On Circuits and Pancyclic Line Graphs

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ABSTRACT

Clark proved that $L(G)$ is hamiltonian if $G$ is a connected graph of order $n \geq 6$ such that $\text{deg } u + \text{deg } v \geq n - 1 - p(n)$ for every edge $uv$ of $G$, where $p(n) = 0$ if $n$ is even and $p(n) = 1$ if $n$ is odd. Here it is shown that the bound $n - 1 - p(n)$ can be decreased to $(2n + 1)/3$ if every bridge of $G$ is incident with a vertex of degree 1, which is a necessary condition for hamiltonicity of $L(G)$. Moreover, the conclusion that $L(G)$ is hamiltonian can be strengthened to the conclusion that $L(G)$ is pancyclic. Lesniak-Foster and Williamson proved that $G$ contains a spanning closed trail if $\delta(G) \geq 2$ and $\text{deg } u + \text{deg } v \geq n - 1$ for every pair of nonadjacent vertices $u$ and $v$. The bound $n - 1$ can be decreased to $(2n + 3)/3$ if $G$ is connected and bridgeless, which is necessary for $G$ to have a spanning closed trail.

1. TERMINOLOGY

We use [4] for basic terminology and notation, but speak of vertices and edges instead of points and lines. Accordingly we denote the edge set of a graph $G$ by $E(G)$. In [7] a circuit was defined as a nontrivial closed trail. Here the following subtle variation on this definition will be more convenient. A circuit $C$ of a graph $G$ is a nontrivial eulerian subgraph of $G$. Alternatively, $C$ is a circuit if
and only if $C$ is a nontrivial connected subgraph such that every vertex of $C$ has even degree in $C$. If $C$ is a circuit of $G$, then $\beta(C)$ denotes the number of edges of $G$ incident with at least one vertex of $C$. A spanning circuit, or briefly $S$-circuit, of a graph $G$ is a circuit that contains all vertices of $G$. A dominating circuit or $D$-circuit of $G$ is a circuit such that every edge of $G$ is incident with at least one vertex of the circuit. If $H$ is a subgraph of $G$, then vertices of $G - V(H)$ which are adjacent to at least one vertex of $H$ are called neighbors of $H$. We denote the neighbors of $H = \{v\}$ by $N(v)$. A graph of order $n$ is pancyclic if it contains a cycle of length $i$ for each $i$ with $3 \leq i \leq n$. A chord of a cycle $C$ in $G$ is an edge in $E(G) - E(C)$ whose ends are in $C$. A connected graph $G$ is said to be almost bridgeless if every bridge of $G$ is incident with a vertex of degree 1. If $x$ is a real number, then $\lfloor x \rfloor$ and $\lceil x \rceil$ denote, respectively, the greatest integer smaller than or equal to $x$ and the smallest integer greater than or equal to $x$.

2. DOMINATING CIRCUITS AND PANCYCLIC LINE GRAPHS

In [5] the following relation between $D$-circuits in graphs and hamiltonian cycles in line graphs is established.

**Theorem 1.** (Harary and Nash-Williams [5]). The line graph $L(G)$ of a graph $G$ contains a hamiltonian cycle if and only if $G$ has a $D$-circuit or $G$ is isomorphic to $K_{1,s}$ for some $s \geq 3$.

In [3], Clark proved that the line graph $L(G)$ of a graph $G$ is hamiltonian if $G$ is connected, $|V(G)| = n \geq 6$ and $\deg u + \deg v \geq n - 1 - p(n)$ for every edge $uv$ of $G$, where $p(n) = 0$ if $n$ is even and $p(n) = 1$ if $n$ is odd. The graphs showing that Clark's result is best possible all contain a bridge which is not incident with a vertex of degree 1. If a graph $G$ contains a bridge $uv$ with $\deg u \neq 1 \neq \deg v$, then the vertex of $L(G)$ corresponding to $uv$ is a cut vertex of $L(G)$, so that $L(G)$ is nonhamiltonian. Hence a necessary condition for $L(G)$ to have a hamiltonian cycle, and for $G$ to have a $D$-circuit, is that $G$ is almost bridgeless. Using Theorem 1 we will show how Clark's bound $n - 1 - p(n)$ can be decreased if $G$ is additionally required to be almost bridgeless. Before presenting our result we state two lemmas, the first of which is easily proved and frequently used in [2] and [3].

**Lemma 2.** Let $G$ be a connected graph and $C$ a circuit of $G$ with maximum number of vertices. Then $G$ contains no circuit $C'$ satisfying $V(C') \cap V(C) \neq \emptyset \neq V(C') \cap V(G) - V(C)$ and $|E(C') \cap E(C)| \leq 1$.

**Lemma 3.** Let $G$ be a connected graph, $C$ a circuit of $G$ with maximum number of vertices, $K$ a component of $G - V(C)$ and $u_1$ and $u_2$ two neighbors of $K$ on $C$. Then the following assertions hold.
a. $u_1$ and $u_2$ are nonadjacent.

b. If $w \in N(u_1) \cap N(u_2) - V(K)$, then none of the vertex pairs \{u_1, w\} and \{u_2, w\} has a common neighbor.

c. If $w_1 \in N(u_1) - V(K)$, $w_2 \in N(u_2) - V(K)$ and $w_1w_2 \in E(G)$, then at most one of the pairs \{u_1, w_1\}, \{u_2, w_2\}, and \{w_1, w_2\} has a common neighbor.

d. If $v \in V(K)$ and $w \in N(u_1) \cap N(u_2) - V(K)$, then $v$ and $w$ are nonadjacent and have no common neighbor in $G - (V(K) \cup \{u_1, u_2\})$.

e. If $w_1, w_2 \in N(u_1) \cap N(u_2) - V(K)$, then $w_1$ and $w_2$ are nonadjacent and have no common neighbor in $G - \{u_1, u_2\}$.

Proof. Let $G$ be a connected graph, $C$ a circuit of $G$ of maximum order, $K$ a component of $G - V(C)$ and $u_1$ and $u_2$ two neighbors of $K$ on $C$. Throughout the proof $P$ will denote a $u_1$-$u_2$ path with $P \not= V(P) - \{u_1, u_2\} \subset V(K)$.

a. Suppose $u_1u_2 \in E(G)$. Then the cycle with edge set $E(P) \cup \{u_1u_2\}$ contradicts the assertion of Lemma 2. Hence $u_1$ and $u_2$ are nonadjacent.

b. Let $w$ be a vertex of $N(u_1) \cap N(u_2) - V(K)$. If $u_1w \not\in E(C)$ or $u_2w \not\in E(C)$ then the cycle with edge set $E(P) \cup \{u_1w, u_2w\}$ contradicts Lemma 2. Hence $u_1w, u_2w \in E(C)$. Suppose, for example, $u_1$ and $w$ have a common neighbor $v$. From Lemma 2 we deduce that $v \in V(C)$ and at least one of the edges $u_1v$ and $uvw$ is in $E(C)$. Depending on whether or not each of the edges $u_1v$ and $uvw$ is in $E(C)$ we now define a subgraph $C'$ of $G$ by specifying $E(C') - E(C)$ and $E(C) - E(C')$; $V(C')$ will be the set of vertices of $G$ with at least one edge of $E(C')$. In the table below there is a column for each of the edges $u_1v$ and $uvw$; a one in such a column means that the relevant edge is in $E(C)$, while a zero means that it is in $E(G) - E(C)$.

<table>
<thead>
<tr>
<th>$u_1v$</th>
<th>$uvw$</th>
<th>$E(C') - E(C)$</th>
<th>$E(C) - E(C')$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>$E(P)$</td>
<td>${u_1w, u_2w}$</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>$E(P) \cup {vw}$</td>
<td>${u_1v, u_2w}$</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>$E(P) \cup {u_1v}$</td>
<td>${vw, u_2w}$</td>
</tr>
</tbody>
</table>

If, for example, $u_1v \in E(C)$ and $uvw \not\in E(C)$, then $C'$ is defined as the subgraph of $G$ with $V(C') = V(C) \cup V(P)$ and $E(C') = E(C) \cup E(P) \cup \{vw\} - \{u_1v, u_2w\}$, as indicated in the second row of the table. In all cases the fact that $C$ is connected implies that $C'$ is connected. Furthermore, since all vertices of $C$ have even degree in $C$, all vertices of $C'$ have even degree in $C'$. It follows that $C'$ is a circuit with $|V(C')| = |V(C) \cup V(P)| > |V(C)|$, contradicting the choice of $C$ and completing the proof of (b).

c. Let $w_1$ and $w_2$ be vertices of $G$ such that $w_1 \in N(u_1) - V(K)$, $w_2 \in N(u_2) - V(K)$ and $w_1w_2 \in E(G)$. By Lemma 2 at least two of the edges $u_1w_1$, $w_1w_2$ and $u_2w_2$ are in $E(C)$. If one of the three edges is in $E(G) -$


\[ E(C), \text{ then a slight variation on the arguments used in (a) yields that the vertices incident with each of the remaining edges have no common neighbor. Hence assume } u_{1}w_{1}, w_{1}w_{2}, u_{2}w_{2} \in E(C). \text{ Suppose that at least two of the pairs } \{u_{1}, w_{1}\}, \{w_{1}, w_{2}\} \text{ and } \{u_{2}, w_{2}\} \text{ have a common neighbor. We derive contradictions in two cases.} \]

**Case 1.** There exists a vertex \( w \) of \( G \) which is adjacent to at least three of the vertices \( u_{1}, u_{2}, w_{1}, w_{2} \).

From Lemma 2 and (b) we deduce that \( w \in V(C) - \{u_{1}, u_{2}, w_{1}, w_{2}\} \) and \( w \) is adjacent to \( w_{1}, w_{2} \) and exactly one of the vertices \( u_{1} \) and \( u_{2} \), \( u_{1} \) say. Lemma 2 also implies that at least one of the edges \( w_{1}u_{1}, w_{2}u_{2} \) is in \( E(C) \). In all possible cases we now specify, like in the proof of (b), a circuit \( C' \) of \( G \) with \( |V(C')| > |V(C)| \), contradicting the choice of \( C \).

<table>
<thead>
<tr>
<th>( wu_{1} )</th>
<th>( ww_{1} )</th>
<th>( ww_{2} )</th>
<th>( E(C') - E(C) )</th>
<th>( E(C) - E(C') )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>( E(P) )</td>
<td>( {u_{1}w_{1}, u_{2}w_{2}, w_{1}w_{2}} )</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>( E(P) \cup {ww_{2}} )</td>
<td>( {wu_{1}, u_{2}w_{2}} )</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>( E(P) \cup {ww_{1}} )</td>
<td>( {wu_{1}, w_{1}w_{2}, u_{2}w_{2}} )</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>( E(P) \cup {wu_{1}} )</td>
<td>( {wu_{2}, u_{2}w_{2}} )</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>( E(P) \cup {ww_{2}} )</td>
<td>( {wu_{1}, u_{2}w_{2}} )</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>( E(P) \cup {wu_{1}} )</td>
<td>( {wu_{2}, u_{2}w_{2}} )</td>
</tr>
</tbody>
</table>

**Case 2.** Each vertex of \( G \) is adjacent to at most two of the vertices \( u_{1}, u_{2}, w_{1}, w_{2} \).

We assume that \( u_{i} \) and \( w_{i} \) have a common neighbor \( v_{i} (i = 1, 2) \); the remaining subcases are similar. From Lemma 2 we deduce that \( v_{1}, v_{2} \in V(C) \) and at least one of the edges \( u_{i}v_{1}, v_{1}w_{1}, u_{2}v_{2} \) and \( v_{2}w_{2} \) is in \( E(C) \). Again a circuit \( C' \) of \( G \) with \( |V(C')| > |V(C)| \) can be specified in all possible cases. We only treat two representative cases.

<table>
<thead>
<tr>
<th>( u_{1}v_{1} )</th>
<th>( v_{1}w_{1} )</th>
<th>( u_{2}v_{2} )</th>
<th>( v_{2}w_{2} )</th>
<th>( E(C') - E(C) )</th>
<th>( E(C) - E(C') )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>( E(P) \cup {v_{2}w_{2}} )</td>
<td>( {u_{1}w_{1}, w_{1}w_{2}, u_{2}v_{2}} )</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>( E(P) \cup {u_{1}v_{1}, v_{1}w_{1}, u_{2}v_{2}} )</td>
<td>( {w_{1}w_{2}, v_{2}w_{2}} )</td>
</tr>
</tbody>
</table>

**d.** Let \( v \) be a vertex of \( K \) and \( w \) a vertex in \( N(u_{1}) \cap N(u_{2}) - V(K) \). For \( i = 1, 2 \), let \( P_{i} \) be a \( u - v \) path with all internal vertices in \( K \). From Lemma 2 it follows that \( vw \notin E(G) \) and \( u_{1}w, u_{2}w \in E(C) \). Suppose \( v \) and \( w \) have a common neighbor \( u \) in \( G - (V(K) \cup \{u_{1}, u_{2}\}) \). Then \( uv \in E(C) \) by Lemma 2. If \( w \) is not a cut vertex of \( C \) or if \( u_{1}, u_{2} \) and \( u \) are in the same component of \( C - w \), then the subgraph \( C' \) of \( G \) with \( V(C') = V(C) \cup V(P_{1}) \) and \( E(C') = E(C) \cup E(P_{1}) \cup \{uv\} - \{uw, u_{1}w\} \) is connected, implying that \( C' \) is a circuit of \( G \) with \( |V(C')| > |V(C)| \). Hence assume that \( w \) is a cut vertex of \( C \) and, for example, \( u \) and \( u_{2} \) are in different components \( H_{1} \) and \( H_{2} \) of \( C - w \), respectively. Let \( C_{i} \) be the subgraph of \( C \)
induced by \( V(H_i) \cup \{w\} \) \((i = 1, 2)\). Then \( C_1 \) and \( C_2 \) are subcircuits of \( C \). In particular, \( C_1 \) and \( C_2 \) are bridgeless, so \( C_1 - uw \) and \( C_2 - u_2w \) are connected subgraphs of \( C \). It follows that \( C - \{uw, u_2w\} \) is connected.

But then the circuit \( C' \) with \( V(C') = V(C) \cup V(P_2) \) and \( E(C') = E(C) \cup E(P_2) \cup \{uv\} - \{uw, u_2w\} \) contradicts the choice of \( C \).

e. Let \( w_1 \) and \( w_2 \) be two vertices in \( N(u_1) \cup N(u_2) - V(K) \). Then \( u_iw_j \in E(C) \) by Lemma 2 \((i = 1, 2; j = 1, 2)\). The table below shows that a circuit \( C' \) with \( |V(C')| > |V(C)| \) can be constructed if \( w_1w_2 \in E(G) \).

<table>
<thead>
<tr>
<th>( w_1w_2 )</th>
<th>( E(C') - E(C) )</th>
<th>( E(C) - E(C') )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( E(P) )</td>
<td>( {u_1w_1, u_2w_1} )</td>
</tr>
<tr>
<td>0</td>
<td>( E(P) \cup {w_1w_2} )</td>
<td>( {u_1w_1, u_2w_2} )</td>
</tr>
</tbody>
</table>

Suppose \( w_1 \) and \( w_2 \) have a common neighbor \( v \) in \( G - \{u_1, u_2\} \). Again a circuit \( C' \) with \( |V(C')| > |V(C)| \) can be specified. Note that in the fourth row of the table below \( v \) may be a vertex of \( P \).

<table>
<thead>
<tr>
<th>( vw_1 )</th>
<th>( vw_2 )</th>
<th>( E(C') - E(C) )</th>
<th>( E(C) - E(C') )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>( E(P) )</td>
<td>( {u_1w_1, u_2w_1} )</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>( E(P) \cup {vw_2} )</td>
<td>( {u_1w_1, vw_1, u_2w_2} )</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>( E(P) \cup {vw_1} )</td>
<td>( {u_1w_1, vw_2, u_2w_2} )</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>( E(P) \cup {vw_1, vw_2} )</td>
<td>( {u_1w_1, u_2w_2} )</td>
</tr>
</tbody>
</table>

**Theorem 4.** Let \( G \) be a nontrivial connected, almost bridgeless graph of order \( n \) with \( G \not\cong K_{1,n-1} \). If \( \deg u + \deg v \geq (2n + 1)/3 \) for every edge \( uv \) of \( G \), then \( G \) contains a \( D \)-circuit.

**Proof.** Let \( G \) be a connected, almost bridgeless graph of order \( n \) with \( G \not\cong K_{1,n-1} \). Assuming that \( G \) contains no \( D \)-circuit, we will exhibit two adjacent vertices with degree-sum at most \( \frac{3}{2}n \). Since \( G \) is almost bridgeless and \( G \not\cong K_{1,n-1} \), deletion of all vertices of degree 1 yields a nontrivial bridgeless graph, implying that \( G \) contains a circuit. Let \( C \) be a circuit of \( G \) such that \( |V(C)| \) is maximum and \( \beta(C) \geq \beta(C') \) for every circuit \( C' \) with \( |V(C')| = |V(C)| \). Since \( C \) is not a \( D \)-circuit, \( G - V(C) \) has a nontrivial component \( K \). From Lemma 2 and the fact that \( G \) is almost bridgeless we conclude that \( K \) has at least two neighbors on \( C \). We distinguish three cases.

**Case 1.** \( K \) has two neighbors on \( C \) which are joined by a path of length 2 contained in \( G - V(K) \).

Let \( u_1 \) and \( u_2 \) be two neighbors of \( K \) on \( C \) which are joined by the path \( u_1w_1u_2 \), where \( w_1 \not\in V(K) \). Let \( P \) be a \( u_1 - w_2 \) path with \( \emptyset \neq V(P) - \{u_1, u_2\} \subset V(K) \) such that \( |V(P)| \) is minimum. Define \( v_1 \) as the immediate successor of \( u_1 \) on \( P \). If \( V(P) - \{u_1, u_2\} = \{v_1\} \), let \( v_2 \) be an arbitrary neighbor of \( v_1 \) in \( K \), otherwise let \( v_2 \) be the successor of \( v_1 \) on \( P \). Finally, let \( H \) be the
induced subgraph \((V(P) \cup v_2, w_1)\) of \(G\). From Lemmas 2, 3(b) and 3(d) it follows that
\[
N(u_i) \cap N(v_i) \cap (V(G) - V(H)) = N(u_i) \cap N(w_i) \cap (V(G) - V(H))
= N(v_i) \cap N(w_i) \cap (V(G) - V(H)) = \emptyset. \tag{1}
\]

We next show that
\[
V(G) - (V(H) \cup N(u_i) \cup N(v_i) \cup N(w_i)) \neq \emptyset. \tag{2}
\]

Since each vertex of \(C\) has even degree in \(C\), \(u_2\) has a neighbor \(w_2\) on \(C\) with \(w_2 \neq w_1\). If \(u_1w_2 \notin E(G)\), then, by Lemmas 2 and 3(b), \(w_2\) is not adjacent to any of the vertices \(u_1, v_1\) and \(w_1\), implying (2). Now assume \(u_1w_2 \in E(G)\). Then by Lemma 2 we have \(u_1w_2, u_2w_2 \in E(C)\) and \(v_2w_2 \notin E(G)\). There exists a vertex \(w\) in \(G - V(H)\) which is adjacent to \(w_2\), otherwise the circuit \(C'\) with \(V(C') = V(C) \cup V(P) - \{w_2\}\) and \(E(C') = E(C) \cup E(P) - \{u_1w_2, u_2w_2\}\) satisfies \(|V(C')| \geq |V(C)|\) and \(\beta(C') > \beta(C)\), contradicting the choice of \(C\). By Lemma 2, \(w \notin V(K)\). Application of Lemmas 3(b), 3(d) and 3(e) yields that \(w\) is adjacent to none of the vertices \(u_1, v_1\) and \(w_1\), implying (2).

Equation (1) expresses that each vertex of \(G - V(H)\) is adjacent to at most one of the vertices \(u_1, v_1\) and \(w_1\). Together with (2) we obtain
\[
\deg u_1 + \deg v_1 + \deg w_1 \leq n - |V(H)| - 1 + \deg_H u_1 + \deg_H v_1 + \deg_H w_1. \tag{3}
\]

Similarly,
\[
\deg u_1 + \deg v_2 + \deg w_1 \leq n - |V(H)| - 1 + \deg_H u_1 + \deg_H v_2 + \deg_H w_1. \tag{4}
\]

Summation of the inequalities (3) and (4) yields
\[
2(\deg u_1 + \deg w_1) + \deg v_1 + \deg v_2
\leq 2(n - |V(H)| - 1 + \deg_H u_1 + \deg_H w_1) + \deg_H v_1 + \deg_H v_2. \tag{5}
\]

From Lemma 2, Lemma 3(a) and the minimality of \(|V(P)|\) we conclude that every vertex of \(H - \{v_1, v_2\}\) has degree 2 in \(H\). Furthermore, \(\deg_H u_1 = \deg_H v_2 = 2\) if \(v_2 \in V(P)\), while \(\deg_H u_1 = 3\) and \(\deg_H v_2 = 1\) otherwise. Observing that \(|V(H)| \geq 5\) we now deduce from (5) that
\[
2(\deg u_1 + \deg w_1) + \deg v_1 + \deg v_2 \leq 2n.
\]

It follows that either \(\deg u_1 + \deg w_1 \leq \frac{2}{3}n\) or \(\deg v_1 + \deg v_2 \leq \frac{2}{3}n\), settling Case 1.
Case 2. Case 1 does not apply and $K$ has two neighbors on $C$ which are joined by a path of length 3 contained in $G - V(K)$.

Let $u_1$ and $u_2$ be two neighbors of $K$ on $C$ which are joined by the path $u_1w_1w_2u_2$, where $w_1, w_2 \in V(K)$. Define $P$, $v_1$ and $v_2$ as in Case 1 and put $H = \langle V(P) \cup \{v_2, w_1, w_2\} \rangle$. By Lemma 3(c) at least one of the pairs $\{u_1, w_1\}$ and $\{u_2, w_2\}$, $\{u_1, w_1\}$ say, has no common neighbor. In particular,

$$N(u_1) \cap N(w_1) \cap (V(G) - V(H)) = \emptyset. \quad (6)$$

By Lemma 2, $v_1$ and $w_1$ have no common neighbor outside $C$. Suppose $v_1$ and $w_1$ have a common neighbor $u$ on $C$ with $u \neq u_1$. Then Case 1 applies to the neighbors $u$ and $u_1$ of $K$ on $C$, contrary to assumption. We conclude that

$$N(v_1) \cap N(w_1) \cap (V(G) - V(H)) = \emptyset. \quad (7)$$

Another application of Lemma 2 gives us

$$N(u_1) \cap N(u_2) \cap (V(G) - V(H)) = \emptyset. \quad (8)$$

From (6), (7), and (8) we deduce that

$$\deg u_1 + \deg v_1 + \deg w_1 \leq n - |V(H)| + \deg_H u_1 + \deg_H v_1 + \deg_H w_1. \quad (9)$$

Similarly,

$$\deg u_1 + \deg v_2 + \deg w_1 \leq n - |V(H)| + \deg_H u_1 + \deg_H v_2 + \deg_H w_1. \quad (10)$$

Summation of (9) and (10) yields

$$2(\deg u_1 + \deg w_1) + \deg v_1 + \deg v_2$$
$$\leq 2(n - |V(H)| + \deg_H u_1 + \deg_H w_1) + \deg_H v_1 + \deg_H v_2. \quad (11)$$

By Lemmas 2, 3(a), 3(b) and the minimality of $|V(P)|$, every vertex of $H - \{v_1, v_2\}$ has degree 2 in $H$, while $\deg_H v_1 + \deg_H v_2 = 4$. Observing that $|V(H)| \geq 6$, we deduce from (11) that

$$2(\deg u_1 + \deg w_1) + \deg v_1 + \deg v_2 \leq 2n,$$

implying that either $\deg u_1 + \deg w_1 \leq \frac{2}{3}n$ or $\deg v_1 + \deg v_2 \leq \frac{2}{3}n$.

Case 3. Neither Case 1 nor Case 2 applies.

Let $u_1$ and $u_2$ be two arbitrary neighbors of $K$ on $C$ and $w$ a vertex in $N(u_2) - V(K)$. Define $P$, $v_1$ and $v_2$ as in Case 1 and put $H = \langle V(P) \cup \{v_2, w\} \rangle$. 
By Lemma 2 and by assumption we have

\[ N(u_1) \cap N(v_1) \cap (V(G) - V(H)) = N(u_2) \cap N(v_1) \cap (V(G) - V(H)) \]
\[ = N(u_1) \cap N(u_2) \cap (V(G) - V(H)) = \emptyset, \]

implying that

\[ \deg u_1 + \deg v_1 + \deg u_2 \leq n - |V(H)| + \deg_H u_1 + \deg_H v_1 + \deg_H u_2 \]
\[ \leq n - 5 + 1 + 3 + 2 = n + 1. \]

Suppose \( \deg u_1 + \deg v_1 + \deg u_2 = n + 1 \). Then, putting \( U_1 = N(u_1) \cap V(C), U_2 = N(u_2) \cap V(C) \) and \( V_1 = N(v_1) \cap V(C) - \{u_1, u_2\} \), we have \( U_1 \neq \emptyset \neq U_2 \) and each vertex of \( C - \{u_1, u_2\} \) is in exactly one of the sets \( U_1, U_2 \) and \( V_1 \). Since \( C \) is connected, there exists an edge \( uv \) of \( C \) with \( u \in U_1 \) and \( v \in U_2 \cup V_1 \). If \( v \in V_1 \), then Case 1 applies to the neighbors \( u_1 \) and \( v \) of \( K \) on \( C \), contrary to assumption. If \( v \in U_2 \), then Case 2 applies to \( u_1 \) and \( u_2 \), again contrary to assumption. We conclude that

\[ \deg u_1 + \deg v_1 + \deg u_2 \leq n. \]  

By Lemma 2,\( N(v_1) \cap N(w) \cap (V(G) - V(C)) = N(u_1) \cap N(w) \cap (V(G) - V(C)) = \emptyset. \) Assuming that \( N(v_1) \cap N(w) \cap V(C) - \{u_2\} \neq \emptyset \) or \( N(u_1) \cap N(w) \cap V(C) \neq \emptyset \), we reach the contradiction that Case 1 or Case 2 applies. Hence

\[ N(v_1) \cap N(w) \cap (V(G) - V(H)) = N(u_1) \cap N(w) \cap (V(G) - V(H)) = \emptyset. \]

Together with (12) we obtain

\[ \deg u_1 + \deg v_1 + \deg w \leq n - |V(H)| + \deg_H u_1 + \deg_H v_1 + \deg_H w \]
\[ \leq n - 5 + 1 + 3 + 1 = n. \]  

Summation of (13) and (15) yields

\[ 2(\deg u_1 + \deg v_1) + \deg u_2 + \deg w \leq 2n, \]

so that either \( \deg u_1 + \deg v_1 \leq \frac{2}{3}n \) or \( \deg u_2 + \deg w \leq \frac{2}{3}n. \) ■

**Corollary 5.** Let \( G \) be a connected, almost bridgeless graph of order \( n \geq 4 \) such that \( \deg u + \deg v \geq \frac{2(n + 1)}{3} \) for every edge \( uv \) of \( G \). Then \( L(G) \) is hamiltonian. Moreover, if \( G \neq C_4, C_5 \), then \( L(G) \) is pancyclic.
Proof. Let $G$ be a connected, almost bridgeless graph of order $n \geq 4$ such that $\deg u + \deg v \geq (2n + 1)/3$ for every edge $uv$ of $G$. The existence of a hamiltonian cycle in $L(G)$ immediately follows from the combination of Theorems 1 and 4. If $G \cong K_{1,n-1}$, then $L(G)$ is complete and hence pancyclic. Now assume $G \neq C_4, C_5, K_{1,n-1}$ and $L(G)$ is not pancyclic. Let $k = \max\{i \mid L(G) \text{ does not contain } C_i\}$.

We have $\Delta(G) \geq 3$, so $k \geq 4$. Let $D = u_1 u_2 \ldots u_p u_1$ be a shortest cycle in $G$ and suppose $p \geq 5$. Then every vertex of $G - V(D)$ is adjacent to at most one vertex of $D$, implying that

$$p(2n + 1)/6 \leq \sum_{i=1}^{p} \deg u_i \leq n - p + 2p,$$

so that $n \leq \lfloor 5p/(2p - 6) \rfloor \leq 6$. However, it is easily checked that every graph of order at most 6 satisfying our assumptions has a cycle of length at most 4. Hence, in fact, $p \leq 4$ and

$$\beta(D) \geq p + \sum_{i=1}^{p} (\deg u_i - 2) \geq \lceil -p + p(2n + 1)/6 \rceil = \lceil (2n - 5)/6 \rceil = n - 2.$$  \hspace{1cm} (16)

Observing that, for any circuit $C$ of $G$, $L(G)$ contains a cycle of length $i$ for every $i$ with $|E(C)| \leq i \leq \beta(C)$, we conclude that $k \geq n - 1$.

$L(G)$ is hamiltonian, so $k < |E(G)|$ and $L(G)$ contains $C_{k+1}$. Hence $G$ contains a circuit $C$ with $|E(C)| \leq k + 1 \leq \beta(C)$. In fact $|E(C)| = k + 1$, otherwise $L(G)$ contains $C_4$. Since $C$ is a circuit, there exists edge-disjoint cycles $D_1, D_2, \ldots, D_r$ such that $C = \bigcup_{i=1}^{r} D_i$. We now derive contradictions in two cases.

Case 1. $r = 1$.

Since $|E(C)| = k + 1 \geq n$, $C$ is a hamiltonian cycle of $G$ and $k = n - 1$. Let $D'$ be a shortest cycle among all cycles of $G$ that contain exactly one chord of $C$. Let $D'$ have length $q$. If $q = 3$, then $G$, and hence $L(G)$ too, contains $C_{n-1}$, a contradiction. If $q \geq 4$, then $n \geq 6$ and as in (16) we obtain

$$\beta(D') \geq \lceil q(2n - 5)/6 \rceil \geq \lceil 4(2n - 5)/6 \rceil \geq n - 1,$$

again implying the contradiction that $L(G)$ contains $C_{n-1}$.

Case 2. $r \geq 2$.

Let $H$ be the graph with $V(H) = \{D_1, D_2, \ldots, D_r\}$ and $D_i D_j \in E(H)$ if and only if $V(D_i) \cap V(D_j) \neq \emptyset$. Since $H$ is connected, at least two vertices of $H$ are not cut vertices of $H$. Equivalently, there are at least two values of $j$ for which $\bigcup_{1 \leq i \leq n, i \neq j} D_i$ is a connected subgraph of $G$ and hence a circuit of $G$. Assume
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without loss of generality that \( C' = \bigcup_{i=0}^{k-1} D_i \) and \( C'' = D_1 \cup \bigcup_{i=0}^{k-1} D_i \) are circuits of \( G \). If \( E(D_2 - V(C'')) = \emptyset \), then \( |E(C'')| < |E(C)| = k + 1 \leq \beta(C'') \), so that \( L(G) \) contains \( C_k \). Hence there exists an edge \( uv \) of \( D_2 \) with \( u, v \notin V(C'') \). Let \( E_1 \) be the set of edges of \( D_2 \) incident with at least one vertex of \( C' \) and \( E_2 = E(D_2) - E_1 \). Then

\[
\beta(C') \geq |E(C')| + |E_1| + \deg u - 2 + \deg v - 2 \geq |E(C)| - |E_2|
\]
\[
+ (2n + 1)/3 - 4.
\]

On the other hand, since \( L(G) \) does not contain \( C_k \),

\[
\beta(C') \leq k - 1 = |E(C)| - 2.
\]

It follows that \( |E_2| \geq (2n - 5)/3 \). Hence \( |V(D_1 - V(C'))| \geq (2n - 2)/3 \) and similarly \( |V(D_2 - V(C''))| \geq (2n - 2)/3 \). But then

\[
n = |V(G)| \geq |V(D_1 - V(C'))| + |V(D_2 - V(C''))| + 1
\]
\[
\geq 2(2n - 2)/3 + 1 > n,
\]
a contradiction. \( \Box \)

We do not know any connected, almost bridgeless graph \( G \) of order \( n \) without a \( D \)-circuit such that \( G \neq K_{1,n-1} \) and \( \deg u + \deg v \geq \frac{n}{3} \) for every edge \( uv \) of \( G \). We conjecture that, for \( n \) sufficiently large, the bound \( (2n + 1)/3 \) in Theorem 4 and Corollary 5 can be decreased to \( (2n - 9)/5 \). If true, this conjecture is best possible. To see this, construct for \( i \geq 3 \) a graph \( G(i) \) as follows: take five disjoint copies of \( K_n \), label them \( G_1, \ldots, G_5 \); choose three vertices \( u_1, u_2, u_3 \) in \( G_1 \), three vertices \( v_1, v_2, v_3 \) in \( G_2 \), two vertices \( x_1, x_2 \) in \( G_3 \), two vertices \( y_1, y_2 \) in \( G_4 \) and two vertices \( z_1, z_2 \) in \( G_5 \); obtain \( G(i) \) as \( \bigcup_{j=1}^5 G_j + \{u_1x_1, u_2y_1, u_3z_1, v_1x_2, v_2y_2, v_3z_2\} \). Then \( G(i) \) is 2-connected and \( \deg u + \deg v \geq (2|V(G(i))| - 10)/5 \) for every edge \( uv \) of \( G(i) \), while \( G(i) \) contains no \( D \)-circuit and hence \( L(G(i)) \) is nonhamiltonian.

Although Corollary 5 may not be best possible, it is strong enough to contain Clark’s result.

**Corollary 6.** (Clark [3]). Let \( G \) be a connected graph of order \( n \geq 6 \). If \( \deg u + \deg v \geq n - 1 - p(n) \) for every edge \( uv \) of \( G \), where \( p(n) = 0 \) if \( n \) is even and \( p(n) = 1 \) if \( n \) is odd, then \( L(G) \) is hamiltonian.

**Proof.** Let \( G \) be a connected graph of order \( n \geq 6 \) such that \( \deg u + \deg v \geq n - 1 - p(n) \) for every edge \( uv \) of \( G \). Since \( n \geq 6 \), \( n - 1 - p(n) \geq (2n + 1)/3 \). Hence we are done by Corollary 5 if \( G \) is shown to be almost bridgeless. Suppose \( G \) contains a bridge \( u_1u_2 \) with \( \deg u_1 \neq 1 \neq \deg u_2 \)
deg \( u_2 \). Let \( H_i \) be the component of \( G - u_i u_2 \) containing \( u_i \) \((i = 1, 2)\). Assume without loss of generality that \( |V(H_i)| \leq |V(H_2)| \), so that \( |V(H_i)| \leq (n - p(n))/2 \). Since \( |V(H_1)| \geq 2 \), \( H_1 - u_1 \) contains a vertex \( u \). If \( u \) has a neighbor \( v \) with \( v \neq u_1 \), then \( \deg u + \deg v \leq 2(|V(H_1)| - 1) \leq n - p(n) - 2 \), a contradiction. If \( u \) has no neighbor in \( H - u_1 \), then \( uu_1 \in E(G) \) and \( \deg u = 1 \), so that \( \deg u + \deg u_1 \leq 1 + |V(H_1)| \leq 1 + (n - p(n))/2 \). For \( n \geq 6 \) we have \( 1 + (n - p(n))/2 \leq n - 2 - p(n) \). Thus \( \deg u + \deg u_1 \leq n - 2 - p(n) \), again a contradiction.

The bound \((2n + 1)/3\) in Corollary 5 can be decreased in case only hamiltonian graphs are considered.

**Theorem 7.** Let \( G \) be a hamiltonian graph of order \( n \geq 13 \). If \( \deg u + \deg v \geq n/2 \) for every edge \( uv \) of \( G \), then \( L(G) \) is pancyclic.

For the proof of Theorem 7 we refer to [1].

### 3. SPANNING CIRCIRCTS

In [6] Lesniak-Foster and Williamson proved that a graph \( G \) contains an S-circuit if \( |V(G)| = n \geq 6 \), \( \delta(G) \geq 2 \) and \( \deg u + \deg v \geq n - 1 \) for every pair of nonadjacent vertices \( u \) and \( v \). All graphs showing that this result is best possible contain a bridge. For a graph \( G \) to have an S-circuit it is necessary that \( G \) is connected and contains no bridges. We now show how the above result can be improved by additionally imposing these necessary conditions.

**Theorem 8.** Let \( G \) be a connected bridgeless graph of order \( n \geq 3 \). If \( \deg u + \deg v \geq (2n + 3)/3 \) for every pair of nonadjacent vertices \( u \) and \( v \), then \( G \) contains an S-circuit.

**Proof.** Let \( G \) be a connected bridgeless graph of order \( n \geq 3 \). Assuming that \( G \) contains no S-circuit, we will exhibit two nonadjacent vertices with degree-sum smaller than \((2n + 3)/3\). Since \( G \) is bridgeless, \( G \) contains a circuit. Let \( C \) be a circuit of \( G \) of maximum order and \( K \) a component of \( G - V(C) \). By Lemma 2 and the fact that \( G \) is bridgeless, \( K \) has at least two neighbors on \( C \). We distinguish three cases.

**Case 1.** \( K \) has two neighbors on \( C \) which are joined by a path of length 2 contained in \( G - V(K) \).

Let \( u_1 \) and \( u_2 \) be two neighbors of \( K \) on \( C \) which are joined by the path \( u_1 w_1 u_2 \), where \( w_1 \notin V(K) \). Let \( P \) be a \( u_1 - u_2 \) path with \( \emptyset \neq V(P) - \{u_1, u_2\} \subset V(K) \) such that \( |V(P)| \) is minimum and let \( v \) be an arbitrary vertex in \( V(P) \cap V(K) \). We distinguish two subcases.
Case 1.1. \( u_1 \) and \( u_2 \) have a common neighbor \( w_2 \in V(G) - (V(K) \cup \{w_1\}) \).

Put \( H = \langle V(P) \cup \{w_1, w_2\} \rangle \). Lemmas 2, 3(d) and 3(e) imply that \( \{v, w_1, w_2\} \) is an independent set and each vertex of \( G - V(H) \) is adjacent to at most one of the vertices \( v, w_1, \) and \( w_2 \). Together with the minimality of \( |V(P)| \) we obtain

\[
\deg v + \deg w_1 + \deg w_2 \leq n - |V(H)| + \deg_H v + \deg_H w_1 + \deg_H w_2 \\
\leq n - 5 + 2 + 2 + 2 = n + 1.
\]

It follows that at least one of the nonadjacent vertex pairs \( \{v, w_1\}, \{v, w_2\} \) and \( \{w_1, w_2\} \) has degree-sum at most \( 2(n + 1)/3 \), settling Case 1.1.

Case 1.2. \( u_1 \) and \( u_2 \) have no common neighbor in \( V(G) - (V(K) \cup \{w_1\}) \).

Put \( H = \langle V(P) \cup \{w_1\} \rangle \). By Lemmas 2, 3(b) and 3(d), each vertex of \( G - V(H) \) is adjacent to at most one of the vertices \( u_1, u_2, v, w_1 \), so that

\[
\deg u_1 + \deg u_2 + \deg v + \deg w_1 \leq n - |V(H)| + \deg_h u_1 + \deg_h u_2 \\
+ \deg_h v + \deg_h w_1 \leq n - 4 + 2 + 2 + 2 = n + 4.
\]

It follows that at least one of the nonadjacent vertex pairs \( \{u_1, u_2\} \) and \( \{v, w_1\} \) has degree-sum at most \( (n + 4)/2 \). If \( n > 6 \), then \( (n + 4)/2 < (2n + 3)/3 \) and we are done. Now assume \( n \leq 6 \). Since \( \deg_C u_i \geq 2 \), \( u_i \) has a neighbor \( v_i \) on \( C \) with \( v_i \neq w_1, w_2, w_3 \). By assumption \( v_1 \) and \( v_2 \) do not coincide, so that \( n \geq 6 \) and hence \( n = 6 \). By Lemmas 2 and 3(b), \( (N(v) \cup N(w_1)) \cap (\{v_1, v_2\} = \emptyset \). Thus \( \deg v = \deg w_1 = 2 \), so that \( \deg v + \deg w_1 = 4 < 5 = (2n + 3)/3 \).

Case 2. Case 1 does not apply and \( K \) has two neighbors on \( C \) which are joined by a path of length 3 contained in \( G - V(K) \).

Let \( u_1 \) and \( u_2 \) be two neighbors of \( K \) on \( C \) which are joined by the path \( u_1 w_1 w_2 u_2 \), where \( w_1, w_2 \notin V(K) \). Define \( P \) and \( v \) as in Case 1 and put \( H = \langle V(P) \cup \{w_1, w_2\} \rangle \). By Lemma 3(c) at least one of the following three subcases applies.

Case 2.1. \( N(u_1) \cap N(v) = N(u_2) \cap N(w_2) = \emptyset \).

By Lemma 2 and the fact that Case 1 does not apply, each vertex of \( G - V(H) \) is adjacent to at most one of the vertices \( u_1, v \) and \( w_1 \). Hence

\[
\deg u_1 + \deg v + \deg w_1 \leq n - |V(H)| + \deg_H u_1 + \deg_H v + \deg_H w_1 \\
\leq n - 5 + 2 + 2 + 2 = n + 1.
\]

Similarly,

\[
\deg u_2 + \deg v + \deg w_2 \leq n + 1.
\]
Assuming without loss of generality that $\deg w_1 \leq \deg w_2$ we deduce from (17) and (18) that

$$2(\deg v + \deg w_1) + \deg u_1 + \deg u_2 \leq 2 \deg v + \deg w_1 + \deg w_2 + \deg u_1 + \deg u_2 \leq 2n + 2.$$ 

Hence one of the nonadjacent vertex pairs $\{v, w_1\}$ and $\{u_1, u_2\}$ has degree-sum at most $(2n + 2)/3$.

**Case 2.2** $N(u_1) \cap N(w_1) = N(w_1) \cap N(w_2) = \emptyset$.

Similar arguments as used in Case 2.1 now yield

$$\deg u_1 + \deg v + \deg w_1 \leq n + 1$$

and

$$\deg v + \deg w_1 + \deg w_2 \leq n + 1,$$

implying that

$$2(\deg v + \deg w_1) + \deg u_1 + \deg w_2 \leq 2n + 2.$$ 

Hence either $\deg v + \deg w_1 \leq (2n + 2)/3$ or $\deg u_1 + \deg w_2 \leq (2n + 2)/3$.

**Case 2.3.** $N(u_2) \cap N(w_2) = N(w_1) \cap N(w_2) = \emptyset$.

This case is symmetric to Case 2.2.

**Case 3.** Neither Case 1 nor Case 2 applies.

Let $u_1$ and $u_2$ be two neighbors of $K$ on $C$ and, for $i = 1, 2$, $w_i$ a vertex in $N(u_i) \setminus V(K)$. Define $P$ and $v$ as in Case 1 and put $H = \langle V(P) \cup \{w_1, w_2\} \rangle$. By Lemma 2 and the fact that neither Case 1 nor Case 2 applies, each vertex of $G \setminus V(H)$ is adjacent to at most one of the vertices $u_1$, $v$ and $w_2$. Hence

$$\deg u_1 + \deg v + \deg w_2 \leq n - |V(H)| + \deg_H u_1 + \deg_H v + \deg_H w_2$$

$$\leq n - 5 + 2 + 2 + 1 = n.$$ 

Similarly,

$$\deg u_2 + \deg v + \deg w_1 \leq n.$$ 

Assuming without loss of generality that $\deg w_1 \leq \deg w_2$, we obtain

$$2(\deg v + \deg w_1) + \deg u_1 + \deg u_2 \leq 2n.$$ 

Hence either $\deg v + \deg w_1 \leq \frac{2}{3}n$ or $\deg u_1 + \deg u_2 \leq \frac{2}{3}n$. 

The graph $K_{2,3}$ is the only known example of a connected bridgeless graph of order $n \geq 3$ without an $S$-circuit such that $\deg u + \deg v \geq (2n + 2)/3$ for every pair of nonadjacent vertices $u$ and $v$. We conjecture that the bound in Theorem 8, too, can be decreased to $(2n - 9)/5$ if $n$ is sufficiently large. Such an improvement would be best possible in view of the graphs $G(i)$ defined in Section 2.

Theorem 8 implies the result of Lesniak-Foster and Williamson mentioned above.

Corollary 9. (Lesniak-Foster and Williamson [6]). Let $G$ be a graph with $|V(G)| = n \geq 6$ and $\delta(G) \geq 2$. If $\deg u + \deg v \geq n - 1$ for every pair of nonadjacent vertices $u$ and $v$, then $G$ contains an $S$-circuit.

Proof. Let $G$ be a graph with $|V(G)| = n \geq 6$ and $\delta(G) \geq 2$ such that $\deg u + \deg v \geq n - 1$ for every pair of nonadjacent vertices $u$ and $v$. It is easily seen that $G$ must be connected. Since $n \geq 6$, $n - 1 \geq (2n + 3)/3$. In view of Theorem 8 it remains to be shown that $G$ is bridgeless. Suppose $G$ contains a bridge $u_1u_2$. Let $H_i$ be the component of $G - u_1u_2$ containing $u_i$ $(i = 1, 2)$. Since $\delta(G) \geq 2$, $H_i$ is nontrivial, say that $v_i \in V(H_i) - \{u_i\}$ $(i = 1, 2)$. Then $v_1v_2 \notin E(G)$ and $\deg v_1 + \deg v_2 \leq |V(H_1)| - 1 + |V(H_2)| - 1 = n - 2$, a contradiction. 

4. DOMINATING CIRCUITS REVISITED

A slight variation on the proof of Theorem 8 gives us the following counterpart of Theorem 4.

Theorem 10. Let $G$ be a connected, almost bridgeless graph of order $n \geq 3$. If $\deg u + \deg v \geq (2n + 1)/3$ for every pair of nonadjacent vertices $u$ and $v$, then $G$ contains a $D$-circuit.

Proof outline. Let $G$ be a connected, almost bridgeless graph of order $n \geq 3$. We will exhibit a nonadjacent vertex pair with degree-sum smaller than $(2n + 1)/3$ under the assumption that $G$ contains no $D$-circuit. Let $C$ be a circuit of $G$ of maximum order and $K$ a nontrivial component of $G - V(C)$. $K$ has at least two neighbors on $C$.

Distinguish the same cases as in the proof of Theorem 8. In each case define $P$ as a shortest $u_1 - u_2$ path with $\emptyset \neq V(P) - \{u_1, u_2\} \subseteq V(K)$ and $v_1$ as the successor of $u_1$ on $P$. If $V(P) - \{u_1, u_2\} = \{v_1\}$, let $v$ be an arbitrary neighbor of $v_1$ in $K$, otherwise let $v$ be the successor of $v_1$ on $P$. Now all upper bounds on degree-sums in the proof of Theorem 8 can be decreased to obtain a vertex pair as desired.

Without proof we mention that the corresponding counterpart of Corollary 5 also holds.
Corollary 11. Let $G$ be a connected, almost bridgeless graph of order $n \geq 3$ such that $\deg u + \deg v \geq (2n + 1)/3$ for every pair of nonadjacent vertices $u$ and $v$. Then $L(G)$ is hamiltonian. Moreover, if $G \neq C_4, C_5$, then $L(G)$ is pancyclic.

Again we conjecture, as a best possible improvement of Theorem 10 and Corollary 11, that the bound $(2n + 1)/3$ can be decreased to $(2n - 9)/5$ for $n$ sufficiently large.

Note added in proof. A graph $G$ is cyclically 2-edge-connected if no two cycles of $G$ can be separated by the removal of at most one edge. Suppose $G$ has order $n \geq 5$ with $\deg u + \deg v \geq (2n + 1)/3$ for every edge $uv$ of $G$. Then $G$ is connected and almost bridgeless if and only if $G$ is cyclically 2-edge-connected and has no isolated vertices. Consequently, a corollary of Theorem 4 is the following: Let $G$ be a nontrivial cyclically 2-edge-connected graph of order $n$ with no isolated vertices. If $\deg u + \deg v \geq (2n + 1)/3$ for every edge $uv$ of $G$, then $G$ contains a $D$-circuit. Here the bound $(2n + 1)/3$ is best possible, as the following example shows. Let $u$ be any vertex in $K_{(n,3)-1}$, $v$ the center of the star $K_{1,\{2n/3\}}$, and $G = (K_{(n,3)-1} \cup K_{1,\{2n/3\}}) + uv$. Then $G$ satisfies the above conditions with $(2n + 1)/3$ replaced with $2n/3$ but $G$ has no $D$-circuit, since $L(G)$ is not hamiltonian.

References