Orientations of Hamiltonian Cycles in Large Digraphs

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ABSTRACT

We prove that, with some exceptions, every digraph with \( n \geq 9 \) vertices and at least \( (n - 1)(n - 2) + 2 \) arcs contains all orientations of a Hamiltonian cycle.

1. INTRODUCTION

Heydemann, Sotteau, and Thomassen proved in [6] that every digraph \( D \) with \( n \geq 5 \) vertices and at least \( (n - 1)(n - 2) + 3 \) arcs contains every orientation of a cycle of length \( n \) except possibly the directed cycle. We characterize digraphs with \( n \geq 9 \) vertices and at least \( (n - 1)(n - 2) + 2 \) arcs which do not contain every orientation of a Hamiltonian cycle (Theorem 1 and Corollary 9). We use some known results about graphs and digraphs.

The terminology is the same as in [4]. For a digraph \( D \), we denote by \( V(D) \) the vertex set of \( D \), and by \( E(D) \) the arc set of \( D \). We write \( e(D) = |E(D)| \) and \( \nu(D) = |V(D)| \). For \( A \subset V(D) \), \( D(A) \) denotes the subgraph of \( D \) induced by \( A \). For subsets \( A \) and \( B \) of \( V(D) \) by \( e(A \rightarrow B) \) we denote the number of arcs from \( A \) to \( B \), \( e(A, B) = e(A \rightarrow B) + e(B \rightarrow A) \).

\( G(D) \) is the undirected graph with the same vertex set as \( D \), such that the edges of \( G(D) \) correspond with the 2-cycles (symmetric arcs) of \( D \). We shall make occasional use of the obvious fact that \( e(G(D)) \geq e(D) - (\ell) \) for a digraph \( D \) with \( n \) vertices. For digraphs \( D \) and \( F \), by \( D \bowtie F \) we denote the digraph with \( V(D \bowtie F) = V(D) \cup V(F) \) and arc set consisting of the arcs of \( D \) and \( F \), and all arcs from \( D \) to \( F \).

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$C^*_k$ is the symmetric cycle of length $k$, $U_k$ denotes $C^*_k$ minus one arc, $X_k, Y_k, Z_k$ denote the digraphs obtained from $C^*_k$ by deleting two consecutive arcs as shown in Figure 1.

Every maximal directed path contained in an orientation of a cycle $C$ is called a segment of $C$.

$I_n$ is the digraph with $n$ vertices and $(n - 1)(n - 2) + 2$ arcs containing one pendent vertex. $J_n$ is the digraph with $n$ vertices and $(n - 1)(n - 2) + 2$ arcs containing a vertex of in- and outdegree 1 joined to two different vertices (see Figure 2).

Let $u$ be a vertex of $K^*_n-2$, the complete symmetric digraph of order $n - 2$. Obtain Digraph $H_n$ by adding two new vertices $u$ and $w$ each of which dominates all $n - 2$ vertices of $K^*_n-2$ and is dominated only by $u$. $H_n$ and its converse $H'_n$ are depicted in Figure 3.

$M$ and $N$ are the digraphs depicted in Figure 4.

![Figure 1](image1.png)

![Figure 2](image2.png)

![Figure 3](image3.png)
2. AUXILIARY RESULTS

We shall use the following results:

**Theorem A.** (Woodall [9], Corollary 11.1). If $G$ is an undirected graph on $n \geq 2r + 3$ vertices with at least $\binom{n-2}{2} + \binom{r+2}{2} + 1$ edges, then $G$ contains a cycle of length $d$ for each $d$ such that $3 \leq d \leq n - r$.

**Theorem B.** Let $G$ be an undirected graph on $n \geq 8$ vertices with at least $\binom{n-2}{2} + 5$ edges and minimum degree at least 2. Then $G$ is Hamiltonian unless $n = 9$ and $G$ is isomorphic to $K_4 + K_5$.

**Theorem C.** If $G$ is an undirected graph on $n \geq 4$ vertices with at least $\binom{n-2}{2} + 2$ edges, then $G$ is Hamiltonian connected unless $G$ is isomorphic to $K_2 + (K_1 \cup K_{n-3})$ or $K_3 + K_3$.

Theorems B and C are special cases of a result (Theorem 3) proved in [8]. They also follow from a result of Chvátal [5].

**Theorem D.** ([3], Théorème 2.7). The only non-Hamiltonian strong digraphs with $n$ vertices and at least $(n-1)(n-2) + 2$ arcs are $I_n, H_n, H_n', M$ and $N$.

**Theorem E** [2]. If $D$ has $n$ vertices and at least $(n-1)(n-2) + 2$ arcs, then $D$ contains every Hamiltonian cycle composed of exactly two segments unless $D$ is isomorphic to $I_n$ or $M$.

3. NON DIRECTED HAMILTONIAN CYCLES

In this section we shall prove the following

**Theorem 1.** Let $D$ be a digraph on $n \geq 9$ vertices and at least $(n-1) \cdot (n-2) + 2$ arcs. Then $D$ contains every non-strong orientation of a Hamiltonian cycle unless $D$ is isomorphic to $I_n$ or $J_n$. 
Theorem 1 follows from Lemmas 5 and 6 stated below. Observe that for digraphs with \( n \) vertices and at least \((n - 1)(n - 2) + 2\) arcs we have \( e(D) \leq 2n - 4\). Hence
\[
\frac{n^2 - 5n + 8}{2} \geq \frac{n^2 - 7n + 26}{2} = \binom{n - 3}{2} + \binom{4}{2} + 1.
\]
So, we have by Theorem A the following Lemma 2.

Lemma 2. Let \( D \) be a digraph of order \( n \geq 9 \) such that \( e(D) \geq (n - 1) \cdot (n - 2) + 2 \). Then \( D \) contains every symmetric cycle \( C_k^* \) for \( 2 \leq k \leq n - 2 \).

Corollary 3. If \( D \) is a digraph with \( n \geq 9 \) vertices and at least \((n - 1) \cdot (n - 2) + 2\) arcs, then \( D \) contains every orientation of a cycle of length at most \( n - 1 \) except the directed cycle of length \( n - 1 \) in case \( D = K_{n-2}^* \) or \( D = K_{n-2}^* \sim K_2^* \).

Proof. In view of Lemma 2, we need only consider cycles of length \( n - 1 \). By the same lemma, \( D \) contains \( C_{n-2}^* \). If either of the remaining two vertices is joined to consecutive vertices of this cycle by more than 2 arcs, then \( D \) contains \( U_{n-1} \) and hence all orientations of a cycle of length \( n - 1 \). By counting arcs, we see that this must happen unless \( D \) consists of \( K_{n-2}^* \) and \( K_2^* \) with exactly \( 2(n - 2) \) arcs joining them, and (since any two vertices of a complete graph are consecutive in some Hamiltonian cycle) there is exactly one arc between each vertex in \( K_{n-2}^* \) and each vertex in \( K_2^* \). It is now easy to check that the result follows.

The following lemma is evident.

Lemma 4. Let \( D \) be a digraph of order \( k + 1 \) containing a symmetric cycle \( C_k^* \) of length \( k \) and no symmetric Hamiltonian cycle. Let \( a \) be the vertex of \( D \) which is not in \( C_k^* \). Then the total degree of \( a \) in \( D \) satisfies \( d(a, D) \leq [3k/2] \). Moreover, if equality holds, then the vertices of \( C_k^* \) can be labelled in such a way that \( C_k^* = (x_1, x_2, \ldots, x_k, x_1) \), every odd vertex (except \( x_k \) if \( k \) is odd) is joined to \( a \) by a symmetric arc and every even vertex (and \( x_k \) if \( k \) is odd) is joined to \( a \) by an antisymmetric arc.

Lemma 5. Let \( D \) be a digraph of order \( n (n \geq 9) \) and size at least \((n - 1) \cdot (n - 2) + 2\) containing a symmetric cycle of length \( n - 1 \). Then \( D \) contains every non-strong orientation of a Hamiltonian cycle unless \( D \) is isomorphic to \( I_n \) or \( n \) is even and \( D \) is isomorphic to \( J_n \).
Proof. Let $D$ be a digraph satisfying the assumptions of the lemma, and suppose that there is a non-strong cycle $C_n$ of length $n$ which is not contained in $D$.

Let $C_{n-1}^* = (x_1, \ldots, x_{n-1}, x_1)$ be a symmetric cycle of length $n - 1$ contained in $D$ and let $a$ be the vertex of $D$ which is not in $C_{n-1}^*$. $D$ contains no $X_n$ or $Y_n$, hence $e(a \to x_i) + e(a \to x_{i+1}) \leq 1$ and $e(x_i \to a) + e(x_{i+1} \to a) \leq 1$, for every $i, 1 \leq i \leq n - 1$ (reducing suffices modulo $n - 1$). So

$$d(a, D) = \sum_{i=1}^{n-1} e(x_i \to a) + \sum_{i=1}^{n-1} e(a \to x_i) \leq n - 1.$$ 

Assume first that equality holds. Then $e(x_i \to a) + e(x_{i+1} \to a) = e(a \to x_i) + e(a \to x_{i+1}) = 1$ for every $i, 1 \leq i \leq n - 1$, and $n$ is odd. So, $C_n$ is not the antirected cycle. Hence we have $e(x_{i+1} \to a) + e(a \to x_i) \leq 1$, for every $i, 1 \leq i \leq n - 1$, and therefore we can assume that $x_a$ is a symmetric arc of $D$ if $i$ is odd and a symmetric arc of $D$ if $i$ is even. Since $U_n$ is not contained in $D$, we have $(x_i, x_j) \not\in E(D)$ for $i, j$ even. So,

$$e(D) \geq \frac{n - 1}{2} \frac{n - 3}{2} + n - 1 > 2n - 4,$$

a contradiction.

Now assume that $d(a, D) \leq n - 2$. Since $e(D - a) \geq (n - 1)(n - 2) + 2 - (n - 2)$, $e(G(D - a)) \geq e(D - a) - \binom{n - 1}{2} = \binom{n^2 - 5n + 10}{2} = \binom{n - 2}{2} + 2$.

By Theorem $C$ the graph $G(D - a)$ is Hamiltonian connected unless isomorphic to $K_2 + (K_1 \cup K_{n-4})$.

If $G(D - a)$ is Hamiltonian connected then $d^-(a, D) = d^-(a, D) = 1$, hence either $D$ is isomorphic to $I_n$ or $n$ is even, $D$ is isomorphic to $J_n$ and $C_n$ is the antirected Hamiltonian cycle. The case that $G(D - a)$ is isomorphic to $K_2 + (K_1 \cup K_{n-4})$ is left for the reader. 

Lemma 6. Let $D$ be a digraph of order $n, n \geq 9$, and size at least $(n - 1) \cdot (n - 2) + 2$ that does not contain a symmetric cycle of length $n - 1$. Then $D$ contains every non-strong orientation of a Hamiltonian cycle.

Sketched proof. Observe that, by Theorem $E$, it is sufficient to prove that if $D$ does not contain a symmetric cycle of length $n - 1$ then $D$ contains every orientation of a Hamiltonian cycle containing at least four segments.

By Lemma 2, $D$ contains a symmetric cycle of length $n - 2, C_n^* = (x_1, \ldots, x_{n-2}, x_1)$ say. Let $V(D) - V(C_n^*) = \{a, b\}$.

Case 1. $a$ and $b$ are joined by a symmetric arc.
Then we can assume that \( e(a, x_i) + e(b, x_{i+1}) \leq 2 \) for every \( i, 1 \leq i \leq n - 2 \).

Hence \( D(V(C_{n-2}^*) \) is complete and, for every \( i \neq j \), \( e(a, x_i) + e(b, x_j) = 2 \).

Since \( D \) does not contain \( C_{n-1}^* \), the only possibility is that \( e(a, x_i) = e(b, x_i) = 1 \) for every \( i \) \( (1 \leq i \leq n - 2) \). The result clearly holds if there are vertices \( x_i \neq x_j \) such that \( e(x_i \rightarrow a) + e(a \rightarrow x_j) = 1 \). A similar remark holds for \( b \). This leaves only three digraphs still to be considered and these are left to the reader.

Case 2. \( a \) or \( b \), \( a \) say, is incident to more than one symmetric arc and Case 1 does not apply.

By Lemma 4 we can assume that \( d(a, D - b) \leq [3(n - 2)/2] \) and \( d(b, D - a) \leq [3(n - 2)/2] \).

For \( d(b, D - a) = [3(n - 2)/2] \) we consider two subcases.

Subcase 2.1. \( d(a, D - b) = [3(n - 2)/2] \).

Then, applying Lemma 4 one can easily prove that \( D \) contains every non-strong orientation of a Hamiltonian cycle.

(The case when the symmetric arcs join \( a \) to \( x_i \) (odd) and \( b \) to \( x_j \) (even) is immediate, since then \( D \) contains a symmetric Hamiltonian cycle. The case when the symmetric arcs join \( a \) and \( b \) to the same vertices of \( D - a - b \) is slightly longer).

Subcase 2.2. \( d(a, D - b) \leq [3(n - 2)/2] - 1 \).

Since \( [3(n - 2)/2] \geq n > 2 \), and \( D - a - b \) contains a symmetric Hamiltonian cycle, we have \( \delta(G(D - a)) \geq 2 \). Case 1 does not apply, so

\[
e(D - a) \geq (n - 1)(n - 2) + 2 - \left[ \frac{3(n - 2)}{2} \right] + 1 - 1,
\]

and therefore

\[
e(G(D - a)) \geq e(D - a) - \left( \frac{n - 1}{2} \right) \geq \frac{(n - 1)(n - 2)}{2} - \left[ \frac{3n - 10}{2} \right] \\
= \left[ \frac{n^2 - 6n + 13}{2} \right] \geq \left[ \frac{n^2 - 7n + 22}{2} \right] = \left( \frac{n - 3}{2} \right) + 5.
\]

Thus one can apply Theorem B to see that \( G(D - a) \) is Hamiltonian unless \( n = 10 \) and \( G(D - a) \) is isomorphic to \( K_4 + \overline{K_5} \). If \( G(D - a) \) is isomorphic to \( K_4 + \overline{K_5} \), then one can easily check that \( D \) contains \( X_n \) or \( Y_n \).

So, assume \( d(b, D - a) \leq [3(n - 2)/2] - 1 \). Since \( \delta(G(D - b)) \geq 2 \) by the hypothesis of Case 2, one can argue as in Subcase 2.2 above to show that \( G(D - b) \) contains a Hamiltonian cycle unless \( G(D - b) \) is isomorphic to \( K_4 + \overline{K_5} \) and \( D \) contains every non-strong orientation of a Hamiltonian cycle.
Having settled Cases 1 and 2, we may assume that \( G(D - \{a, b\}) \) is a graph with \( n - 2 \) vertices and at least \( \binom{n}{2} - 1 \) edges. In particular we may assume that \( G(D - \{a, b\}) \) is Hamiltonian connected.

**Case 3.** \( a \) and \( b \) are adjacent, \( (a, b) \in E(D) \) say, and each of them is joined to a vertex of \( V(D) - \{a, b\} \) by a symmetric arc.

Let \( x_i \) and \( x_J \) be the vertices joined by a symmetric arc to \( a \) and \( b \), respectively. If \( i \neq j \), then \( U_n \) is contained in \( D \). So assume \( i = j \). If for a vertex \( x_k \neq x_i \), then \( (x_k, a) \in E(D) \) or \( (x_k, b) \in E(D) \), then \( D \) contains \( X_n \) or \( Y_n \) and hence every non-strong orientation of a Hamiltonian cycle. So assume \( (a, x_i), (x_k, b) \notin E(D) \) for every \( k \neq i \). Then for every but at most one \( x \in V(D) - \{a, b\} \) we have \( (x, a), (b, x) \in E(D) \). Clearly, \( D \) contains every orientation of a Hamiltonian cycle which is not antidirected. Moreover, for \( n \geq 9 \), there exists an \( m \), such that \( (x_m, a), (x_m, b) \in E(D) \) and \( (b, x_{m+2}), (b, x_{m+3}) \in E(D) \), implying the existence of an antidirected Hamiltonian cycle contained in \( D \) in case \( n \) is even.

**Case 4.** The cases 1, 2 and 3 do not apply.

Then \( D - \{a, b\} \) is complete and each vertex of \( D - \{a, b\} \) is adjacent to \( a \) and \( b \).

There exist four vertices \( x_1, \ldots, x_4 \) in \( D - \{a, b\} \) such that \( (x_1, a), (x_2, a) \in E(D) \) or \( (a, x_1), (a, x_2) \in E(D) \) and \( (x_3, b), (x_4, b) \in E(D) \) or \( (b, x_3), (b, x_4) \in E(D) \). The existence of every orientation (with at least four segments) of a Hamiltonian cycle is evident.

### 3. DIRECTED HAMILTONIAN CYCLES

**Proposition 7.** Let \( D \) be a non-strong digraph with \( n \) vertices and at least \( n(n - 1) - k(n - k) \) arcs, \( 0 \leq k \leq n/2 \). Then \( D \) contains a strong component with at most \( k \) vertices. Moreover, if \( D \) has a strong component with exactly \( k \) vertices, then \( D \) is isomorphic to \( K^*_k \approx K^*_{a-k} \) or \( K^*_k \approx K^*_{b-k} \).

**Proof.** Let \( C \) be a strong component of \( D \), such that \( v(C) = m \leq n/2 \). Then \( m(n - m) \leq e(D) \leq k(n - k) \), and therefore \( m \leq k \). Furthermore, if \( k = m \), then \( e(D) = k(n - k) \) and \( D(V(C)) \) and \( D(V(D - C)) \) are complete.

Taking \( k = 2 \), Proposition 7 implies the following Corollary 8.

**Corollary 8.** Let \( D \) be a digraph with \( n \) vertices and at least \( (n - 1) \cdot (n - 2) + 2 \) arcs. Then \( D \) is strong unless \( D \) contains a sink or a source, or \( D \) is isomorphic to \( K^*_2 \approx K^*_{a-2} \) or \( K^*_2 \approx K^*_{b-2} \).

Combining Corollary 8 with Theorem D one obtains immediately Corollary 9.
Corollary 9. Let $D$ be a digraph with $n$ vertices and at least $(n - 1) \cdot (n - 2) + 2$ arcs. Then $D$ is Hamiltonian unless $D$ contains a sink or a source, or $D$ is isomorphic to one of the following digraphs: $K_2^* \bowtie K_{n-2}^*$, $K_{n-2}^* \bowtie K_2^*$, $I_n, H_n, H'_n, M$ and $N$.

Now a result of Lewin [7] can be improved as follows.

Corollary 10. If a digraph $D$ with $n$ vertices and at least $(n - 1) \cdot (n - 2) + 3$ arcs is non-Hamiltonian, then $D$ contains a sink or a source.

D. Amar, I. Fournier, and A. Germa [1] proved recently that every digraph satisfying $d^+(x, D) \geq k$, $d^-(x, D) \geq k$ and $e(D) \geq n(n - 1) - (k + 1)(n - k - 1) + 1$ is Hamiltonian, hence Corollary 10 for $k = 1$.

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References