate: the semi-analytical solutions available from [17] or [23] do not satisfy either the Euler equations or the solid-wall boundary conditions exactly. DGFEM allows relatively easy $hp$-refinement of the system. $p$-refinement was already used in all presented numerical test cases and it appears that the quadratic polynomial approximations in the local elements provide sufficient order of accuracy for capturing the physi-
Figure 11: $u$ and $v$ components of the three-dimensional velocity vector field in the simulation of an inertial wave with eigenfrequency $\sigma = 0.477$. Rotation vector is in $z$-direction. A $32 \times 16 \times 16$ mesh and time step $\Delta t = T/20$ is used.
Figure 12: $w$ component of the three-dimensional velocity vector and scalar pressure $p$ for the inertial wave simulation.
Figure 13: Energy and discrete divergence-free velocity field during 100 time periods of a simulation of inertial waves in a cuboid.

cal phenomena of inertial waves in rotating domains. Moreover, preliminary simulations for inertial waves in a rotating domain with one slanted wall suggest that for capturing sufficient details of the originating attractors one needs to introduce the $hp$-refinement of the domain near the zones of the attraction. Further work will also include the incorporation of a free surface in the presented numerical scheme for incompressible fluid flows.

Appendix A. Constrained Hamiltonian continuum dynamics

The first calculation concerns the derivation from (20a) to (21)

$$0 = \{ \mathcal{F}[\rho], \mathcal{H} \} + \int_D \lambda_\Delta(x') \{ \mathcal{F}[\rho], \Delta(x') \} dx'$$

$$= -\int_D \frac{\delta \mathcal{F}}{\delta \rho} \nabla \cdot (\rho_0 u) \, dx - \int_{D,D'} \lambda_\Delta(x) \frac{\delta \mathcal{F}}{\delta \rho(x')} \nabla' \cdot \frac{\delta \nabla \cdot u(x)}{\delta u(x')} \, dx \, dx'.$$

(A.2)
The last term in (A.2) can be reworked to

\[
- \int_{D,D'} \int_{\mathbf{x}, x'} \lambda_{\Delta}(\mathbf{x}) \frac{\delta F}{\delta \rho(x')} \nabla' \cdot \left( \frac{\delta \mathbf{u}(\mathbf{x})}{\delta \mathbf{u}(\mathbf{x}')} \right) \, d\mathbf{x} d\mathbf{x}'
\]

\[
= \int_{D,D'} \int_{\mathbf{x}, x'} \lambda_{\Delta}(\mathbf{x}) \nabla' \delta \mathbf{u}(\mathbf{x}) \frac{\delta F}{\delta \rho(x')} \, d\mathbf{x} d\mathbf{x}'
\]

\[
- \int_D \int_{\partial D'} \lambda_{\Delta}(\mathbf{x}) \frac{\delta F}{\delta \rho(x')} \hat{n}' \cdot \left( \frac{\delta \mathbf{u}(\mathbf{x})}{\delta \mathbf{u}(\mathbf{x}')} \right) \, dS' d\mathbf{x}
\]

\[
= - \int_D \nabla \lambda_{\Delta} \cdot \nabla \frac{\delta F}{\delta \rho} \, d\mathbf{x} + \int_{\partial D} \int_{D'} \lambda_{\Delta}(\mathbf{x}) \nabla' \frac{\delta F}{\delta \rho(x')} \hat{n}' \cdot \frac{\delta \mathbf{u}(\mathbf{x})}{\delta \mathbf{u}(\mathbf{x}')} \, dS d\mathbf{x}'
\]

\[
= \int_D \frac{\delta F}{\delta \rho} \nabla^2 \lambda_{\Delta} \, d\mathbf{x} - \int_{\partial D} \frac{\delta F}{\delta \rho} \nabla \lambda_{\Delta} \cdot \hat{n} \, dS \quad (A.3)
\]

in which the first boundary term emerging is zero, and similarly the second boundary term emerging is zero, because

\[
n_i' \partial_j \left( \frac{\delta u_i(\mathbf{x})}{\delta u_i(\mathbf{x}')} \right) = \partial_i \partial_j \left( \frac{n_i' \delta u_i(\mathbf{x}')}{\delta u_i(\mathbf{x})} \right) = 0, \quad (A.4)
\]

since \( \delta u_i(\mathbf{x})/\delta u_i(\mathbf{x}') = \delta u_i(\mathbf{x}')/\delta u_i(\mathbf{x}) = \delta (\mathbf{x} - \mathbf{x}') \), which follows by analysing \( \delta u_i(\mathbf{x}) \) and \( \delta u_i(\mathbf{x}') \) as functionals, and \( n_i \delta u_i = \delta (n_i u_i) = 0 \) at \( \partial D \). The index \( i \) denotes the velocity component and summation over repeated indices is understood. Recombining the above yields the result stated in the main text

\[
0 = \int_D \frac{\delta F}{\delta \rho} \left( - \nabla \cdot (\rho_0 \mathbf{u}) + \nabla^2 \lambda_{\Delta} \right) \, d\mathbf{x} - \int_{\partial D} \frac{\delta F}{\delta \rho} \hat{n} \cdot \nabla \lambda_{\Delta} \, dS. \quad (A.5)
\]

Recall that the primary constraint is \( \nabla \cdot (\rho_0 \mathbf{u}) = \rho_0 \nabla \cdot \mathbf{u} \). The solution of (A.5) is therefore

\[
\lambda_{\Delta} = \text{constant}. \quad (A.6)
\]

The second derivation concerns the step from (20b) to (24). From (23), it follows that

\[
\frac{\delta F[\Delta]}{\delta \mathbf{u}} = - \nabla \frac{\delta F[\Delta]}{\delta \Delta}. \quad (A.7)
\]
Detailed analysis of (20b) entails

\[
0 = \{\mathcal{F}[\Delta], \mathcal{H}\} + \int_D \lambda_\rho(x') \{\mathcal{F}[\Delta], \rho(x')\} dx' \\
+ \int_D \lambda_\Delta(x') \{\mathcal{F}[\Delta], \Delta(x')\} dx'
\]

(A.8)

\[
= \int_D 2\Omega \times u \cdot \nabla \frac{\delta \mathcal{F}}{\delta \Delta} dx + \int \int_{D,D'} \lambda_\rho(x) \nabla' \cdot \left( \frac{\delta \mathcal{F}[\Delta]}{\delta u(x')} \right) \frac{\delta \rho(x)}{\delta \rho(x')} dx dx' \\
+ \int \int_{D,D'} \lambda_\Delta(x) \frac{2\Omega}{\rho_0} \times \frac{\delta \mathcal{F}[\Delta]}{\delta u(x')} \frac{\delta \rho(x)}{\delta \rho(x')} dx dx'.
\]

(A.9)

The third term in (A.9) can be shown to be zero as follows

\[
\int \int_{D,D'} \lambda_\Delta(x) \frac{2\Omega}{\rho_0} \times \frac{\delta \mathcal{F}[\Delta]}{\delta u(x')} \frac{\delta \rho(x)}{\delta \rho(x')} dx dx' = 0,
\]

since \(\lambda_\Delta = \text{cst}\) (cf. (A.6)) and by using (A.4).

The second term in (A.9) can be reworked as follows

\[
\int \int_{D,D'} \lambda_\rho(x) \nabla' \cdot \left( \frac{\delta \mathcal{F}[\Delta]}{\delta u(x')} \right) \frac{\delta \rho(x)}{\delta \rho(x')} dx dx' = \int \lambda_\rho \nabla \cdot \left( \frac{\delta \mathcal{F}[\Delta]}{\delta u} \right) dx \\
= \int \nabla \lambda_\rho \cdot \nabla \left( \frac{\delta \mathcal{F}[\Delta]}{\delta \Delta} \right) dx - \int_{\partial D} \lambda_\rho \hat{n} \cdot \nabla \left( \frac{\delta \mathcal{F}[\Delta]}{\delta \Delta} \right) dS \\
= -\int \nabla^2 \lambda_\rho \cdot \frac{\delta \mathcal{F}[\Delta]}{\delta \Delta} dx + \int_{\partial D} \nabla \lambda_\rho \cdot \hat{n} \frac{\delta \mathcal{F}[\Delta]}{\delta \Delta} dS.
\]

(A.11)

The second last boundary term \(\hat{n} \cdot \nabla (\delta \mathcal{F}/\delta \Delta)\) in (A.11) is zero because it is imposed as extra gauge or boundary condition, cf. an earlier remark.

When we combine (A.11) and (A.9), the final result 24 in the main text
is reached

\[ 0 = - \int_D \frac{\delta F}{\delta \Delta} \left( \nabla \Delta \lambda + \nabla \cdot (2\Omega \times u) \right) d\mathbf{x} + \int_{\partial D} \frac{\delta F}{\delta \Delta} \mathbf{n} \cdot (\nabla \lambda + 2\Omega \times u) dS. \]

(A.12)

Acknowledgments

We would like to acknowledge the financial support of the Netherlands Foundation for Technical Research (STW) for the project “A numerical wave tank for complex wave and current interactions”.

References


