Generating All 3-Connected 4-Regular Planar Graphs from the Octahedron Graph

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ABSTRACT

We prove that all 3-connected 4-regular planar graphs can be generated from the Octahedron Graph, using three operations. We generated these graphs up to 15 vertices inclusive. Moreover, by including a fourth operation we obtain an alternative to a procedure by Lehel to generate all connected 4-regular planar graphs from the Octahedron Graph. © 1993 John Wiley & Sons, Inc.

1. INTRODUCTION

Our interest in (the generation of all) 3-connected 4-regular planar graphs is motivated by a question from chemistry communicated by King [5]. It appears to be so that transition metal clusters can be modeled by 3-connected planar graphs. These clusters tend to avoid vertices of degree five or higher, because this would lead either to excessive angular strain or a very high coordination number. Closed boron hydride and carborane cages utilize when
possible 3-connected planar graphs with all vertices of degree four and five. For the chemical details we refer to [6].

In this paper we show that all 3-connected 4-regular planar graphs can be generated from the Octahedron Graph, using three operations. These three operations, together with a fourth one, were proposed by Manca [8] to generate all connected 4-regular planar graphs from the Octahedron Graph. As noted by Lehel [7], the main result of Manca is not true, but can be corrected by including a fifth operation. By extending the proof of our main result to graphs of connectivity one and two, it can be shown that all connected 4-regular planar graphs can be generated from the Octahedron Graph, using four operations (including the three operations to generate the 3-connected ones).

Generating all 3-connected planar graphs with vertices of degree four and five appears to be much more complicated. We hope to report on this in the near future.

2. TERMINOLOGY AND NOTATION

We use [1] for basic terminology and notation and consider simple graphs only. We use OG to denote the Octahedron Graph of Figure 1. As in [1], we sometimes refer to a planar embedding of a planar graph as a plane graph. A plane graph $G$ partitions the rest of the plane into a number of connected regions; the closures of these regions are called the faces of $G$. A 3-connected 4-regular plane graph will be called a tfp-graph.

Let $G$ be a tfp-graph different from OG. If $G$ contains one of the configurations of type $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E},$ or $\mathcal{F}$ of Figure 2 (where dotted lines indicate edges of the complement of $G$), then we may define the corresponding reduction operations $\phi_A, \ldots, \phi_F$ as follows.

$\phi_A$: deletion of vertex $x$ (and all incident edges) from a configuration of type $\mathcal{A}$ and insertion of the edges $ad$ and $bc$.

$\phi_B$: deletion of the vertices $x, y, z$ from $\mathcal{B}$, addition of a new vertex $u$ with edges $ua, ub, uc, and ud$, and insertion of the edge $ab$.

FIGURE 1. The Octahedron Graph.
\( \phi_C \): deletion of the vertices \( v, w, x, y, \) and \( z \) from \( C \) and addition of a new vertex \( u \) with edges \( ua, ub, uc, \) and \( ud \).

\( \phi_D \): deletion of the vertices different from \( a \) and \( b \) in \( D \) and insertion of the edge \( ab \).

\( \phi_E \): deletion of the vertices \( w, x, y, \) and \( z \) from \( E \) and addition of a new vertex \( u \) with edges \( ua, ub, uc, \) and \( ud \).

\( \phi_F \): deletion of the vertices \( v \) and \( w \) from \( F \) and insertion of the edges \( ax, by \) and \( cz \) (with edge by “in the middle of” \( ax \) and \( cz \)).

If a connected 4-regular plane graph contains one of the configurations of the types of Figure 2, it can be reduced to a connected 4-regular plane graph of a lower order by applying the corresponding reduction operation to a configuration of the same type. For types \( B, C, D, E, \) or \( F \) this is obvious. For type \( A \) this can be shown as follows.

Suppose the connected 4-regular plane graph \( G \) contains a configuration of type \( A \) with vertices \( a, b, c, d, x \) as in Figure 2 and suppose the application of \( \phi_A \) to this configuration yields a disconnected graph \( G' \). Then \( x \) is a cut vertex of \( G \). Since \( G \) is 4-regular, \( G' \) has precisely 2 components and \( x \) has 2 neighbors in each component, namely \( a \) and \( d \) in one component and \( b \) and \( c \) in the other. But then \( G \) contains another configuration of type \( A \) (with the roles of \( b \) and \( d \) exchanged) such that the application of \( \phi_A \) to this configuration yields a 4-regular plane graph of a lower order.

It is also obvious that each of these reduction operations has an inverse operation. We denote these extension operations by \( \overline{\phi}_A, \ldots, \overline{\phi}_F \).

![Figure 2](image-url)
3. RESULTS

In [8] Manca claims that all connected 4-regular planar graphs can be generated from OG by successively using one of the operations $\phi_A$, $\phi_B$, $\phi_C$, and $\phi_D$. Lehel [7] pointed out an error in the generating procedure and corrected it by including the operation $\phi_E$. It is not difficult to see that a connected 4-regular planar graph containing configurations of type $\mathcal{C}$ or $\mathcal{C}$ is not 3-connected. Therefore, one might expect that all tfp-graphs can be generated from OG by successively using one of $\phi_A$, $\phi_B$, and $\phi_C$. This is not obvious, since the application of one of $\phi_A$, $\phi_B$, and $\phi_C$ to reduce a tfp-graph different from OG might yield a 4-regular planar graph of connectivity two or one. However, we prove the following result.

**Theorem 1.** Let $G$ be a tfp-graph different from OG. Then $G$ can be reduced to a tfp-graph of lower order using one of $\phi_A$, $\phi_B$, and $\phi_C$.

It is possible to prove this theorem in the following way: First show that any tfp-graph different from OG contains one of the configurations of type $\mathcal{A}$, $\mathcal{B}$, or $\mathcal{C}$ (starting with a triangular face and considering all possible extensions of it, as it is done in the proof of [8, Theorem 3]). Then show that any tfp-graph containing $\mathcal{A}$ can be reduced to a tfp-graph of lower order using $\phi_A$. Finally, show that any tfp-graph containing $\mathcal{B}$ or $\mathcal{C}$ and not containing $\mathcal{A}$ can be reduced to a tfp-graph of lower order using $\phi_B$ or $\phi_C$.

In the next section we give an alternative proof, using a distinction between the cases where the tfp-graph under consideration has connectivity three or four. By extending this proof technique to tfp-graphs of connectivity one and two, it is possible to prove the following result, which yields an alternative procedure to generate all connected 4-regular planar graphs from OG.

**Theorem 2.** Any connected 4-regular planar graph different from OG can be reduced to a connected 4-regular planar graph of lower order using one of $\phi_A$, $\phi_B$, $\phi_C$, and $\phi_F$.

Since we are mainly interested in tfp-graphs, we omit the proof of Theorem 2; it can also be proved by extending the arguments given in [7] and [8].

4. PROOF OF THEOREM 1

To be short, we call a tfp-graph $G$ $A$-reducible ($B$-reducible, $C$-reducible) if $G$ can be reduced to a tfp-graph of lower order using $\phi_A$ ($\phi_B$, $\phi_C$).

Throughout this section, let $G$ be a tfp-graph different from OG and let $S$ be a vertex cut of $G$ of minimum cardinality. Clearly, $|S| = 3$ or $|S| = 4$. 
Before we distinguish these two cases, we make a few observations, followed by proofs.

(1) Each vertex of \( S \) has a neighbor in each component of \( G - S \).

This is an obvious consequence of the minimality of \( |S| \).

(2) \( G - S \) consists of two components.

Suppose, to the contrary, that \( G_1, G_2, \) and \( G_3 \) are three different components of \( G - S \). By (1) each vertex of \( S \) has neighbors in \( G_1, G_2, \) and \( G_3 \). But then \( G \) contains a subdivision of \( K_{3,3} \), contradicting the planarity of \( G \) (See, e.g. [1, Theorem 9.10]).

(3) The subgraph \( G[S] \) of \( G \) induced by \( S \) contains no triangle.

Suppose, to the contrary, that \( G[S] \) contains a triangle \( T \). By (1), (2), and the nonplanarity of any subdivision of \( K_5 \) ([1, Theorem 9.10]), precisely one of the components of \( G - S \) lies inside \( T \). By (1) and since \( G \) is 4-regular, this component contains an odd number of vertices of odd degree, a contradiction.

Let \( G_1 \) and \( G_2 \) be the two components of \( G - S \) and denote by \( e(G_i, s) \) the number of neighbors of \( s \in S \) in \( G_i \) \((i = 1, 2)\).

**Case 1.** \( |S| = 3 \).

Without loss of generality we assume \( |V(G_1)| \leq |V(G_2)| \) and we assume \( S \) is chosen in such a way that \( |V(G_1)| \) is as small as possible. Let \( S = \{x, y, z\} \). We make two observations, the first of which is an easy consequence of the choice of \( S \).

(4) The subgraph of \( G \) induced by \( V(G_1) \cup S \) to which the edges \( xy, yz \), and \( xz \) are added (if not present), is 4-connected.

(5) At least one of the vertices of \( S \) has 2 neighbors in both \( G_1 \) and \( G_2 \).

This can be seen as follows. \( \sum_{s \in S} e(G_i, s) \) is even since \( G \) is 4-regular \((i = 1, 2)\). By the choice of \( S \), \( e(G_1, s) \geq 2 \), hence \( \sum e(G_1, s) = 6 \) or 8, and in both cases it is easy to find a vertex of \( S \) with precisely 2 neighbors in both \( G_1 \) and \( G_2 \).

By (5), we may assume \( e(G_1, x) = e(G_2, x) = 2 \). Let \( a \) and \( b \) be the neighbors of \( x \) in \( G_1 \), and \( c \) and \( d \) be the neighbors of \( x \) in \( G_2 \), such that \( \{x, a, b, c, d\} \) is a configuration of type \( A \). Let \( G' \) be the graph obtained from \( G \) by applying \( \phi_A \) to \( \{x, a, b, c, d\} \). It suffices to prove \( \kappa(G') \geq 3 \). Suppose, to the contrary, \( \kappa(G') \leq 2 \). Then \( \kappa(G') = 2 \) since \( \kappa(G - x) = 2 \).

Let \( \{p, q\} \) be a vertex cut of \( G' \) and let \( G'_1 \) and \( G'_2 \) denote the components of \( G' - \{p, q\} \). (By an observation similar to (2), \( G' - \{p, q\} \) consists of 2 components). By (4) we may assume \( \{a, b\} \subseteq V(G'_1) \cup \{p, q\} \). If \( \{a, b, c, d\} \subseteq V(G'_i) \cup \{p, q\} \) for \( i = 1 \) or 2, then \( \{p, q\} \) is a vertex cut.
of $G$, a contradiction. Hence, without loss of generality, we may assume $a = p$. We consider two subcases.

**Case 1.1.** $c \in V(G'_2)$.

Clearly, $a$ has a neighbor in $V(G'_2) - \{c\}$, since otherwise $\{x, q\}$ is a vertex cut of $G$. Since both $a$ and $c$ and $b$ and $d$ are disconnected in $G - S$, without loss of generality, assume $y \in V(G'_2)$ and $z \in V(G'_1) \cup \{q\}$. Then the vertex cut $\{x, a, q\}$ contradicts the choice of $S$.

**Case 1.2.** $c \in V(G'_1)$.

Now $d \in V(G'_2)$ and $b = q$. Since both $a$ and $c$ and $b$ and $d$ are disconnected in $G - S$, without loss of generality assume $y \in V(G'_2)$ and $z \in V(G'_1)$. Then the vertex cut $\{x, a, z\}$ contradicts the choice of $S$.

This settles Case 1.

**Case 2.** $|S| = 4$.

Let $e_s$ denote the number of edges in $G[S]$. Since the set of neighbors $N(v)$ of any vertex $v$ of $G$ is a vertex cut of $G$, we may assume $S = N(v)$ and is chosen in such a way that $e_s$ is maximum. One of the components of $G - S$ consists of the vertex $v$; let $G_1$ be the other component of $G - S$. If every vertex of $S$ has at most two neighbors in $G_1$, then $e_s \geq 2$. If there exists a vertex $s \in S$ with three distinct neighbors $x, y, z$ in $G_1$, then $G$ is $A$-reducible unless at least two of $xy, xz, yz$ are edges of $G$. In the latter case $S_1 = N(s)$ is a vertex cut of $G$ with $e_{S_1} \geq 2$.

Hence, we may assume $e_s \geq 2$. If $e_s \geq 4$, it is not difficult to see that $G$ is $C$-reducible (By (3) and since $G \neq OG$). We distinguish two subcases.

**Case 2.1.** $e_s = 3$.

By (3), $G[S]$ is a path: $a_1b_1b_2a_2$. Now $G$ is $A$-reducible unless $a_i$ and $b_i$ have a common neighbor $c_i$ in $G_1$ ($i = 1, 2$). In the latter case, since $G$ is 4-connected and $G \neq OG$, $b_1c_2 \notin E(G)$ and $G$ is $B$-reducible.

**Case 2.2.** $e_s = 2$.

If $G[S]$ contains a perfect matching, then $G$ is $A$-reducible. Next assume $abc$ is a path in $G[S]$. Then $G$ is again $A$-reducible unless $ab$ and $bc$ are in a triangle with the neighbor $x$ of $b$ in $G_1$. In the latter case $N(b)$ contains four adjacent pairs of vertices, contradicting the choice of $S$.

**5. GENERATION OF tfp-GRAPHS**

In the previous section we proved that all $tfp$-graphs can be generated from the Octahedron Graph by successively using one of $\Phi_A$, $\Phi_B$, and $\Phi_C$. 
We generated all $tfp$-graphs up to 15 vertices inclusive on a SUN workstation using the functional language TWENTEL (TWENTEL is operational on the IBMPC compatibles, on DEC's microVAX and on SUN workstations). As a representation of a planar graph, embedded on the sphere, we chose a set of codes of the faces of the graph. By a code of a face we mean a list of vertices that occur while walking along the boundary anticlockwise. See also [2]. This representation, which we call the code of the graph, has the advantage that the planarity of the generated graphs is captured in a natural way. Moreover, with the help of this code it is easy to obtain the successive codes of the generated graphs from the code of the original graph. During the generation procedure, $\overline{\phi}_A$ has been applied to every pair of nonadjacent edges of the boundary of every face, $\overline{\phi}_B$ to every triangular face, and $\overline{\phi}_C$ to every vertex of the graph. It is clear that the result of this generation may give isomorphic graphs. This generation may also lead to non-3-connected graphs. Duplicate graphs and non-3-connected graphs have been removed. Duplicate removal from a set of graphs has been achieved by assigning a unique identification number to a graph with the property that isomorphic graphs have equal identification numbers and nonisomorphic graphs have different identification numbers. An algorithm for calculating such an identification number is described in [3]. This algorithm also gives the order of the automorphism group of the graph. Using the number of $tfp$-graphs and the orders of their automorphism groups, we also computed the number of rooted 4-regular planar graphs (The notion of a rooted planar graph was introduced by Tutte [10]). For each $tfp$-graph $G$ on $n$ vertices with order automorphism group $h$, there are $4|E(G)|/h = 8n/h$ corresponding rooted 4-regular planar graphs (see also [4]). These numbers are listed below.

### Table 1. The Number of $tfp$-Graphs on $n \leq 15$ Vertices, the Order of Their Automorphism Group, and the Corresponding Number of Rooted Graphs

<table>
<thead>
<tr>
<th>No. Vertices</th>
<th>No. $tfp$-Graphs</th>
<th>No. Rooted Graphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>[1, 48]</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>[1, 16]</td>
<td>4</td>
</tr>
<tr>
<td>9</td>
<td>[1, 12]</td>
<td>6</td>
</tr>
<tr>
<td>10</td>
<td>[1, 4], [1, 16], [1, 20]</td>
<td>29</td>
</tr>
<tr>
<td>11</td>
<td>[1, 2], [2, 4]</td>
<td>88</td>
</tr>
<tr>
<td>12</td>
<td>[5, 2], [2, 4], [2, 12], [1, 24], [1, 48]</td>
<td>310</td>
</tr>
<tr>
<td>13</td>
<td>[4, 1], [11, 2], [3, 4]</td>
<td>1066</td>
</tr>
<tr>
<td>14</td>
<td>[16, 1],[29, 2], [9, 4], [1, 8], [2, 16], [1, 28]</td>
<td>3700</td>
</tr>
<tr>
<td>15</td>
<td>[86, 1], [41, 2], [5, 4], [4, 6], [2, 12], [1, 20]</td>
<td>13036</td>
</tr>
</tbody>
</table>

**Reading Example.** There are 139 $tfp$-graphs on 15 vertices, namely 86 of order automorphism group 1, 41 of order 2, 5 of order 4, 4 of order 6, 2 of order 12, and 1 of order 20. The number of rooted graphs corresponding to these graphs is $10320 + 2460 + 150 + 80 + 20 + 6 = 13036$. 

As a check, it may be possible to enumerate these numbers by an alternative method, e.g., via so-called quadrangulations (see [9]).

ACKNOWLEDGMENT

We thank the referees for pointing out some mistakes in an earlier version of the paper.

References