LONG CYCLES IN GRAPHS WITH LARGE DEGREE SUMS

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A number of results are established concerning long cycles in graphs with large degree sums. Let \( G \) be a graph on \( n \) vertices such that \( d(x) + d(y) + d(z) \geq s \) for all triples of independent vertices \( x, y, z \). Let \( c \) be the length of a longest cycle in \( G \) and \( \alpha \) the cardinality of a maximum independent set of vertices. If \( G \) is 1-tough and \( s \geq n \), then every longest cycle in \( G \) is a dominating cycle and \( c \geq \min(n, n + \frac{1}{3}s - \alpha) \geq \min(n, \frac{1}{2}n + \frac{1}{3}s) \geq \frac{2}{3}n \). If \( G \) is 2-connected and \( s \geq n + 2 \), then also \( c \geq \min(n, n + \frac{1}{3}s - \alpha) \), generalizing a result of Bondy and one of Nash-Williams. Finally, if \( G \) is 2-tough and \( s > n \), then \( G \) is hamiltonian.

1. Terminology

We consider only finite undirected graphs without loops or multiple edges. Our terminology is standard except as indicated. A good reference for any undefined terms is [7]. We need a few definitions and some convenient notation. Let \( \omega(G) \) denote the number of components of a graph \( G \). As introduced by Chvátal [10], a graph \( G \) is \( t \)-tough if \( |S| \geq t\omega(G - S) \) for any subset \( S \) of the vertex set \( V \) of \( G \) with \( \omega(G - S) > 1 \). The toughness of \( G \), denoted \( t(G) \), is the maximum value of \( t \) for which \( G \) is \( t \)-tough \((t(K_n) = \infty \) for all \( n \geq 1 \)). We will denote by \( \alpha \) the cardinality of a maximum set of independent vertices of \( G \). A cycle \( C \) of \( G \) is a dominating cycle if every edge of \( G \) has at least one of its vertices on \( C \). A cycle \( C \) of \( G \) is a dominating cycle if every edge of \( G \) has at least one of its vertices on \( C \). If \( C \) is a cycle of \( G \) we denote by \( C \) the cycle \( C \) with a given orientation. If \( u, v \in V(C) \), then \( uCv \) denotes the consecutive vertices on \( C \) from \( u \) to \( v \) in the direction specified by \( C \). The same vertices, in reverse order, are given by \( vCu \). We use \( u^* \)

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to denote the successor of \( u \) on \( C \) and \( u^- \) to denote its predecessor. If \( v \in V \) then \( N(v) \) is the set of all vertices in \( V \) adjacent to \( v \). If \( A \subseteq V(C) \), then \( A^+ = \{ v^+ \mid v \in A \} \). The set \( A^- \) is analogously defined.

2. Results

Our work was motivated by two recent conjectures of Ainouche and Christofides [1].

**Conjecture 1.** Let \( G \) be a 1-tough graph on \( n \geq 3 \) vertices such that \( d(x) + d(y) + d(z) \geq n \) for all independent sets of vertices \( x, y, z \). Then \( G \) is hamiltonian.

**Conjecture 2.** Let \( G \) be a 1-tough graph on \( n \geq 3 \) vertices such that \( d(x) + d(y) \geq q \) for all distinct nonadjacent vertices \( x, y \). Then \( G \) has a cycle of length at least \( \min(n, q + 2) \).

The following class of graphs, given in [1], shows that each conjecture, if true, would be best possible. For \( n = 3r + 1 \geq 7 \), construct the graph \( H_n \) from \( 3K_r + K_1 \) by choosing one vertex from each copy of \( K_r \), say \( u, v \) and \( w \), and adding the edges \( uv, uw \) and \( vw \). The graph \( H_n \) is 1-tough on \( n = 3r + 1 \) vertices, satisfies \( d(x) + d(y) \geq 2r \) for all distinct nonadjacent vertices \( x, y \) and also satisfies \( d(x) + d(y) + d(z) \geq n - 1 \) for all sets of independent vertices \( x, y, z \). Yet a longest cycle in \( H_n \) has length only \( 2r + 2 \).

Conjecture 2 was recently proven to be true [5]. For convenience we state it as a theorem below.

**Theorem 1.** Let \( G \) be a 1-tough graph on \( n \geq 3 \) vertices such that \( d(x) + d(y) \geq q \) for all distinct nonadjacent vertices \( x, y \). Then \( G \) has a cycle of length at least \( \min(n, q + 2) \).

Conjecture 1, however, is false as indicated by the following class of graphs. For odd \( n \geq 15 \), construct the graph \( G_n \) from

\[ \tilde{K}_{\frac{n}{3}(n-1)} \cup K_m \cup K_{\frac{n}{3}(n+1)-m}, \text{ where } \frac{1}{3}n \leq m \leq \frac{1}{3}(n-5), \]

by joining every vertex in \( K_m \) to all other vertices and by adding a matching between all vertices in \( K_{\frac{n}{3}(n+1)-m} \) and \( \frac{1}{3}(n+1) - m \) vertices in \( \tilde{K}_{\frac{n}{3}(n-1)} \). Note that \( G_n \) has minimum degree \( m \). It is easily seen that \( G_n \) is 1-tough but not hamiltonian. If \( \frac{1}{3}(n+1) - m \) is odd (even) then a longest cycle in \( G_n \) has length \( \frac{1}{3}(3n+1) + \frac{1}{3}m \) \( (\frac{1}{3}(3n+3) + \frac{1}{3}m) \). A variation of the graph \( G_n \), with \( K_m \) replaced by \( K_{\frac{1}{3}(n-5)} \), has already appeared in the literature [8, 13]. It can be used to show that the following theorem of Jung [11] is best possible.
Theorem 2. Let $G$ be a 1-tough graph on $n \geq 11$ vertices such that $d(x) + d(y) > n - 4$ for all distinct nonadjacent vertices $x, y$. Then $G$ is hamiltonian.

Although Conjecture 1 is false its hypothesis justifies the following conclusion, which follows immediately from Theorem 9 below.

Theorem 3. Let $G$ be a 1-tough graph on $n \geq 3$ vertices such that $d(x) + d(y) + d(z) > s > n$ for all independent sets of vertices $x, y, z$. Then $G$ contains a cycle of length at least $\min(n, \frac{1}{2}n + \frac{1}{3}s)$.

Corollary 4. Let $G$ be a 1-tough graph on $n \geq 3$ vertices with minimum degree $\delta \geq \frac{1}{3}n$. Then $G$ contains a cycle of length at least $\frac{1}{2}n$.

Theorem 3 is a little surprising in the following sense. If, for example, $\delta = \frac{1}{4}n$ we conclude from Theorem 1 (which is “best possible”) that $G$ has a cycle of length at least $\frac{1}{2}n + 2$. From Corollary 4 we deduce that $G$ has a cycle of length at least $\frac{1}{3}n$. Apparently for 1-tough graphs $G$, as $\delta$ crosses the threshold of $\frac{1}{3}n$, the length of a longest cycle that is forced in $G$ jumps from $\frac{1}{2}n + 2$ to at least $\frac{1}{3}n$. If Conjecture 3, mentioned in Section 4, is true then $G$ is forced to have a cycle of length at least $\frac{1}{12}(11n + 3)$.

The proof of Theorem 3, as well as the proofs of our other results, depends on the intermediate conclusion that every longest cycle in $G$ is a dominating cycle. This is established by our next theorem, whose proof is given in Section 3.

Theorem 5. Let $G$ be a 1-tough graph on $n$ vertices such that $d(x) + d(y) + d(z) \geq n$ for all independent sets of vertices $x, y, z$. Then every longest cycle in $G$ is a dominating cycle.

Theorem 5 generalizes the following theorem of Bigalke and Jung [8].

Theorem 6. Let $G$ be a 1-tough graph on $n$ vertices with $\delta \geq \frac{1}{2}n$. Then every longest cycle in $G$ is a dominating cycle.

The graphs $H_n$ with $n \geq 10$ show that both Theorem 5 and Theorem 6 are best possible. We remark that for $n \geq 5$ the condition in Theorem 5 that $G$ be 1-tough can in fact be replaced by the weaker condition that the deletion of any nonempty proper subset $S$ of $V$ yields a graph with at most $|S|$ nontrivial components. This weaker condition is necessary for a graph to have a dominating cycle [14]. Thus, if the condition that $G$ be 1-tough is replaced by the above weaker condition, we obtain a result that also generalizes the following theorem of Bondy [9].

Theorem 7. Let $G$ be a 2-connected graph on $n$ vertices such that $d(x) + d(y) + d(z) \geq n + 2$ for all independent sets of vertices $x, y, z$. Then every longest cycle in $G$ is a dominating cycle.
The next key lemma, proved in Section 3, is the basis for many of the results that follow.

**Lemma 8.** Let \( G \) be a graph on \( n \) vertices such that \( \delta \geq 2 \) and \( d(x) + d(y) + d(z) \geq n \) for all independent sets of vertices \( x, y, z \). Let \( G \) contain a longest cycle \( C \) which is a dominating cycle. If \( v_0 \in V - V(C) \) and \( A = N(v_0) \), then \( (V - V(C)) \cup A^+ \) is an independent set of vertices.

Lemma 8 has a number of applications. The next two theorems are obtained by combining Lemma 8 with Theorems 5 and 7, respectively. A proof of Theorem 10 and an outline proof of Theorem 9 are given in Section 3.

**Theorem 9.** Let \( G \) be a 1-tough graph on \( n \geq 3 \) vertices such that \( d(x) + d(y) + d(z) \geq s \geq n \) for all independent sets of vertices \( x, y, z \). Then \( G \) contains a cycle of length at least
\[
\min(n, n + 3s - \alpha).
\]

Since \( \alpha \leq \frac{1}{3}n \) for 1-tough graphs, Theorem 3 follows immediately from Theorem 9.

**Theorem 10.** Let \( G \) be a 2-connected graph on \( n \) vertices such that \( d(x) + d(y) + d(z) \geq s \geq n + 2 \) for all independent sets of vertices \( x, y, z \). Then \( G \) contains a cycle of length at least
\[
\min(n, n + 3s - \alpha).
\]

Theorem 10 is best possible in two different ways. The graph \( K_{p,q} \), with \( 2 \leq p \leq q \leq 2p - 2 \) and \( q \geq 3 \) has a longest cycle of length exactly \( n + \frac{1}{3}s - \alpha = 2p \).
The graph \( H = 3K_1 + 2K_1 \) has \( d(x) + d(y) + d(z) \geq s \geq n + 1 \) for all independent sets of vertices \( x, y, z \) and has a longest cycle of length \( 2t + 2 \), which is less than \( \min(n, n + \frac{1}{3}(n + 1) - 3) = n \) (\( t \geq 2 \)).

It is easily seen that if \( \alpha \geq 3 \), the hypothesis of Theorem 10 implies \( \alpha \leq n - \frac{1}{3}s \).
Hence Theorem 10 generalizes the following result of Bondy [9].

**Theorem 11.** Let \( G \) be a 2-connected graph on \( n \) vertices such that \( d(x) + d(y) + d(z) \geq s \geq n + 2 \) for all independent sets of vertices \( x, y, z \). Then \( G \) has a cycle of length at least \( \min(n, \frac{3}{2}s) \).

Theorem 10 also generalizes the following result of Nash-Williams [12].

**Theorem 12.** Let \( G \) be a 2-connected graph on \( n \) vertices with \( \delta \geq \max(\frac{1}{3}(n + 2), \alpha) \). Then \( G \) is hamiltonian.

Bigalke and Jung [8] also generalized Theorem 12.

**Theorem 13.** Let \( G \) be a 1-tough graph on \( n \geq 3 \) vertices with \( \delta \geq \max(\frac{1}{3}n, \alpha - 1) \). Then \( G \) is hamiltonian.
Note that Theorem 9 is only a partial generalization of Theorem 13. Theorem 9 allows us to draw conclusions concerning long, but not necessarily hamiltonian, cycles in $G$. However if $\delta = \alpha - 1 \geq \frac{1}{n}$ we cannot conclude from Theorem 9 that $G$ is hamiltonian. It is possible, however, to combine Lemma 8 with a suitably modified proof of Theorem 13 to obtain the following.

**Theorem 14.** Let $G$ be a 1-tough graph on $n \geq 3$ vertices with $\delta \geq \frac{3}{2}n$. Then $G$ contains a cycle of length at least $\min(n, n + \delta - \alpha + 1)$.

The proof of Theorem 14 is lengthy and will appear elsewhere [6]. Note that this result yields a slight strengthening of Corollary 4. We can actually conclude that $G$ has a cycle of length at least $\frac{5}{3}n + 1$.

Theorem 14 completely generalizes Theorem 13 and, like Theorem 10, is best possible in two ways. If $m = \frac{1}{2}(n - 5)$, the graph $G_n$ has $n + \delta - \alpha + 1 = n - 1$ and $G_n$ is not hamiltonian; in view of Conjecture 3 in Section 4, however, we do not believe that Theorem 14 is best possible for values of $\delta$ less than $\frac{3}{4}(n - 5)$. The graph $H_n$ has $\delta \geq \frac{3}{4}(n - 1)$ and has a longest cycle of length $\frac{5}{4}(n - 1) + 2$, less than $\min(n, n + \delta - \alpha + 1) = \min(n, n + \frac{3}{4}(n - 1) - 2) = n$.

We now turn our attention to graphs with $t(G) = \tau \geq 2$. The inequality $\alpha \leq \frac{3}{2}n$, used to prove Theorem 3 from Theorem 9, suggests that our conclusions can be strengthened if $\tau > 1$. Since obviously $\alpha \leq n/(\tau + 1)$, Theorem 9 immediately implies our next result.

**Corollary 15.** Let $G$ be a graph on $n \geq 3$ vertices with $t(G) = \tau \geq 1$. If $d(x) + d(y) + d(z) > s \geq n$ for all independent sets of vertices $x, y, z$, then $G$ has a cycle of length at least $\min(n, n/(\tau + 1) + \frac{1}{3}s)$.

A special case of Corollary 15 may be a first small step toward proving the well-known conjecture that 2-tough graphs are hamiltonian [10].

**Corollary 16.** Let $G$ be a 2-tough graph on $n \geq 3$ vertices. If $d(x) + d(y) + d(z) \geq n$ for all independent sets of vertices $x, y, z$, then $G$ is hamiltonian.

3. **Proofs**

**Proof of Theorem 5.** Let $C$ be a longest cycle of $G$ with a fixed orientation. Assume $C$ is not a dominating cycle of $G$. Then $G - V(C)$ has a nontrivial component $H$. Set $A = \bigcup_{v \in V(H)} N(v) - V(H)$ and let $u_1, \ldots, u_k$ be the elements of $A$, occurring on $\tilde{C}$ in consecutive order. Since $G$ is 1-tough, $G$ is 2-connected in particular, so $k \geq 2$. For $i = 1, \ldots, k$, set $u_i = u_i^+$ and $w_i = u_i^+$ (indices modulo $k$). Since $C$ is a longest cycle, $u_i \neq u_{i+1}$ ($i = 1, \ldots, k$). If $v$ is a vertex in $u_i\tilde{C}w_i$ such that $u_iv^+ \in E$, then $v$ will be called an $i$-vertex; in particular, $u_i$ is an
i-vertex \((i = 1, \ldots, k)\). Since \(G\) is 2-connected, there exist integers \(r\) and \(s\) with \(1 < r < s < k\) such that \(v_i\) and \(v_s\) are connected by a path \(P_{r,s}\) of length at least 3 with all internal vertices in \(H\). We make a number of observations.

(1) If \(x_i\) is an \(r\)-vertex and \(x_s\) an \(s\)-vertex, then there exists no \((x_i, x_s)\)-path which is internally disjoint from \(C\); in particular, \(x_i x_s \notin E\).

Assuming the contrary to (1), let \(P\) be an \((x_i, x_s)\)-path, internally disjoint from \(C\). Since \(x_i, x_s \notin A\), we have \(V(P) \cap V(H) = \emptyset\). Now \(v_i P_{r,s} v_s \hat{C} x_i^r x_s P x_i \hat{C} u_i x_i^r \hat{C} v_i\), denoting the cycle having as consecutive vertices the vertices of \(P_{r,s}\), \(v_i \hat{C} x_i^r\), \(u_i \hat{C} u_i\), and \(x_i^r \hat{C} v_i\), respectively, has length at least \(|V(C)| + 2\). This contradiction proves (1).

(2) If \(x_i\) is an \(r\)-vertex and \(x_s\) an \(s\)-vertex, then \(x_i u_i^r, u_i^r x_s \notin E\). If the contrary to (2) is assumed, a cycle longer than \(C\) can be indicated as in (1). The only difference is that now this cycle has length at least \(|V(C)| + 1\) instead of \(|V(C)| + 2\), since it omits the vertex \(u_i\) or \(u_s\) of \(C\).

(3) Let \(x_i\) be an \(r\)-vertex and \(x_s\) an \(s\)-vertex. If \(v \in x_i \hat{C} x_i^r, x_i v \in E\) and \(x_i v^+ \notin E\). To prove (3) assume, e.g. \(v \in x_i \hat{C} x_i^r, x_i v \in E\) and \(x_i v^+ \in E\). By (1), \(v \neq x_i\) and \(v^+ \neq u_i, x_i\) (since \(u_i\) is also an \(s\)-vertex). If \(v^+ \in x_i^r \hat{C} v_i\), then the cycle \(v_i P_{r,s} v_s \hat{C} v^+ x_i \hat{C} u_i x_i^r \hat{C} v_i\) has length at least \(|V(C)| + 2\), a contradiction. If \(v^+ \in u_i^r \hat{C} x_i^r\), then the cycle \(v_i P_{r,s} v_s \hat{C} x_i^r \hat{C} v^+ x_i \hat{C} u_i x_i^r \hat{C} v_i\) yields a similar contradiction.

(4) Let \(x_i\) be an \(r\)-vertex and \(x_s\) an \(s\)-vertex. If \(v \in x_i \hat{C} x_i^r\) and \(x_i v \in E - E(C)\), then \(x_i v^{++} \notin E\). Similarly, if \(v \in x_i \hat{C} x_i^r\) and \(x_i v \in E - E(C)\), then \(x_i v^{++} \notin E\).

The proof of (4) is similar to the proof of (3), except now the longer cycle has length \(|V(C)| + 1\) instead of \(|V(C)| + 2\).

Using observations (1) through (4) we now derive an upper bound for \(d(u_0) + d(x_r) + d(x_s)\), where \(x_r\) is an \(r\)-vertex, \(x_s\) an \(s\)-vertex and \(u_0\) an arbitrary vertex of \(H\). Define

\[
R_1(x_i) = \{v \in x_i \hat{C} x_i^r \mid x_i v^+ \in E\},
\]

\[
S_1(x_i) = \{v \in x_i \hat{C} x_i^r \mid x_i v \in E\}.
\]

\[
R_2(x_i) = \{v \in x_i \hat{C} x_i^r \mid x_i v \in E\},
\]

\[
S_2(x_i) = \{v \in x_i \hat{C} x_i^r \mid x_i v^+ \in E\}.
\]

\[
R_3(x_i) = \{v \in V - V(C) \mid x_i v \in E\},
\]

\[
S_3(x_i) = \{v \in V - V(C) \mid x_i v \in E\},
\]

\[
B(x_i, x_s) = R_1(x_i) \cup S_1(x_i) \cup R_2(x_i) \cup S_2(x_i).
\]

By (3), \(R_1(x_i) \cap S_1(x_s) = R_2(x_i) \cap S_2(x_s) = \emptyset\). By (1) and the fact that \(x_i, x_s \notin A\),
Long cycles in graphs with large degree sums

$R_s(x_s) \cap S_t(x_t) = V(H) \cap (R_s(x_s) \cup S_t(x_t)) = \emptyset$. Furthermore, for $i \in \{1, \ldots, k\} - \{r, s\}$, either $u_i$ or $v_i$ is not in $B(x_r, x_s)$. To see this, suppose e.g. $u_i \in R_i(x_i) \cup S_i(x_i)$. Then $x_i u_i E \in E$, since the assumption that $x_i u_i E \in E$ implies the existence of a cycle longer than $C$, containing the vertices of a $(u_i, v_i)$-path of length at least 2 with all internal vertices in $H$ (cf. (1)). But then, by (4) with $v = v_i$, $x_i v_i E \in E$. Also, like $x_i u_i$, $x_i u_i E \in E$. It follows that $v_i \notin R_i(x_i) \cup S_i(x_i)$.

We conclude that

$$d(u_0) + d(x_r) + d(x_s) = d(u_0) + |R_1(x_1)| + |R_2(x_2)| + |R_3(x_3)| + |S_1(x_1)| + |S_2(x_2)| + |S_3(x_3)| \leq (k + |V(H)| - 1) + (|V(C)| - (k - 2)) + |R_3(x_3)| + |S_3(x_3)|$$

$$\leq (k + |V(H)| - 1) + (|V(C)| - (k - 2)) + (|V| - |V(C)|) = n + 1.$$

On the other hand, since $(u_0, x_r, x_s)$ is an independent set,

$$d(u_0) + d(x_r) + d(x_s) = n.$$

It follows that $u_0$, and hence every vertex of $H$, is adjacent to all but at most one of the vertices in $A$. This implies the existence of a $(u_i, v_i)$-path $P_{i,j}$ of length at least 3 with all internal vertices in $H$ for all $i, j \in \{1, \ldots, k\}$ with $i \neq j$. A number of conclusions now follow. We first note that (1) through (6) actually hold for arbitrary $r$ and $s$ with $1 \leq r < s \leq k$. Furthermore, $u_i \neq w_i$ ($i = 1, \ldots, k$). Also, it follows immediately from (2) that for $1 \leq r < s \leq k$ and $i \in \{1, \ldots, k\} - \{r, s\}$, $u_i$ (instead of $u_r$ or $v_r$) is not in $B(x_r, x_s)$, where $x_r$ is an $r$-vertex and $x_s$ an $s$-vertex.

From (5) and (6) we also deduce the following:

(7) If $x_r$ is an $r$-vertex and $x_s$ an $s$-vertex, then at most one vertex of $V(C) - \{u_i \mid i \in \{1, \ldots, k\}, i \neq r, s\}$ is not in $B(x_r, x_s)$ ($1 \leq r < s \leq k$).

The next three observations, where $s \in \{1, \ldots, k\}$, will facilitate the remainder of the proof.

(8) If $v \in u_{s+1, \tilde{C}u_s}$ and $u_s v \in E$, then $w_s v \notin E$.

Assuming the contrary, the cycle $w_s v \notin \tilde{C}u_{s+1, P_{s+1, u_s} \tilde{C}u_{s+1, \tilde{C}w_s}}$ has length at least $|V(C)| + 2$, a contradiction.

(9) If $v \in u_{s+1, \tilde{C}u_s}$ and $u_s v \in E$, then $w_s v \notin \tilde{w_s} v \notin E$.

The proof of (9) is similar to the proof of (8).

(10) If $v \in u_{s+1, \tilde{C}u_s}$ and $u_s v \in E$, then $w, v \notin \tilde{w_s} v \notin E$.

Assuming the contrary, the cycle $w, v \notin \tilde{C}u_{s+1, u_s \tilde{C}w_s}$, where $w = w_s$ or $w = w^\tau_s$, yields a contradiction.

Using observations (1) through (10) we now derive contradictions in all cases distinguished below. If $v \in V$, then by $N'(v)$ we denote the set of vertices $x$ such that there is a $(v, x)$-path of length at least 1 with all internal vertices in $V - V(C)$. In particular, $N(v) \subseteq N'(v)$.
Case 1. For all \( i \in \{1, \ldots, k\} \),
\[
N'(u_i) \cap V(C) \subseteq \{u_i \} \cup A \quad \text{and} \quad N'(w_i) \cap V(C) \subseteq \{u_i \} \cup A.
\]
Suppose there exist integers \( r, s \) and vertices \( x, y \) such that \( 1 \leq r < s \leq k \),
\( x \in u_r^+ \), \( y \in u_s^+ \), and \( xy \in E \). Since by the hypotheses of Case 1 \( u_x, u_y \notin E \), either \( u_x y^+ \) or \( u_y y^+ \) is in \( E \), otherwise \( x, y \notin B(u_x, u_y) \), contradicting (7).
Assume, without loss of generality, that \( u_x y^+ \in E \), i.e. \( x \) is an \( r \)-vertex. By (3) and (4), \( u_x y^+ \notin E \), and hence \( y, y^+ \notin B(u_x, u_y) \). This contradiction with (7) shows that in this case no edge, and similarly no path with all internal vertices in \( V - V(C) \), joins two vertices in different sets of the collection \( \{w_i \mid 1 \leq i \leq k\} \).
But then \( \omega(G - A) \geq |A| + 1 \), contradicting the fact that \( G \) is 1-tough.

Case 2. For some \( i \in \{1, \ldots, k\} \),
\[
N'(u_i) \cap V(C) \notin \{u_i \} \cup A \quad \text{or} \quad N'(w_i) \cap V(C) \notin \{w_i \} \cup A.
\]
Assume e.g. \( y \in N'(u_i) \), where \( y \in u_i \), \( r < s \), and \( |u_i \cdot y| \) is minimum. For convenience we also assume \( u_x, y \in E \); in case \( u_x \) and \( y \) are connected by a path of length at least 2 with all internal vertices in \( V - V(C) \), completely analogous arguments apply, since the path must be disjoint from \( H \). Note that by (1) and (2), \( y \neq u_x, u_x^+ \). Let \( x \) be the \( r \)-vertex in \( u_i \cdot y^+ \) that minimizes \( |x, y^+| \); possibly \( x = u_x \). Either \( x^+ = y^+ \) or \( x^+ = y \), otherwise \( x^+, x^+ y^+ \notin B(u_x, u_y) \), contradicting (7). We distinguish two subcases.

Case 2.1. \( x^+ = y^+ \).
Then \( u_x, y \notin E \). By (4), \( u_x y^+ \notin E \) and by (8) and (10), \( w_x y^+, w_y y^+ \notin E \). Hence \( w_x \) is not an \( s \)-vertex, otherwise \( y, y^+ \notin B(w_x, w_y) \), contradicting (7). Thus \( u_x, w_x \notin E \).
But then \( u_x, w_x \in E \), otherwise \( y, y^+ \notin B(u_x, u_y) \). Now \( x, w_x \notin E \), otherwise the cycle \( x, w_x, u_x, y, w_x, x \) is longer than \( C \). Also, by (9), \( x, w_x \notin E \). It follows that \( w_x, w_x^+ \notin B(x, u_x) \), a contradiction.

Case 2.2. \( x^+ = y \).

Case 2.2.1. \( u_x w_x^+ \notin E \).
By (8) and (9), \( x, w_x, x, w_x \notin E \). Thus \( u_x, w_x \in E \), otherwise \( w_x^-, w_x \notin B(x, u_x) \). In other words, \( w_x \) is an \( s \)-vertex. By (3) and (4), \( x, y^+, x, y^+ \notin E \). Recalling that \( x, w \notin E \) by (8), we conclude that \( u_x, y \notin E \), since otherwise \( y, y^+ \notin B(x, u_x) \). Now by (9) and (10), \( w_x y^+, w_x y^+ \notin E \). It follows that \( y, y^+ \notin B(x, w_x^+) \), a contradiction.

Case 2.2.2. \( u_x w_x^+ \in E \).
Then \( w_x \) is an \( s \)-vertex. Recall that, by (3) and (4), \( x, y^+, x, y^+ \notin E \). By (10), \( w_x y^+, w_x y^+ \notin E \). Hence \( w_x y^+ \in E \), otherwise \( y, y^+ \notin B(x, w_x) \). Now \( x, w \notin E \), otherwise
the cycle $x_i w_i \bar{C} y w y \bar{C} u_{s+1} P_{s+2, s+1} v_{s+1} \bar{C} x_r$ is longer than $C$. Also, $w_w \notin E$, otherwise the cycle $w_w \bar{C} u_{s+1} P_{s+2, s+1} v_{s+1} \bar{C} w_w$ is longer than $C$. It follows that $y_r^+, w_r^+ \notin B(x_r, w_r)$. Hence, by (7), $y_r^+ = w_r^+$. We now show that

(11) $u_r$ is adjacent to all vertices in $u_r^+ \bar{C} u_{s+1}$.

Assuming the contrary, let $v$ be the vertex in $u_r^+ \bar{C} u_{s+1}$ such that $u_r v \notin E$ and $|v u_{s+1}|$ is minimum. Then $v \in u_r^+ \bar{C} w_r$ and $u_r v^+ \in E$. By (4), $u_r v^+ \notin E$ and by (8), $u_r y_r^+ \notin E$. Hence $v^+, y_r^+ \notin B(u_r, u_r)$. This contradiction proves (11). Similarly we have

(12) $u_i$ is adjacent to all vertices in $u_i^+ \bar{C} y_i$.

By (9), $u_i y_i^+ \notin E$. Recalling that $u_i y_i^+ \notin E$ we now note that for all $i \in \{1, \ldots, k\}$ the assumption $u_i y_i^+ \in E$ or $u_i y_i^+ \notin E$ leads to a contradiction by applying the above arguments with $i$ in place of $s$. Thus $u_i y_i^+, u_i y_i^+ \notin E$ for all $i \in \{1, \ldots, k\}$ except $r$. By (3) and (4), $u_i y_i^+, u_i y_i^+ \notin E$. Hence $u_i y_r \in E$, for otherwise $y_r, y_r^+ \notin B(u_r, u_r) (i \in \{1, \ldots, k\})$. It now follows that (11) remains true if $s$ is replaced by $i (i \in \{1, \ldots, k\} \setminus \{r\})$. By (7), $B(u_r, u_r) = V(C) - \{(y_r^+ \cup u_i) | i \in \{1, \ldots, k\} \setminus \{r\}\}$, implying that

$$\begin{align*}
N(u_r) \cap V(C) &= u_r^+ \bar{C} y_r \cup A \\
N(u_i) \cap V(C) &= u_i^+ \bar{C} u_{s+1} \cup A (i \in \{1, \ldots, k\} \setminus \{r\}).
\end{align*}$$

In particular, every vertex of $V(C) - \{(A \cup \{y\})\}$ is an $i$-vertex for some $i \in \{1, \ldots, k\}$. Using (1), (3) and (4) we conclude that no edge, and similarly no path with all internal vertices in $V - V(C)$, joins two vertices in different sets of the collection $\{u_i \bar{C} w_i | 1 \leq i \leq k, i \neq r\} \cup \{u_i \bar{C} y_r^+ \cup \{y_r^+, y_r^+\}\}$. But then $\omega(G - \{(A \cup \{y\})\}) \geq |A \cup \{y\}| + 1$, our final contradiction. □

**Proof of Lemma 8.** By assumption $V - V(C)$ is an independent set and a standard argument shows that $A^+$ is an independent set. Hence it suffices to show that no vertex in $V - V(C)$ is adjacent to a vertex in $A^+$. Let $A = N(v_0)$ consist of distinct vertices $x_1, x_2, \ldots, x_k (k \geq 2)$ on $C$ such that $x_{i+1} x_i \bar{C} x_{i+2}$, $1 \leq i \leq k$ (indices mod $k$). Clearly $v_0$ is not adjacent to any vertex in $A^+$, i.e. $A \cap A^+ = \emptyset$. Suppose $v_1 \in V - V(C)$ and $v_1 x_i^+ \notin E$. Consider the following sets of vertices.

$$\begin{align*}
A_1 &= \{v \in x_1^+ \bar{C} x_k | v v^+ \in E\}, \\
A_2 &= \{v \in x_1^+ \bar{C} x_k | x_k^+ v \in E\}, \\
B_1 &= \{v \in x_1^+ \bar{C} x_1 | v_1 v \in E\}, \\
B_2 &= \{v \in x_1^+ \bar{C} x_1 | x_1^+ v^+ \in E\}, \\
D &= \{v \in V - V(C) | v^+ v \in E\}.
\end{align*}$$

Observe that for each $i, 1 \leq i \leq k - 1$, $x_i^+ \notin A_1$. This is clear if $i = 1$. Assuming $x_i^+ \in A_1$ for some $i \in \{2, \ldots, k - 1\}$, the cycle $v_1 x_i^+ \bar{C} x_1 v_0 x_i \bar{C} x_i v_1$ is a longer cycle than $C$, a contradiction. Since $A^+$ is an independent set, $x_i^+ \notin A_2$ for
It is easy to see that $x_1 \notin B_1$ and $v_0, v_1 \notin D$. But
$A_1 \cap A_2 = \emptyset$, for if $w \in A_1 \cap A_2$, then $x_1^+ v_1 w^+ \tilde{C}_k v_0 x_1^+ \tilde{C}_k w^+ \tilde{C}_k$ is a longer cycle
than $C$, a contradiction. Similarly $B_1 \cap B_2 = \emptyset$. Thus $|D| \leq n - |V(C)| - 2$ and
$|A_1| + |A_2| + |B_1| + |B_2| = |V(C)| - k$. Since $N(v_1) = A_1^+ \cup \{x_1^+\} \cup B_1$ and $v_1 x_k^+ \notin E$, $d(v_1) = |A_1| + |B_1| + 1$. Also $d(x_k^+) = |A_2| + |B_2| + |D|$. Thus
$$d(v_0) + d(v_1) + d(x_k^+) = k + |A_1| + |A_2| + |B_1| + |B_2| + |D| + 1 \leq n - 1.$$
However, $v_0, v_1$ and $x_k^+$ are independent, thus contradicting our assumption and
proving the lemma. \(\square\)

Outline proof of Theorem 9. Let $C$ be a longest cycle in $G$. By Theorem 5, $C$ is a
dominating cycle. Assume $C$ is chosen such that $\text{max} \{d(v) \mid v \in V - V(C)\}$ is
maximum. If $V - V(C) = \emptyset$ there is nothing to prove. Thus we assume $V - V(C) = \{v_0, v_1, \ldots, v_i\}$, $d(v_0) \geq d(v_1) \geq \cdots \geq d(v_i)$. Let $A = N(v_0) = \{x_1, x_2, \ldots, x_k\}$, where $k \geq 2$ and $x_{i+1} \in x_i \tilde{C}_k x_{i+2}$, $1 \leq i < k$ (indices mod $k$). From
Lemma 8 we have $|V - V(C)| + |A^+| \leq \alpha$. Hence $|V(C)| \geq n + |A^+| - \alpha = n + d(v_0) - \alpha$. Thus it suffices to show that $d(v_0) \geq \frac{3s}{2}$. This is clearly true if $t \geq 2$.

Suppose $t = 1$, $d(v_0) < \frac{3s}{2}$ and consider $x_1^+$. Suppose $x_1^+ x_j^+ \in E$, where $2 \leq j \leq k$. Then the cycle $C' = x_1^+ x_j^+ \tilde{C}_k x_0 x_0 \tilde{C}_k x_1^+$ has $|V(C')| = |V(C)|$ but includes $v_0$ and
omits $x_i^+$. However, $v_0, v_1$ and $x_i^+$ are independent and thus $s \leq d(v_0) + d(v_1) + d(x_i^+)$. This implies $d(x_i^+) > \frac{3s}{2} > d(v_0)$, contradicting the choice of $C$. Thus we may assume $x_1^+ x_j^+ \notin E$ for $2 \leq j \leq k$. Since $x_1^+$ is not adjacent to any vertex in $A^+$ and $A^+ \cap A^{++} = \emptyset$, $d(x_1^+) \leq |V(C)| - 2d(v_0) + 1$. But then $n \leq d(x_1^+) + d(v_0) + d(v_1) = |V(C)| + 1 \leq n - 1$, a contradiction.

Finally, the proof for $t = 0$ is modelled along the lines of the proof of Theorem 5. Whenever a contradiction is obtained in the proof of Theorem 5 by finding a
longer cycle, we now find a contradiction either in the same way, or by finding a
cycle $C'$ such that $|V(C')| = n - 1$ and $u_0 \in V - V(C')$ has $d(u_0) > d(v_0)$. The
argument, although quite lengthy and involved, is tedious and is thus omitted
here. The full proof can be found in the appendix of $[4]$.

Proof of Theorem 10. If $G$ is hamiltonian we are done. Otherwise, as in the
proof of Theorem 9, let $C$ be a longest cycle in $G$ such that $\text{max} \{d(v) \mid v \in V - V(C)\}$ is maximum. By Theorem 7, $C$ is a dominating cycle. Let $v_0$ be a vertex in $V - V(C)$ having maximum degree among all vertices of $V - V(C)$ and set $A = N(v_0) = \{x_1, x_2, \ldots, x_k\}$, where $k \geq 2$ and $x_{i+1} \in x_i \tilde{C}_k x_{i+2}$, $1 \leq i < k$ (indices mod $k$). As in the proof of Theorem 9, Lemma 8 implies that $|V(C)| \geq n + d(v_0) - \alpha$, so that it suffices to show that $d(v_0) \geq \frac{3s}{2}$. Suppose $d(v_0) < \frac{3s}{2}$ and
assume, without loss of generality, that $\min \{d(x_i^+) \mid 1 \leq i \leq k\} = d(x_i^+)$. Then for $i = 2, \ldots, k$ we have $d(x_i^+) \geq \frac{3s}{2}$, since $d(v_0) + d(x_i^+) + d(x_i^+) \geq s$. It follows that $x_i^+ x_i^+ \notin E$ for $i = 3, \ldots, k$, otherwise a cycle $C'$ with $|V(C')| = |V(C)|$ exists that includes $v_0$ and omits $x_i^+$, contradicting the choice of $C$. Thus, defining $B(x_1^+, x_2^+)$.  

\(\square\)
as in the proof of Theorem 5, we have that $x_1^+, \ldots, x_k^+ \notin B(x_1^+, x_2^+)$. But then
\[
d(v_v) + d(x_v^+) + d(x_v^+) = k + |B(x_v^+, x_v^+)| = k + (|V(C)| - (k - 2))
\]
\[= |V(C)| + 2 \leq n + 1 < s,
\]
a contradiction. \(\square\)

4. Conjectures

We begin with the following conjecture.

**Conjecture 3.** Let $G$ be a 1-tough graph on $n \geq 3$ vertices such that $d(x) + d(y) + d(z) \geq s \geq n$ for all independent sets of vertices $x, y, z$. Then $G$ contains a cycle of length at least $\min(n, \frac{1}{2}(3n + 1) + \frac{1}{2}s)$.

The graphs $G_n$ show that Conjecture 3, if true, is best possible. Conjecture 3 would also imply the following generalization of Jung's Theorem (Theorem 2).

**Conjecture 4.** Let $G$ be a 1-tough graph on $n \geq 13$ vertices such that $d(x) + d(y) + d(z) \geq \frac{1}{2}(3n - 14)$ for all independent sets of vertices $x, y, z$. Then $G$ is hamiltonian.

The graph obtained from $H_{13}$ by deleting a vertex of degree 4 shows that the requirement that $n \geq 13$ cannot be released.

We close by noting that an application of Theorem 7 and Lemma 8 leads to a simple proof of Jung's Theorem for graphs on at least 16 vertices. By applying Theorem 5 instead of Theorem 7 it is possible to obtain a new proof of Jung's entire theorem ($n \geq 11$). Details will appear elsewhere [2, 3].

References

70

D. Bauer et al.


