Queueing Analysis of Wireless Network Coding

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ABSTRACT

We consider a wireless communication device that uses network coding and acts as a relay for two connections. We analyze a two-dimensional continuous-time queueing model of the system and show that steady-state performance can be expressed in the solution of a Riemann-Hilbert boundary value problem. From this solution we derive the expected energy consumption and expected packet delay.

1. INTRODUCTION

Network coding is a technique, introduced in [1], that in addition to routing, i.e., forwarding packets, allows to combine packets from different connections, hence mixing data streams. To illustrate the difference between classical packet forwarding and network coding, we consider a wireless network in which devices $A$ and $C$ need to exchange bits $x$ and $y$ using a relay $B$. First, as illustrated in Figure 1a, consider the routing case. Four transmissions – separated in time, frequency or signal space – are required. Most importantly, note that the relay is transmitting twice and that each of the transmissions is useful to only one of the other nodes. An example use of network coding is depicted in Figure 1b. The relay computes $z = x + y$, the exclusive or of the bits $x$ and $y$, and transmits $z$, which is again a single bit. Node $A$ recovers $y$ by taking the exclusive or of $z$ and $x$. Node $C$ can recover $x$ in similar fashion, hence the transmission of a single bit by the relay is useful to both other nodes. We refer the reader to a recent monograph [2] on network coding and the references therein for an overview of results established in the field.

From the above example it is clear that compared to routing, network coding has the potential to reduce the number of transmissions in a wireless communication network and hence increase the performance. Of particular interest in this paper are energy consumption and packet delay. Results in the literature on the performance of network coding [3, 4] depend on the assumption that relay nodes always have packets from all connections to transmit, i.e., queues are saturated. Initial results on queueing models that are not based on this assumption appear in [5]. Whereas approximate results on the performance have been provided in [5], the aim of this paper is to provide exact analytical results.

We model wireless networks as continuous-time Markov processes and consider the homogeneous process on $\mathbb{N}_0^2$ associated with the relay from Figure 1. We use analytical methods, developed in, for instance, [6, 7, 8], to derive expressions for steady-state performance. In particular, we will derive a relation for the generating function of the stationary distribution, leading to a Riemann-Hilbert boundary value problem. The solution to this boundary value problem can be used to find steady-state performance measures. Various methods to derive a boundary value problem from a relation on the generating function of the stationary distribution are known in the literature. Our method is closely related to [8].

In this paper we make the following contributions: 1) A queueing model for a wireless communication network with network coding is specified, 2) A Riemann-Hilbert boundary value problem for this communication network is derived, 3) Energy consumption and packet delay are expressed in terms of the solution to this boundary value problem.

The remainder of this paper is organized as follows. In Section 2 we specify the continuous-time Markov chain that will be analyzed and the performance measures of interest. The boundary value problem is derived in Section 3. In Section 4 we demonstrate how the performance measures of interest can be obtained. Finally, in Section 5 we outline ongoing and future work.

2. MODEL AND PROBLEM STATEMENT

We consider a single node in a wireless network that is acting as a relay for two sessions and develop a continuous-time queueing model. Packets from both sessions arrive at the node according to independent Poisson processes with rate $\lambda_1$ and $\lambda_2$. The time required to transmit a packet, i.e., to provide service for a packet, is exponentially distributed with rate $\mu$.

A separate queue is kept for each session, leading to a two-dimensional model in which the state variables $N$ and $M$ denote the number of packets contained in each of the queues. Network coding is employed by transmitting linear combinations of two packets, one packet from each queue in a combination. This means that a service completion will reduce the number of packets in both queues by one. If only one queue has a packet it is transmitted uncoded.
and a service completion will remove only one packet from a queue. Since transmitting an uncoded packet is unfavorable in terms of, for instance, energy consumption, we allow for an operating policy in which uncoded packets will not always be transmitted if opportunity arises.

If there is the opportunity to transmit a packet from the first queue, while the second queue is empty, this packet will be transmitted with probability $\gamma_1$. Similarly, packets from the second queue will be transmitted uncoded with probability $\gamma_2$.

The above description leads to a continuous-time Markov chain $Q$ on state space $\mathbb{N}_0^2$ with transition rates $q$ defined as

\[ q_{n,m}(i,j) = \begin{cases} 
\lambda_1, & \text{if } i = 1, \ j = 0, \ n \geq 0 \ m \geq 0, \\
\lambda_2, & \text{if } i = 0, \ j = 1, \ n \geq 0 \ m \geq 0, \\
\mu, & \text{if } i = -1, j = -1, \ n, m > 0, \\
\gamma_1\mu, & \text{if } i = 1, \ j = 0, \ n > 0, \ m = 0, \\
\gamma_2\mu, & \text{if } i = -1, \ j = -1, \ n = 0, \ m > 0, \\
0, & \text{otherwise.} 
\end{cases} \]

where $q_{n,m}(i,j)$ denotes the transition rate from state $(n,m)$ to state $(n+i, m+j)$. The transition structure is depicted in the transition diagram of Figure 2.

To simplify the notation in the remainder of the paper let

\[ \rho_1 = \frac{\lambda_1}{\mu}, \ \ \ \rho_2 = \frac{\lambda_2}{\mu}. \]  \hfill (2)

Remember that $\gamma_1$ and $\gamma_2$ denote probabilities and take values in the interval $[0,1]$. We assume $\lambda_1 > 0$, $\lambda_2 > 0$ and $\mu > 0$, ensuring irreducibility and aperiodicity of $Q$. In addition we assume that the ergodicity conditions given in [5] are satisfied.

**Theorem 1** ([5]). The process $Q$ is ergodic if and only if $\rho_1 < 1$, $\rho_2 < 1$, $\gamma_1 > (\rho_1 - \rho_2)/(\rho_1 - 1)$ and $\gamma_2 > (\rho_2 - \rho_1)/(\rho_2 - 1)$.

Finally, without loss of generality, we assume $\rho_1 \geq \rho_2$.

Our interest is in the steady-state performance of $Q$ and we will analyze the stationary-distribution $\pi(n,m)$. The first performance measure of interest is the packet delay. Let $D_1$ and $D_2$ denote the expected delay of packets of the first respectively second queue. By Little's law it follows that $D_1 = \mathbb{E}[N]/\lambda_1$ and $D_2 = \mathbb{E}[M]/\lambda_2$. The second performance measure is the expected energy consumption per unit time, denoted by $C$. The energy consumed by transmitting a packet is $\mu$ per unit time. Therefore, $C = \mathbb{E}[c(N,M)]$.

where

\[ c(n, m) = \gamma_1\mu \mathbb{I}_{(n>0,m=0)} + \gamma_2\mu \mathbb{I}_{(n=0,m>0)} + \mu \mathbb{I}_{(n>0,m>0)} \]  \hfill (3)

**3. THE BOUNDARY VALUE PROBLEM**

The first result in this section is a relation for the generating function $F(x,y)$ of the stationary distribution $\pi(n,m)$. Later in this section we derive a Riemann-Hilbert boundary value problem, the solution of which gives $F(0,y)/F(0,0)$.

**Lemma 1.** Let $F(x,y) = \sum_{n,m} \pi(n,m)x^ny^m$ denote the generating function of the stationary distribution $\pi$. It satisfies

\[ Q(x,y)F(x,y) = a(x,y)F(x,0) + b(x,y)F(0,y) + c(x,y)F(0,0), \]  \hfill (4)

where

\[ Q(x,y) = (1 + \rho_1 + \rho_2)x - \rho_1 x^2 y - \rho_2 xy^2 - 1, \]

\[ a(x,y) = (1 - \gamma_1)xy + \gamma_1 y - 1, \]

\[ b(x,y) = (1 - \gamma_2)xy + \gamma_2 x - 1, \]

\[ c(x,y) = -(1 - \gamma_1 - \gamma_2)xy - \gamma_2 x - \gamma_1 y + 1. \]

**Proof.** Follows directly from the Kolmogorov forward equations. \hfill \square

Let $Y(x)$ be the algebraic function satisfying $Q(x,y)Y(x) = 0$. It is readily verified that

\[ Y(x) = \frac{1 + \rho_1 + \rho_2 - \rho_1 x}{2\rho_2 x}. \]  \hfill (6)

where

\[ D(x) = x \left( (1 + \rho_1 + \rho_2 - \rho_1 x)^2 x - 4\rho_2 \right). \]  \hfill (7)

**Lemma 2.** The function $Y(x)$ has four real branch points $x_1, x_2, x_3$ and $x_4$ that satisfy

\[ 0 = x_1 < x_2 < 1 < x_3 < x_4. \]  \hfill (8)

Moreover, $D(x) < 0$ on the interval $(x_3, x_4)$.

**Proof.** Existence of four real branch points satisfying (8) follows from [7, Lemma 2.3.8]. The value of $D(x)$, $x_3 < x < x_4$, follows from $D(1) > 0$. \hfill \square

Let $L$ denote the image of $[x_1, x_4]$ under $Y(x)$. From the above lemma it follows that for $x \in (x_3, x_4)$ the two values of $Y(x)$ are complex conjugate. Hence, $L$ is a closed contour which is symmetric with respect to the real line. Let $L$ denote the interior of $L$. The next result states that for $y \in L^+ \cup L$, $F(0,y)/F(0,0)$ can be expressed as the solution of a Riemann-Hilbert boundary value problem [9] on $L$.

**Theorem 2.** The function $F(0,y)$ satisfies for $y \in L$ the condition

\[ \text{Im} \left[ \frac{b \left( \frac{1}{\gamma_2 y}, y \right)}{a \left( \frac{1}{\gamma_2 y}, y \right)} \frac{F(0,y)}{F(0,0)} \right] = -\text{Im} \left[ \frac{c \left( \frac{1}{\gamma_2 y}, y \right)}{a \left( \frac{1}{\gamma_2 y}, y \right)} \right]. \]  \hfill (9)

**Proof sketch.** First note that for any pair $(x,y)$ for which $Q(x,y) = 0$ we have

\[ a(x,y)F(x,0) + b(x,y)F(0,y) + c(x,y)F(0,0) = 0. \]  \hfill (10)
Since, for $x \in [x_3, x_4]$, $F(x, 0)$ is real, we obtain
\[
\text{Im} \left[ \frac{b(x, y)}{a(x, y)} F(0, y) \right] = -\text{Im} \left[ \frac{c(x, y)}{a(x, y)} \right]. \tag{11}
\]

Note, in addition that for $y \in L$, $y$ and its complex conjugate $\bar{y}$ are the two roots a quadratic equation and satisfy the relation
\[
y \bar{y} = \frac{1}{\rho_2 y}. \tag{12}
\]

Substituting (12) in (11) gives the result.

The above is only the sketch of the proof since it remains to discuss the analytical continuation of $F(0, y)$ for $y \in L^+ \cup L^-$.

In the remainder we will need the values of $F(0, y)/F(0, 0)$ and its derivative at $y = 1$. From the next result, which is given without a proof due to space constraints, it follows that these values can be obtained straightforwardly from the solution of the boundary value problem.

**Lemma 3.** If $\rho_1 \geq \rho_2$ then $1 \in L^+ \cup L$.

## 4. PERFORMANCE

In this section we demonstrate how to obtain $C$, $D_1$ and $D_2$ after solving the Riemann-Hilbert boundary value problem of Theorem 2. Note that
\[
C = \gamma_1 \mu \left[ F(0, 1) - F(0, 0) \right] + \gamma_1 \mu \left[ F(1, 0) - F(0, 0) \right] + \mu \left[ 1 - F(1, 0) - F(0, 1) + F(0, 0) \right]. \tag{13}
\]

Therefore, after obtaining $F(0, 1)/F(0, 0)$, $C$ can be computed using the next result.

**Lemma 4.**
\[
\rho_1 = 1 - (1 - \gamma_1) \left( F(0, 1) - F(0, 0) \right) - F(1, 0), \tag{14}
\]
\[
\rho_2 = 1 - (1 - \gamma_2) \left( F(1, 0) - F(0, 0) \right) - F(0, 1). \tag{15}
\]

**Proof.** Consider $X(y)$ such that $a(X(y), y) = 0$. Relation (14) follows by considering (4) for $(X(y), y)$, dividing by $Q(X(y), y)$, and taking the limit $y \to 1$. Relation (15) follows by taking the limit $x \to 1$ for $(x, \hat{Y}(x)$), where $b(x, \hat{Y}(x)) = 0$.

Since the previous lemma also provides the value of $F(0, 0)$, $D_2$ can be obtained using the next result.

**Lemma 5.** The expected delay of packets of the second connection is
\[
D_2 = \frac{\rho_2}{\lambda_2(1 - \rho_2)} F(1, 0) + \frac{\rho_2(1 - \gamma_2)}{\lambda_2(1 - \rho_2)^2} \left[ F(0, 1) - F(0, 0) \right] + \frac{1 - \gamma_2}{\lambda_2(1 - \rho_2)} \left. \frac{d}{dy} F(0, y) \right|_{y=1}. \tag{16}
\]

**Proof.** After substituting $x = 1$ in (4), divide by $Q(1, y)$, take the derivative of the LHS and the RHS with respect to $y$ and consider the limit $y \to 1$.

Finally, $D_1$ follows from the next result.

**Lemma 6.** The expected delay of packets from the first and second connection, $D_1$ respectively $D_2$, satisfy
\[
1 - \lim_{z \to 1} \frac{a(z, z)}{Q(z, z)} \lambda_1 D_1 = \lim_{z \to 1} \frac{d}{dz} \left( \frac{a(z, z)}{Q(z, z)} F(1, 0) + \frac{b(z, z)}{Q(z, z)} F(0, 1) + \frac{c(z, z)}{Q(z, z)} F(0, 0) \right) - \left[ 1 - \lim_{z \to 1} \frac{b(z, z)}{Q(z, z)} \right] \lambda_2 D_2. \tag{17}
\]

**Proof.** Consider $(z, z)$ in (4), divide by $Q(z, z)$, take the derivative of the LHS and the RHS with respect to $z$ and consider the limit $z \to 1$.

## 5. CONCLUSIONS AND FUTURE WORK

Communication networks in which network coding is employed provide an exciting new class of queuing problems. We have analyzed a model for a single wireless device acting as a relay for two connections.

It is known how to solve Riemann-Hilbert boundary value problems [9]. An important aspect of the solution to the boundary value problem given in Theorem 2 is the index of the function $b((\rho_2 y)^{-1}, y)/a((\rho_2 y)^{-1}, y)$. Ongoing work consists of finding this index. In addition, numerical examples will be obtained to illustrate the performance of network coding in wireless communications.

## 6. REFERENCES


