Conditions for the existence of quasi-stationary distributions for birth-death processes with killing

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Abstract. We consider birth-death processes on the nonnegative integers, where \( \{1,2,\ldots\} \) is an irreducible class and 0 an absorbing state, with the additional feature that a transition to state 0 (killing) may occur from any state. Assuming that absorption at 0 is certain we are interested in additional conditions on the transition rates for the existence of a quasi-stationary distribution. Inspired by results of M. Kolb and D. Steinsaltz (Quasimartingale behaviour for one-dimensional diffusions with killing, Annals of Probability, to appear) we show that a quasi-stationary distribution exists if the decay rate of the process is positive and exceeds at most finitely many killing rates. If the decay rate is positive and smaller than at most finitely many killing rates then a quasi-stationary distribution exists if and only if the process one obtains by setting all killing rates equal to zero is recurrent.

Keywords: birth-death process with killing, orthogonal polynomials, quasi-stationary distribution

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1 Introduction and main results

We consider a continuous-time Markov chain \( X := \{X(t), \ t \geq 0\} \) taking values in \( \{0\} \cup S \) where 0 is an absorbing state and \( S := \{1, 2, \ldots\} \). The generator \( Q := (q_{ij}, \ i, j \in S) \) of the (sub)Markov chain on \( S \) satisfies

\[
q_{i,i+1} = \lambda_i, \quad q_{i+1,i} = \mu_{i+1}, \quad q_{ii} = -(\lambda_i + \mu_i + \gamma_i), \quad i \geq 1, \\
q_{ij} = 0, \quad |i - j| > 1,
\]

where \( \lambda_i > 0 \) and \( \gamma_i \geq 0 \) for \( i \geq 1 \), \( \mu_i > 0 \) for \( i > 1 \), and \( \mu_1 = 0 \). The parameters \( \lambda_i \) and \( \mu_i \) are the birth and death rates in state \( i \), while \( \gamma_i \) is the rate of absorption into state 0 (or killing rate). A Markov chain of this type is known as a birth-death process with killing.

We will assume throughout that the parameters of the process are such that absorption at 0 is certain, that is, by [9, Theorem 1],

\[
\sum_{n=1}^{\infty} \frac{1}{\lambda_1 \pi_n} \sum_{j=1}^{n} \gamma_j \pi_j = \infty,
\]

where

\[
\pi_1 := 1 \quad \text{and} \quad \pi_n := \frac{\lambda_1 \lambda_2 \cdots \lambda_{n-1}}{\mu_2 \mu_3 \cdots \mu_n}, \quad n > 1.
\]

Clearly, this assumption implies that \( X \) is nonexplosive (cf. [2, Theorem 8]) and hence uniquely determined by \( Q \). Also, we must have \( \gamma_i > 0 \) for at least one state \( i \in S \).

We write \( P_i(\cdot) \) for the probability measure of the process when the initial state is \( i \), and for any vector \( u = (u_i, \ i \in S) \) representing a distribution over \( S \) we let \( P_u(\cdot) := \sum_{i \in S} u_i P_i(\cdot) \). We also write \( P_{ij}(\cdot) := P_i(X(\cdot) = j) \). It is well known (see, for example, [1, Theorem 5.1.9]) that under our assumptions there exists a parameter \( \alpha \geq 0 \) such that

\[
\alpha = - \lim_{t \to \infty} \frac{1}{t} \log P_{ij}(t), \quad i, j \in S.
\]

The parameter \( \alpha \) plays a key role in what follows and will be referred to as the decay rate of \( X \).
An honest distribution over $S$ represented by the vector $\mathbf{u} = (u_i, i \in S)$ is called a \textit{quasi-stationary distribution} for $\mathcal{X}$ if the distribution of $X(t)$, conditional on non-absorption up to time $t$, is constant over time when $\mathbf{u}$ is the initial distribution. That is, $\mathbf{u}$ is a quasi-stationary distribution if, for all $t \geq 0$,

$$P_u(X(t) = j | T > t) = u_j, \quad j \in S, \quad (5)$$

where $T := \sup\{t \geq 0 : X(t) \in S\}$ is the \textit{absorption time} (or \textit{survival time}) of $\mathcal{X}$, the random variable representing the time at which absorption at 0 occurs.

In what follows we are concerned with conditions for the existence of a quasi-stationary distribution for a birth-death process with killing. Our main results are presented in the following two theorems.

**Theorem 1** If (2) is satisfied and $0 < \alpha < \lim_{i \to \infty} \inf \gamma_i$ then there exists a quasi-stationary distribution for the process $\mathcal{X}$.

**Theorem 2** If (2) is satisfied and $\alpha > \lim_{i \to \infty} \sup \gamma_i$ then a quasi-stationary distribution for the process $\mathcal{X}$ exists if and only if the \textit{unkilled} process – the birth-death process on $S$ one obtains from $\mathcal{X}$ by setting $\gamma_i = 0$ for all $i$ – is recurrent.

These results have been inspired by similar findings for one-dimensional diffusions with killing by Kolb and Steinsaltz [12], extending earlier work of Steinsaltz and Evans [16]. However, our method of proof is different and exploits the integral representation for the transition probabilities of a birth-death process with killing disclosed in [8].

The remainder of this paper is organized as follows. In Section 2 we introduce the orthogonal polynomials that are associated with the birth-death process with killing $\mathcal{X}$, and note some relevant properties. In Section 3 we recall the integral representation for the transition probabilities of $\mathcal{X}$, and derive some further properties of the orthogonal polynomials. These properties subsequently enable us in Section 4 to prove the Theorems 1 and 2. We conclude in Section 5 with some remarks and conjectures.
2 Orthogonal polynomials

The transition rates of the process \( X \) determine a sequence of polynomials \( \{Q_n(x)\} \) through the recurrence relation

\[
\begin{align*}
\lambda_n Q_n(x) &= (\lambda_n + \mu_n + \gamma_n - x)Q_{n-1}(x) - \mu_n Q_{n-2}(x), \quad n > 1, \\
\lambda_1 Q_1(x) &= \lambda_1 + \gamma_1 - x, \quad Q_0(x) = 1.
\end{align*}
\]

By letting

\[
P_0(x) := 1 \quad \text{and} \quad P_n(x) := (-1)^n \lambda_1 \lambda_2 \ldots \lambda_n Q_n(x), \quad n \geq 1,
\]

we obtain the corresponding sequence of monic polynomials, which satisfy the recurrence relation

\[
\begin{align*}
P_n(x) &= (x - \lambda_n - \mu_n - \gamma_n)P_{n-1}(x) - \lambda_{n-1} \mu_n P_{n-2}(x), \quad n > 1, \\
P_1(x) &= x - \lambda_1 - \gamma_1, \quad P_0(x) = 1.
\end{align*}
\]

Since \( \lambda_{n-1} \mu_n > 0 \) for \( n \geq 1 \), it follows (see, for example, Chihara [3, Theorems I.4.4 and II.3.1]) that \( \{P_n(x)\} \), and hence \( \{Q_n(x)\} \), constitutes a sequence of orthogonal polynomials with respect to a bounded, positive Borel measure on \( \mathbb{R} \). Actually, it has been shown in [8] that there exists a probability measure (a positive Borel measure of total mass 1) \( \psi \) on \( [0, \infty) \) such that

\[
\pi_{j+1} \int_0^\infty Q_1(x)Q_j(x)\psi(dx) = \delta_{ij}, \quad i, j \geq 0,
\]

where \( \delta_{ij} \) is Kronecker’s delta and \( \pi_{j+1} \) the constants defined in (3).

It is well known that the polynomials \( Q_n(x) \) have real, positive zeros \( x_{n1} < x_{n2} < \ldots < x_{nm}, \ n \geq 1 \), which are closely related to \( \text{supp}(\psi) \), the support of the measure \( \psi \). In particular we have

\[
\inf \text{supp}(\psi) = \lim_{n \to \infty} x_{n1},
\]

which exists, since the sequence \( \{x_{n1}\} \) is (strictly) decreasing (see, for example, [3, Theorem II.4.5]). Considering that

\[
(-1)^n P_n(x) = \lambda_1 \lambda_2 \ldots \lambda_n Q_n(x) = (x_{n1} - x)(x_{n2} - x)\ldots(x_{nm} - x),
\]

it now follows that

\[
x \leq y \leq \inf \text{supp}(\psi) \iff Q_n(x) \geq Q_n(y) > 0 \quad \text{for all} \quad n > 0,
\]

where
a result that will be used later on. At this point we also note that
\[ \lambda_n \pi_n (Q_n(x) - Q_{n-1}(x)) = \sum_{j=1}^{n} (\gamma_j - x) \pi_j Q_{j-1}(x), \quad n > 0. \]  
(12)
as can easily be seen by induction. Hence we can write, for all \( x \in \mathbb{R} \),
\[ Q_n(x) = 1 + \sum_{k=1}^{n} \frac{1}{\lambda_k \pi_k} \sum_{j=1}^{k} (\gamma_j - x) \pi_j Q_{j-1}(x), \quad n > 0. \]  
(13)

3 Integral representation

It has been shown in [8] that the transition probabilities for the transient states of the process \( \mathcal{X} \) can be represented in the form
\[ P_{ij}(t) = \pi_j \int_{0}^{\infty} e^{-xt} Q_{i-1}(x) Q_{j-1}(x) \psi(dx), \quad i, j \in S, \ t \geq 0, \]  
(14)
where \( \pi_n \) and \( Q_n(x) \) are as defined in (3) and (6), respectively, and \( \psi \) is an orthogonalizing probability measure on \([0, \infty)\) for the polynomial sequence \( \{Q_n(x)\} \). This result generalizes Karlin and McGregor’s [11] classic representation theorem for the pure birth-death process. Note that by setting \( t = 0 \) in (14) we regain (9). The measure \( \psi \) is in fact unique. Indeed, our assumption that absorption in 0 is certain, and hence that the process \( \mathcal{X} \) is nonexplosive, implies that the transition probabilities \( P_{ij}(t) \) constitute the unique solution to the Kolmogorov backward equations. Since the representation (14) reduces to
\[ P_{11}(t) = \int_{0}^{\infty} e^{-xt} \psi(dx), \quad t \geq 0, \]  
(15)
if \( i = j = 1 \), the uniqueness theorem for Laplace transforms implies that the measure \( \psi \) must be unique as well. Certain absorption in state 0 also implies that the transition probabilities \( P_{ij}(t), \ i, j \in S, \) tend to zero as \( t \to \infty \). Hence the representation (14) tells us that the measure \( \psi \) cannot have a point mass at zero, so that \( \psi \) is, in fact, a probability measure on \((0, \infty)\).

Of particular interest in what follows are the quantities \( Q_n(\alpha) \), where \( \alpha \) is the decay rate of \( \mathcal{X} \), defined in (4). It is obvious from (15) that \( \alpha \) must satisfy
\[ \alpha = \inf \text{supp}(\psi), \]  
(16)
so (11) implies that $Q_n(\alpha) > 0$ for all $n \geq 0$. The next lemma is an essential ingredient for the proof of Theorem 1.

**Lemma 3** If $\alpha < \lim_{i \to \infty} \inf \gamma_i$ then $\sum_{n=1}^{\infty} \pi_n Q_{n-1}^2(\alpha) < \infty$.

**Proof** Let $\alpha < \lim_{i \to \infty} \inf \gamma_i$. From (9) we see that the orthonormal polynomials $p_n(x)$ corresponding to $\psi$ are given by $p_n(x) = \sqrt{\pi_n + 1} Q_n(x)$, while a classic result in the theory of orthogonal polynomials (see [15, Corollary 2.6]) tells us that the measure $\psi$ has a point mass at $x$ if and only if $\sum p_n^2(x) < \infty$. So to prove the theorem we must show that $\psi(\{\alpha\}) > 0$. But it follows from [5, Theorem 9] (by choosing $\chi_n = \lambda_n$) that the smallest limit point in the support of $\psi$, if any, is not less than $\lim_{i \to \infty} \inf \gamma_i$. As a consequence $\alpha$ – the smallest point in the support of $\psi$ – is an isolated point, whence $\psi(\{\alpha\}) > 0$. \qed

The final two lemmas in this section pave the way for the proof of Theorem 2.

**Lemma 4** If $\alpha > \lim_{i \to \infty} \sup \gamma_i$ then, for $N$ sufficiently large, the sequence $\{Q_n(\alpha)\}_{n>N}$ is monotone.

**Proof** By (12) we have

$$\lambda_n \pi_n (Q_n(\alpha) - Q_{n-1}(\alpha)) = \sum_{j=1}^{n} (\gamma_j - \alpha) \pi_j Q_{j-1}(\alpha), \quad n > 0. \quad (17)$$

It follows, if $\alpha > \lim_{i \to \infty} \sup \gamma_i$ and $n$ is sufficiently large, that

$$\lambda_{n+1} \pi_{n+1} (Q_{n+1}(\alpha) - Q_n(\alpha)) < \lambda_n \pi_n (Q_n(\alpha) - Q_{n-1}(\alpha))$$

and hence

$$Q_n(\alpha) \leq Q_{n-1}(\alpha) \implies Q_m(\alpha) < Q_{m-1}(\alpha), \quad m > n,$$

implying the statement of the lemma. \qed

To prove Lemma 5 we need the result

$$\alpha \sum_{n \in S} \pi_n Q_{n-1}(\alpha) = \sum_{n \in S} \gamma_n \pi_n Q_{n-1}(\alpha) \leq \infty, \quad (18)$$

which is part of [9, Theorem 2].
Lemma 5  If \( \alpha > \lim_{i \to \infty} \sup \gamma_i \) and \( \sum_{n=1}^{\infty} \pi_n Q_{n-1}(\alpha) < \infty \), then \( Q_n(\alpha) \) increases in \( n \) for \( n \) sufficiently large.

Proof  Let \( \alpha > \lim_{i \to \infty} \sup \gamma_i \), and suppose that \( Q_n(\alpha) \) decreases in \( n \) for \( n \) sufficiently large. Then, by (17),

\[
\sum_{j=1}^{n} (\gamma_j - \alpha) \pi_j Q_{j-1}(\alpha) < 0
\]

for \( n \) sufficiently large. But considering that \( (\gamma_j - \alpha) \pi_j Q_{j-1}(\alpha) < 0 \) for \( j \) sufficiently large, we actually have

\[
\sum_{j=1}^{n} (\gamma_j - \alpha) \pi_j Q_{j-1}(\alpha) < A < 0,
\]

for some real number \( A \) and \( n \) sufficiently large, so that, by (18), we must have \( \sum \pi_n Q_{n-1}(\alpha) = \infty \). Since, by Lemma 4, \( Q_n(\alpha) \) is monotone for \( n \) sufficiently large, this establishes the lemma. \( \square \)

4 Quasi-stationary distributions

It is well known (see, for example, [7]) that a quasi-stationary distribution for \( X \) (actually, for any absorbing, continuous-time Markov chain on \( \{0\} \cup S \)) can exist only if absorption at state 0 is certain and the decay rate \( \alpha \) is positive. Under these conditions then, the following theorem gives a necessary and sufficient condition for a distribution on \( S \) to be a quasi-stationary distribution for \( X \).

Theorem 6  [4, Theorem 6.2] Let \( X \) be a birth-death process with killing for which absorption at 0 is certain and \( \alpha > 0 \). Then the distribution \( (u_j, j \in S) \) is a quasi-stationary distribution for \( X \) if and only if there is a real number \( x \), \( 0 < x \leq \alpha \), such that both

\[
u_j = \frac{\pi_j Q_{j-1}(x)}{\sum_{n \in S} \pi_n Q_{n-1}(x)}, \quad j \in S,
\]

and

\[
x \sum_{n \in S} \pi_n Q_{n-1}(x) = \sum_{n \in S} \gamma_n \pi_n Q_{n-1}(x) < \infty.
\]

(19)
However, we can be more explicit if we are just interested in conditions for the existence of a quasi-stationary distribution.

**Theorem 7**  Let $X$ be a birth-death process with killing with decay rate $\alpha > 0$ and certain absorption at 0. A quasi-stationary distribution for $X$ exists if and only if $\sum_{n \in S} \pi_n Q_{n-1}^{(\alpha)} < \infty$, in which case $(u_j, j \in S)$ with

$$u_j = \frac{\pi_j Q_{j-1}^{(\alpha)}}{\sum_{n \in S} \pi_n Q_{n-1}^{(\alpha)}}, \quad j \in S,$$

constitutes a quasi-stationary distribution.

**Proof**  The result (18) tells us that (19) is satisfied if $\sum \pi_n Q_{n-1}^{(\alpha)} < \infty$ and $x = \alpha$. Hence, by Theorem 6, (20) determines a quasi-stationary distribution if $\sum \pi_n Q_{n-1}^{(\alpha)} < \infty$. On the other hand, $\sum \pi_n Q_{n-1}^{(\alpha)} < \infty$ if $\sum \pi_n Q_{n-1}^{(x)} < \infty$ for some $x$, $0 < x \leq \alpha$, as a consequence of (11) and (16). So, by Theorem 6 again, the existence of a quasi-stationary distribution implies $\sum \pi_n Q_{n-1}^{(\alpha)} < \infty$.

We can finally proceed to the proofs of our main results. Recall that, by (11) and (16), $Q_n^{(\alpha)} > 0$, a fact that will be used throughout.

**Proof of Theorem 1:**  Let (2) be satisfied and $0 < \alpha < \lim_{i \to \infty} \inf \gamma_i$. Let $N$ be such that $\alpha < \gamma_j$ for all $j \geq N$. Then we can rewrite (17) for $n > N$ as

$$\lambda_n \pi_n (Q_n^{(\alpha)} - Q_{n-1}^{(\alpha)}) = \sum_{j=1}^{N} (\gamma_j - \alpha) \pi_j Q_{j-1}^{(\alpha)} + \sum_{j=N+1}^{n} (\gamma_j - \alpha) \pi_j Q_{j-1}^{(\alpha)}.$$

If $\sum \pi_n Q_{n-1}^{(\alpha)} = \infty$, then the second term of the right-hand side of (21) tends to $\infty$ as $n \to \infty$, so that the right-hand side of (21) is positive, and hence $Q_n^{(\alpha)}$ increases in $n$, for $n$ sufficiently large. However, this would imply divergence of $\sum \pi_n Q_{n-1}^{2^{(\alpha)}}$, which is impossible in view of Lemma 3. So we conclude that $\sum \pi_n Q_{n-1}^{(\alpha)} < \infty$, and hence, by Theorem 7, that a quasi-stationary distribution exists. \qed
Theorem 2 involves the unskilled process, the birth-death process one obtains from \( \mathcal{X} \) by setting all killing rates \( \gamma_i = 0 \). We recall that the unskilled process is recurrent if and only if
\[
\sum_{n=1}^{\infty} \frac{1}{\lambda_n \pi_n} = \infty
\]  
(22)
(see, for example, [11]).

**Proof of Theorem 2:** Let (2) be satisfied and \( \alpha > \lim_{i \to \infty} \sup \gamma_i \). First assuming \( \sum \pi_n Q_{n-1} - 1(\alpha) < \infty \), Lemma 5 tells us that \( Q_n(\alpha) \) is increasing, and hence \( Q_n(\alpha) > A > 0 \) for some real number \( A \), for \( n \) sufficiently large. The result (18) therefore implies convergence of \( \sum \gamma_n \pi_n \), so that, in view of (2), \( \sum (\lambda_n \pi_n)^{-1} = \infty \), that is, the unskilled process is recurrent.

Next assuming \( \sum \pi_n Q_{n-1} - 1(\alpha) = \infty \), we write
\[
g_n := \sum_{j=1}^{n} (\alpha - \gamma_j)\pi_j Q_{j-1} - 1(\alpha),
\]  
and note that \( g_n \to \infty \) as \( n \to \infty \), so that \( g_n > A > 0 \) for some real number \( A \) and \( n \) sufficiently large. Moreover, by setting \( x = \alpha \) and letting \( n \to \infty \) in (13) it follows that \( \sum_{k=1}^{\infty} (\lambda_k \pi_k)^{-1} g_k \leq 1 \). Hence \( \sum (\lambda_n \pi_n)^{-1} < \infty \), that is, the unskilled process is transient.

Since, by Theorem 7, a quasi-stationary distribution exists if and only if \( \sum \pi_n Q_{n-1} - 1(\alpha) \) converges, we have established the theorem. \( \Box \)

5 **Concluding remarks**

By way of illustration we will apply our theorems to some specific processes. First, if \( \gamma_1 > 0 \) but \( \gamma_i = 0 \) for \( i > 0 \), then \( \mathcal{X} \) is a pure birth-death process, for which \( \alpha > 0 \) and certain absorption at 0 are known to be necessary and sufficient for the existence of a quasi-stationary distribution (see [6]). This result is in complete accordance with Theorem 2, since certain absorption in the birth-death process \( \mathcal{X} \) is equivalent to recurrence of the unskilled process. Evidently, we can generalize the setting somewhat by allowing finitely many states to have a positive killing rate and still draw the same conclusion. Interestingly, it has
been shown in [4, Theorems 6.5 and 6.6] that in this generalized setting either
the quasi-stationary distribution is unique or there exists an infinite family of
quasi-stationary distributions, depending on whether the series
\[ \sum_{n=1}^{\infty} \frac{1}{\lambda_n \pi_n} \sum_{j=n+1}^{\infty} \pi_j \] (23)
converges or diverges. A challenging question is whether such a dichotomy can
also be established for birth-death processes with killing when the number of
positive killing rates is unbounded.

Next, we consider the example analysed in [4, Section 6], which concerns
the process with constant birth rates \( \lambda_i = \lambda, \ i \geq 1 \), and constant death rates
\( \mu_i = \mu, \ i > 1 \), but killing rates
\[ \gamma_1 = 0 \text{ and } \gamma_i = \gamma > 0, \ i > 1, \]
so that killing may occur from any state except state 1. It is shown in [4]
that if \( \lambda < \mu + \gamma \) then \( \alpha < \gamma \) and there exists a quasi-stationary distribution,
as predicted by Theorem 1. (Actually, there is exactly one quasi-stationary
distribution.) Also, if \( \lambda > \mu + \gamma \) then \( \alpha > \gamma \) and there is no quasi-stationary
distribution, which is consistent with Theorem 2 since the unskilled process is
transient in this case. When \( \lambda = \mu + \gamma \) we have \( \alpha = \gamma \) and there is no quasi-
stationary distribution, a result that cannot be obtained from our theorems.

In the more general setting of continuous-time Markov chains on \( \{0\} \cup S \) for
which absorption at 0 is certain and the decay rate is positive, a sufficient con-
dition for the existence of a quasi-stationary distribution is *asymptotic remoteness*
of the absorbing state, that is
\[ \lim_{i \to \infty} \mathbb{P}_i(T \leq t) = 0 \quad \text{for all } \ t > 0 \]
(see [10] and [14]). In the setting at hand Theorem 2 therefore tells us that if
(2) is satisfied and \( \alpha > \lim_{i \to \infty} \sup \gamma_i \) then asymptotic remoteness implies (22).
Li and Li [13, Theorem 6.2 (i)] have recently shown that asymptotic remoteness
prevails if \( \lim_{i \to \infty} \gamma_i = 0 \) and the series (23) diverges. So under these conditions,
in addition to (2) and \( \alpha > 0 \), (22) must hold true. No direct proof of this fact
is available yet. Parenthetically, for the pure birth-death process ($\gamma_i = 0$ for $i > 1$) asymptotic remoteness is equivalent to divergence of (23) (see [7]).

In [14] Pakes reminds the reader that an outstanding problem in the setting of continuous-time Markov chains on $\{0\} \cup S$ for which absorption at 0 is certain, is to find a weak substitute for the asymptotic-remoteness condition that preserves the conclusion that a quasi-stationary distribution exists if the decay rate of the process is positive. The results presented here furnish this substitute for birth-death processes with killing. It does not seem bold to conjecture that similar results will be valid in more general settings.

References


