Existence of Spanning and Dominating Trails and Circuits

H. J. Veldman
TWENTE UNIVERSITY OF TECHNOLOGY
ENSCHEDE, THE NETHERLANDS

ABSTRACT

Let $T$ be a trail of a graph $G$. $T$ is a spanning trail (S-trail) if $T$ contains all vertices of $G$. $T$ is a dominating trail (D-trail) if every edge of $G$ is incident with at least one vertex of $T$. A circuit is a nontrivial closed trail. Sufficient conditions involving lower bounds on the degree-sum of vertices or edges are derived for graphs to have an S-trail, S-circuit, D-trail, or D-circuit. Thereby a result of Brualdi and Shanny and one mentioned by Lesniak-Foster and Williamson are improved.

1. INTRODUCTION

We use [2] for basic terminology and notations, but speak of vertices and edges instead of points and lines. Accordingly we denote the edge set of a graph $G$ by $E(G)$.

A spanning trail, or briefly S-trail, of a graph $G$ is a trail that contains all vertices of $G$. A dominating trail or D-trail of $G$ is a trail such that every edge of $G$ is incident with at least one vertex of the trail. A nontrivial closed trail will be called a circuit here.

In Section 2 we state a number of sufficient conditions for the existence of S-trails, S-circuits, D-trails, and D-circuits. Special cases of S-circuits (S-trails) are S-cycles (S-paths), better known as hamiltonian cycles (hamiltonian paths). The existence of hamiltonian cycles and paths has received broad attention in the literature. D-cycles and D-paths, special cases of D-circuits and D-trails, respectively, were studied in [6]. The existence of D-circuits is especially interesting in view of the following result.

Theorem A. (Harary and Nash-Williams [3]). The line graph $L(G)$ of a graph $G$ contains a hamiltonian cycle if and only if $G$ has a D-circuit or $G$ is isomorphic to $K_{1,s}$ for some $s \geq 3$. 
In [4] it is remarked that a slight modification in the proof of Theorem A yields the following analogous result, which forms a justification for investigating the existence of D-trails.

**Theorem B.** (Lesniak-Foster and Williamson [4]). The line graph \( L(G) \) of a graph \( G \) contains a hamiltonian path if and only if \( G \) has a D-trail.

By our results a theorem of Brualdi and Shanny [1] and one mentioned by Lesniak-Foster and Williamson [4] are improved.

We will need the following additional concepts, most of which are introduced in [6]. Two subgraphs \( H_1 \) and \( H_2 \) are close in \( G \) if they are disjoint and there is an edge joining a vertex of \( H_1 \) and one of \( H_2 \). If \( H_1 \) and \( H_2 \) are disjoint and not close, then \( H_1 \) and \( H_2 \) are remote. The degree of an edge \( e \) of \( G \), denoted \( \deg_G e \) or \( \deg e \) if no confusion can arise, is the number of vertices of \( G \) close to \( e \) (viewed as a subgraph of order 2). If \( T \) is an oriented trail in a graph and \( u \) and \( v \) are vertices of \( T \), then \( uTv \) denotes the longest subtrail of \( T \) from \( u \) to \( v \); \( vTu \) is the same subtrail in reverse order.

For ease of survey our results stated in Section 2 are not proved there; all proofs have been gathered in Section 3.

### 2. RESULTS

A well-known result in hamiltonian graph theory is the following.

**Theorem C.** (Ore [5]). If \( G \) is a graph with \( n \) vertices \((n \geq 3)\) such that \( \deg u + \deg v \geq n \) for every pair of nonadjacent vertices \( u \) and \( v \), then \( G \) contains an S-cycle.

Theorem C is best possible; also, the lower bound \( n \) for \( \deg u + \deg v \) cannot be decreased in order to obtain the weaker conclusion that \( G \) contains an S-circuit instead of an S-cycle. The truth of both statements is demonstrated by the graph \( K_1 + (K_1 \cup K_{n-2}) \) \((n \geq 3)\), which contains no S-circuit (and hence no S-cycle) while every pair of nonadjacent vertices has degree-sum \( n - 1 \). However, if the necessary condition \( \delta(G) \geq 2 \) is imposed, the bound can be lowered to guarantee the existence of an S-circuit.

**Theorem D.** (Lesniak-Foster and Williamson [4]). If \( G \) is a graph with \( n \) vertices \((n \geq 6)\) and \( \delta(G) \geq 2 \) such that \( \deg u + \deg v \geq n - 1 \) for every pair of nonadjacent vertices \( u \) and \( v \), then \( G \) contains an S-circuit.

In [6] the following analogue of Theorem C was stated.

**Theorem E.** (Veldman [6]). Let \( G \) be a graph with \( n \) vertices, other than a tree. If \( \deg e + \deg f \geq n - 2 \) for every pair of remote edges \( e \) and \( f \), then \( G \) contains a D-cycle.
Again, Theorem E is best possible and the lower bound \( n - 2 \) for \( \deg e + \deg f \) cannot be decreased to justify the weaker conclusion that \( G \) contains a D-circuit. To see this, subdivide in \( K_1 + (K_1 \cup K_{n-3}) \) \((n \geq 5)\) the edge incident with the vertex of degree 1 to obtain a graph without a D-circuit in which every pair of remote edges has degree-sum \( n - 3 \). Again, to guarantee the existence of a D-circuit, the bound can be lowered if a necessary condition is imposed. Let \( G \) be a graph with a D-circuit \( C \) and let

\[
D_1(G) = \{ v \in V(G) \mid \deg v = 1 \}.
\]

If \( v \) is a vertex of \( G \) with a neighbor in \( D_1(G) \), then \( v \) must be on \( C \), so that \( v \) has at least two neighbors on \( C \). In particular \( v \) has at least two neighbors of degree at least 2. Thus, if in \( G \) all vertices of degree 1 are deleted, then the remaining graph has minimum degree at least 2. Now the following result is analogous to Theorem D.

**Theorem 1.** If \( G \) is a graph with \( n \) vertices and \( \delta(G - D_1(G)) \geq 2 \) such that \( \deg e + \deg f \geq n - 3 \) for every pair of remote edges \( e \) and \( f \), then \( G \) contains a D-circuit.

Let \( H \) be the graph \( K_k \cup K_{n-k} \) or the graph \( (K_k \cup K_{n-k}) + e \) (obtained from \( K_k \cup K_{n-k} \) by joining a vertex of \( K_k \) to a vertex of \( K_{n-k} \)), where \( n \geq 6 \) and \( 3 \leq k \leq n - 3 \). Then \( \delta(H - D_1(H)) = \delta(H) \geq 2 \) and every pair of remote edges of \( H \) has degree-sum at least \( n - 4 \) while \( H \) contains no D-circuit, showing that Theorem 1 is best possible. (Note that by considering \( H \), Theorem D also is seen to be best possible).

A consequence of Theorem 1 is the following.

**Corollary 2.** Let \( G \) be a graph with \( n \) vertices \((n \geq 4)\) and at least one edge. If \( G \neq P_4, \ G \neq K_{1,n-1}, \) and \( \deg u + \deg v \geq n - 1 \) for every edge \( uv \) of \( G \), then \( G \) contains a D-circuit.

In view of Theorem A, Corollary 2 improves the following result of Brualdi and Shanny [1]: if \( G \) is a graph with \( n \) vertices \((n \geq 4)\) and at least one edge such that \( \deg u + \deg v \geq n \) for every edge \( uv \) of \( G \), then \( L(G) \) is hamiltonian.

The graphs used to demonstrate that Theorem 1 is best possible also show that Corollary 2 is best possible.

We now turn our attention to S- and D-trails that are not necessarily closed. Lesniak-Foster and Williamson [4] mention that the following can be verified: If \( G \) is a connected graph with \( n \) vertices \((n \geq 5)\) such that \( \deg u + \deg v \geq n - 2 \) for every pair of nonadjacent vertices \( u \) and \( v \), then \( G \) contains an S-trail. This result can be improved as follows.

**Theorem 3.** If \( G \) is a connected graph with \( n \) vertices \((n \geq 5)\) such that \( \deg u + \deg v + \deg w \geq n - 1 \) for every triple \( u, v, w \) of independent vertices, then \( G \) contains an S-trail.
Theorem 3 is best possible: For each \( n \geq 5 \) obtain the graph \( H_n \) from a \( K_{1,3} \) and a \( K_{n-3} \) by identifying one endvertex of \( K_{1,3} \) with a vertex of \( K_{n-3} \); \( H_n \) has no S-trail while every triple of independent vertices has degree-sum at least \( n - 2 \).

An analogous condition is sufficient for the existence of a D-trail instead of an S-trail.

Theorem 4. If \( G \) is a connected graph with \( n \) vertices \( (n \geq 8) \) such that \( \deg e + \deg f + \deg g \geq n - 4 \) for every triple \( e,f,g \) of mutually remote edges, then \( G \) contains a D-trail.

For \( n \geq 8 \) the graph obtained from \( H_{n-2} \) by subdividing both edges incident with a vertex of degree 1 shows that Theorem 4 is best possible.

If \( e \) is an edge of a graph and \( u \) a vertex incident with \( e \), then \( \deg e \geq \deg u - 1 \). As a consequence, the sufficient conditions for D-trails stated in Theorems E, 1 and 4 are weaker than the corresponding analogous conditions for S-trails in Theorems C, D, and 3, respectively, in accordance with the fact that every S-trail is also a D-trail whereas the converse is not true in general.

Finally we state a sufficient condition for the existence of an S-circuit resembling the condition of Corollary 2.

Theorem 5. If \( G \) is a graph with \( n \) vertices and \( \delta(G) > 0 \) such that \( \deg u + \deg v \geq n + 1 \) for every edge \( uv \) of \( G \), then \( G \) contains an S-circuit.

Theorem 5 is seen to be best possible by considering the graph \( K_1 + (K_1 \cup K_{n-2}) (n \geq 3) \) or, for odd \( n \geq 3 \), the graph \( K_{2,n-2} \).

3. PROOFS

Proof of Theorem 1. By contradiction. Suppose \( G \) satisfies the conditions of the theorem without containing a D-circuit. Two edges \( e \) and \( f \) cannot be in different components of \( G \), since otherwise \( \deg e + \deg f \geq n - 4 \) while \( e \) and \( f \) are remote. Hence \( G \) has at most one nontrivial component and we may assume that \( G \) is connected.

By [6, Corollary 9.1], a connected graph with \( n \) vertices has a D-path if the degree-sum of every three mutually remote edges is at least \( n - 3 \). Clearly \( G \) satisfies this condition. Let \( P = v_1v_2 \cdots v_p \) be a longest D-path of \( G \), so that all neighbors of \( v_1 \) and \( v_p \) are on \( P \). Using the assumptions that \( G \) contains no D-circuit, that \( P \) is a longest D-path and that \( \delta(G - D_1(G)) \geq 2 \), it is easily shown that \( p \geq 6 \).

Put \( e = v_1v_2 \) and \( f = v_{p-1}v_p \); then \( e \) and \( f \) are remote. Assuming the contrary, e.g., \( v_2v_{p-1} \in E(G) \), the cycle \( v_2 \overline{P} v_{p-1}v_2 \) is a D-circuit of \( G \), a contradiction. Furthermore, \( v_1 \) and \( v_{p-2} \) are the only vertices of \( G \) that may be close to both \( e \)
and $f$: suppose $v \in V(G) - \{v_3, v_{p-2}\}$ and $v$ is close to both $e$ and $f$, e.g., $vv_1$ and $vv_{p-1}$ are edges of $G$; then $v_1v_{p-1}Pv_1$ is a D-circuit, a contradiction.

Distinguishing three cases we will show that $\deg e + \deg f \leq n - 4$, the final contradiction.

**Case 1.** $v_3$ and $f$ are remote and $v_{p-2}$ and $e$ are remote. Then every vertex of $V(G) - \{v_1, v_2, v_{p-1}, v_p\}$ is close to at most one of the edges $e$ and $f$. Thus $\deg e + \deg f \leq n - 4$.

**Case 2.** $v_3$ and $f$ are close, e.g., $v_3v_p \in E(G)$, and $v_{p-2}$ and $e$ are remote (or, symmetrically, $v_3$ and $f$ are remote and $v_{p-2}$ and $e$ are close). Then certainly $\deg e + \deg f \leq n - 3$. Case 2 will be settled by proving that some vertex of $V(G) - \{v_1, v_2, v_{p-1}, v_p\}$ is close to neither of the edges $e$ and $f$. Define

$$I_1 = \{i \mid 4 \leq i \leq p - 3, v_i \in E(G)\},$$
$$I_2 = \{i \mid 4 \leq i \leq p - 3, v_2v_i \in E(G)\},$$
$$I_3 = \{i \mid 4 \leq i \leq p - 3, \exists u \notin V(P): (v_2u, uv_i) \in E(G)\}.$$

We show that $I_1 \cup I_2 \cup I_3 \neq \emptyset$. Suppose $I_1 = I_2 = \emptyset$. $v_1$ and $v_3$ are nonadjacent, otherwise $v_1v_2v_3Pv_1$ would be a D-circuit of $G$. Thus $\deg v_1 = 1$. Since $\delta(G - D_1(G)) \geq 2$, $v_2$ has, next to $v_3$, another neighbor $u$ of degree at least 2. Since $I_2 = \emptyset$, $v \notin V(P)$. Let $u$ be a neighbor of $u$ other than $v_2$. Since $P$ is a D-path, $u$ belongs to $P$. Using previous arguments we have that $u \notin \{v_1, v_2, v_{p-1}, v_p\}$. Also $u \notin \{v_3, v_{p-2}\}$, since otherwise $v_1v_2v_3Pv_p$ or $v_1v_2v_{p-1}Pv_pPv_{p-1}$ would be a D-path longer than $P$. In conclusion, $I_3 \neq \emptyset$.

Let $m = \min\{i \in I_1 \cup I_2 \cup I_3\}$, e.g., $v_m \in I_2$. Then $m \neq 4$, for otherwise $v_2v_4Pv_pv_3v_2$ would be a D-circuit. Now by definition of $m$ the vertex $v_{m-1}$ is not close to $e$. However, $v_{m-1}$ is close to $f$: If, e.g., $v_{m-1}v_{p-1} \in E(G)$, then $v_2v_{m-1}Pv_{p-1}v_{m-1}v_2$ is a D-circuit. Again we conclude that $\deg e + \deg f \leq n - 4$.

**Case 3.** $v_3$ and $f$ are close and $v_{p-2}$ and $e$ are close, e.g., $v_1v_{p-2} \in E(G)$ and $v_2v_{p-1} \in E(G)$. Then certainly $\deg e + \deg f \leq n - 2$. Case 3 is settled by indicating two vertices in $V(G) - \{v_1, v_2, v_{p-1}, v_p\}$ which are neither close to $e$ nor to $f$. If $p = 6$, then $v_1v_4v_5v_6v_1$ is a D-circuit, so that $p \geq 7$. The vertex $v_4$ is close to neither of the edges $e$ and $f$: If, e.g., $v_4v_k \in E(G)$, then $v_4v_Pv_{p-1}v_4v_2$ is a D-circuit; if, e.g., $v_4v_p \in E(G)$, then $v_1v_{p-2}Pv_4v_pPv_{p-1}v_2v_1$ is a D-circuit. Analogously, $v_{p-3}$ is neither close to $e$ nor to $f$. If $v_4 \neq v_{p-3}$, i.e., if $p > 7$, then $\deg e + \deg f \leq n - 4$. Now assume that $v_4 = v_{p-3}$ or, equivalently, $p = 7$. $v_4$ has a neighbor $v$ outside $P$, otherwise $v_1v_5v_6v_3v_2v_1$ would be a D-circuit. $u$ and $e$ are remote: If, e.g., $v_4v \in E(G)$, then $v_2v_4v_3v_2v_1v_4v_2$ is a D-circuit. Symmetrically, $v$ and $f$ are remote. Hence, as in the case $p > 7$, we found two
vertices \((u_4 \text{ and } u)\) which are close to neither of the edges \(e\) and \(f\). Again it follows that \(\deg e + \deg f \leq n - 4\), completing the proof.

**Proof of Corollary 2.** Let \(G\) satisfy the conditions of Corollary 2 and let \(e = uv\) be an arbitrary edge of \(G\). Then

\[
\deg e \geq \max\{\deg u, \deg v\} - 1 \geq \frac{1}{2}(n - 1) - 1 = \frac{1}{2}(n - 3).
\]

It follows that every pair of edges of \(G\), and hence a fortiori every pair of remote edges, has degree-sum at least \(n - 3\). By Theorem 1 the proof is complete if \(\delta(G - D_1(G))\) is shown to be at least 2. As in the proof of Theorem 1 we may assume that \(G\) is connected. Every pair of adjacent vertices has degree-sum at least \(n - 1\), so every vertex with a neighbor in \(D_1(G)\) has degree at least \(n - 2\). Hence \(\delta(G - D_1(G)) \geq 2\) if \(|V(G - D_1(G))| \geq 4\). Since \(G \not\cong K_{1,n-1}\), \(G - D_1(G) \not\cong K_1\). It remains to be shown that \(G - D_1(G) \not\cong K_2, P_3\).

Suppose \(G - D_1(G) \cong K_2\) and let \(u\) and \(v\) be the vertices of \(G - D_1(G)\). In \(G\) both \(u\) and \(v\) are adjacent to a vertex of degree 1, otherwise \(u\) or \(v\) would be a vertex of \(D_1(G)\). Hence \(\deg_G u \geq n - 2\) and \(\deg_G v \geq n - 2\). It follows that in \(G\) both \(u\) and \(v\) are adjacent to exactly one vertex of degree 1, so that \(G \cong P_4\), contrary to assumption.

Finally suppose that \(G - D_1(G) \cong P_3\). Again both endvertices of \(G - D_1(G)\), \(u_1\) and \(u_2\), say, are adjacent in \(G\) to a vertex of degree 1. But then \(\deg_G u_i \leq n - 3\) \((i = 1, 2)\), contradicting the simultaneous conclusion that \(\deg_G u_i \geq n - 2\).

**Proof of Theorem 3.** By contraposition. Assume that \(G\) is a connected graph on \(n\) vertices \((n \geq 5)\) without an \(S\)-trail. We will exhibit an independent set of three vertices with degree-sum at most \(n - 2\).

Let \(T = u_1u_2\cdots u_p\) be a trail of \(G\) such that \(|V(T)|\) is maximum while \(|E(T)| \leq |E(T')|\) for every trail \(T'\) with \(|V(T')| = |V(T)|\). Since \(G\) is connected and has no \(S\)-trail, there is a vertex \(v \in V(G) - V(T)\) with at least one neighbor on \(T\). \(T\) is not a circuit; assuming the contrary and letting \(w\) be a neighbor of \(v\) on \(T\), the trail \(vwT'w\) has more vertices than \(T\), contradicting the choice of \(T\). More generally \(G\) contains no circuit \(C\) with \(V(C) \supseteq V(T)\).

\(u_1\) is not an internal vertex of \(T\), otherwise \(u_2T'u_p\) would be a trail satisfying \(|V(u_2T'u_p)| = |V(T)|\) and \(|E(u_2T'u_p)| < |E(T)|\), again a contradiction with the choice of \(T\). By the same token \(u_p\) is not an internal vertex of \(T\). Since there is no circuit of \(G\) containing all vertices of \(T\), it follows that \(u_1\) and \(u_p\) are nonadjacent. Furthermore, neither of the vertices \(u_1\) and \(u_p\) is adjacent to \(v\): If, e.g., \(uu_1 \in E(G)\), then the trail \(uu_1T'u_p\) has more vertices than \(T\). We will show that \(\deg u_1 + \deg u_p + \deg v \leq n - 2\).

Clearly, \(|V(T)| \geq 3\). If \(|V(T)| = 3\), then, since \(G\) is connected and \(|V(T)|\) is maximum, every component of \(G - V(T)\) is trivial and every vertex of \(G - V(T)\) is adjacent to \(u_2\). In that case \(G \cong K_{1,n-1}\) and \(\deg u_1 + \deg u_p + \deg v = 3 \leq n - 2\), since \(n \geq 5\). Henceforth assume that \(|V(T)| \geq 4\).
Put $H = G[\{v, u_1, u_2, u_3, u_{p-1}, u_p\}]$ and $U = V(G) - V(H)$. Every vertex of $U$ is adjacent to at most one of the vertices $u_1, u_p, v$. Assuming the contrary, let $u$ be a vertex of $U$ which is adjacent to at least two of the vertices $u_1, u_p, v$ (it is immaterial whether or not $u$ belongs to $T$). If $u \in N(u_1) \cap N(u_p)$, then $C = uu_1 Tu_2u$ is a circuit with $V(C) \supseteq V(T)$, which is impossible. If $u \in N(u_1) \cap N(v)$, then the trail $uu_1 Tu_2u$ contradicts the choice of $T$; so does the trail $u_1 Tu_2u$ in case $u \in N(u_p) \cap N(u)$. Putting

$$
\sigma = \deg_H u_1 + \deg_H u_p + \deg_H v,
$$

we conclude that

$$
\deg_G u_1 + \deg_G u_p + \deg_G v \leq \sigma + |U|.
$$

The proof will be complete if it is shown that $\sigma \leq |V(H)| - 2$. We distinguish three cases.

**Case 1.** $u_2 = u_{p-1}$. Since $|V(T)| \geq 4$, the vertices $u_3$ and $u_p$ do not coincide. Like the vertices in $U$, the vertex $u_3$ is adjacent to at most one of the vertices $u_1, u_p, v$. Hence, if $v$ is not adjacent to $u_2$, we have $\sigma \leq 3 = |V(H)| - 2$. If $v$ is adjacent to $u_2$, then $u_3$ is adjacent to none of the vertices $u_1, u_p, v$: the choice of $T$ is contradicted by the trail $uu_3 u_1 Tu_2u$ if $u_1 u_3 \in E(G)$, by $u_1 u_2 u_3 Tu_2u$ if $u_3 u_3 \in E(G)$ and by $u_1 u_2 u_3 Tu_2u$ if $u_3 \in E(G)$. Again it follows that $\sigma \leq 3 = |V(H)| - 2$.

**Case 2.** $u_3 = u_{p-1}$. The vertex $v$ is adjacent to at most one of the vertices $u_2$ and $u_3$, otherwise the trail $u_1 u_2 u_3 Tu_2u$ would contradict the choice of $T$. Also, at most one of the pairs $u_1, u_3$ and $u_p, u_2$ is adjacent, otherwise $u_1 u_2 Tu_3 u_2 u_1$ would be a circuit containing all vertices of $T$. Hence, if $v$ is neither adjacent to $u_2$ nor to $u_3$, then $\sigma \leq 3 = |V(H)| - 2$. If $u_2 \in E(G)$ and $u_3 \notin E(G)$, then $u_1 u_3 \notin E(G)$ and $u_p u_2 \notin E(G)$. Assuming the contrary, the trails $uu_3 u_1 Tu_2u$ and $u_1 Tu_2u$ respectively, contradict the choice of $T$. Again it follows that $\sigma \leq 3$. This conclusion is reached analogously if $u_3 \notin E(G)$ and $u_3 \in E(G)$.

The case $u_2 = u_{p-2}$ can be handled similarly.

**Case 3.** $u_2 \neq u_{p-1}$ and $u_3 \neq u_{p-1}$ and $u_2 \neq u_{p-2}$. The set of edges of $H$ incident with $u_1, u_p$, or $v$ and different from $u_1 u_2$ and $u_{p-1} u_p$ is a subset of

$$
E' = \{u_1 u_3, u_1 u_{p-1}, u_2 u_p, u_3 u_p, uu_2, uu_3, uu_{p-1}\}.
$$

We show that no triple of elements of $E'$ is a subset of $E(H)$, thereby reaching the conclusion that $\sigma \leq 4 = |V(H)| - 2$. First we reduce the number of triples to be checked by listing pairs of elements of $E'$ that cannot be subsets of $E(H)$. If one of the pairs in the table below is assumed to be a subset of $E(H)$, a trail
$T'$ can be indicated that contradicts the choice of $T$. $T'$ may be a trail with more vertices than $T$ or a circuit containing all vertices of $T$.

Assumed to be a subset of $E(H)$

\[
\begin{align*}
\{u_1, u_2, u_3, u_4\} \\
\{u_1, u_2, u_3, u_4\} \\
\{u_1, u_2, v_2\} \\
\{u_1, u_3, v_3\} \\
\{u_1, u_{p-1}, v_2, u_p\} \\
\{u_1, u_{p-1}, v_3\} \\
\{u_2, u_3, u_4\} \\
\{v_2, v_3\}
\end{align*}
\]

$T'$ contradicting choice of $T$

\[
\begin{align*}
u_1, u_3, u_2, u_1 \\
u_1, u_3, u_2, u_1 \\
u_2, u_3, u_2 \\
u_3, u_2, u_1 \\
u_4, u_3, u_2, u_1 \\
u_1, u_3, u_2, u_1 \\
u_3, u_2, u_1 \\
u_1, u_3, u_2, u_1
\end{align*}
\]

There are four triples of elements of $E'$ which contain none of the above pairs. To complete the proof, in the following table it is shown that none of these triples can be a subset of $E(H)$.

Assumed to be a subset of $E(H)$

\[
\begin{align*}
\{u_1, u_{p-1}, u_2, u_p, u_3, u_4\} \\
\{u_1, u_{p-1}, u_2, u_p, v_2\} \\
\{u_2, u_p, u_3, u_4\} \\
\{u_3, u_p, v_2, u_4\} \\
\{v_2, v_3\}
\end{align*}
\]

$T'$ contradicting choice of $T$

\[
\begin{align*}
\{u_1, u_{p-1}, u_2, u_p, u_3, u_4\} \\
\{u_2, u_p, u_3, u_4\} \\
\{u_3, u_p, v_2, u_4\} \\
\{u_1, u_3, u_2, v_3\} \\
\{u_1, u_3, u_2, v_3\}
\end{align*}
\]

Outline of the proof of Theorem 4. By contraposition. Suppose $G$ is a connected graph on $n$ vertices $(n \geq 8)$ without a D-trail. Let $T = u_1u_2 \cdots u_p$ be a trail of $G$ such that $|E(G - V(T))|$ is minimum while $|E(T)| \leq |E(T')|$ for every trail $T'$ with $|E(G - V(T'))| = |E(G - V(T))|$. Since $G$ is connected and has no D-trail, there is an edge $v_1v_2 \in E(G - V(T))$ which is close to at least one vertex of $T$. There is no circuit of $G$ containing all vertices of $T$, otherwise one could indicate a trail $T'$ with $E(G - V(T')) \subseteq E(G - V(T)) - \{v_1, v_2\}$, contradicting the choice of $T$. Moreover, as in the proof of Theorem 3, neither $u_1$ nor $u_p$ is an internal vertex of $T$. It follows that $u_1$ and $u_p$ are nonadjacent. Clearly neither of the vertices $u_1$ and $u_p$ is close to the edge $v_1v_2$.

The vertex $u_1$ is adjacent to a vertex $u_0 \in V(G) - V(T)$, otherwise $T' = u_2u_3u_p$ would be a trail satisfying $|E(G - V(T'))| = |E(G - V(T))|$ and $|E(T')| < |E(T)|$, again a contradiction with the choice of $T$. By the same token, $u_p$ is adjacent to a vertex $u_{p+1} \in V(G) - V(T)$. Since there is no circuit containing all vertices of $T$, the vertices $u_0$ and $u_{p+1}$ do not coincide.

From the choice of $T$ it follows that the independent edges $u_0u_1, u_pu_{p+1}$ and $v_1v_2$ are, in fact, mutually remote. By inspection of the proof of Theorem 3 one now shows, using completely analogous arguments, that $\deg u_0u_1 + \deg u_pu_{p+1} + \deg v_1v_2 \leq n - 5$, the role of the three independent vertices $u_1, u_p, v$ now being played by the three mutually remote edges $u_0u_1, u_pu_{p+1, v_1v_2}$. 

Proof of Theorem 5. Suppose $G$ satisfies the conditions of the theorem without containing an $S$-circuit. From the conditions one easily deduces that $G$ is connected and $\delta(G) \geq 2$. It follows that $G$ contains a cycle and thus, in particular, a circuit.

Let $C$ be a longest circuit in $G$. Since $C$ is not an $S$-circuit and $G$ is connected, $G$ has an edge $uv$ with $u \not\in V(C)$ and $v \in V(C)$. Suppose $u$ and $v$ have a common neighbor $w$. If $uw \in E(C)$, then the circuit obtained from $C$ by replacing the edge $uw$ by the path $vuw$ is longer than $C$, a contradiction with the choice of $C$. If $uw \notin E(C)$, then the circuit $vuwCv$ contradicts the choice of $C$. Hence $u$ and $v$ have no common neighbors. It follows that $\deg u + \deg v \leq n - 2 + 2 = n$, a contradiction.

Note added in proof: In [J. Graph Theory, 8 (1984), 303–307] L. Clark proves that, if $G$ is a connected graph with $|V(G)| = n \geq 6$ and $\deg u + \deg v \geq n - 1 - p(n)$ for every edge $uv$ of $G$, where $p(n)$ is 0 for $n$ even and 1 for $n$ odd, then $L(G)$ is hamiltonian. Via an extension of the proof of Corollary 2 one can show that Clark's result also is a corollary of Theorem 1. Moreover, the condition that $G$ be connected can be replaced by the condition that $E(G) \neq \emptyset$.

References