Adaptive LQ Control: Conflict Between Identification and Control

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ABSTRACT

We consider one of the fundamental limitations of indirect adaptive control based on the minimization of a quadratic cost criterion and the certainty equivalence principle. We show that the interaction between (closed-loop) identification and optimal control is conflictive in the sense that almost all possible limits of the sequence of parameter estimates induce suboptimal behavior of the adaptively controlled system.

1. INTRODUCTION

Most of the literature on adaptive control is devoted to the study and analysis of one or several specific algorithms. In this paper we do not refer to specific algorithms, but a study is made of one of the fundamental limitations of a class of adaptive-control algorithms.

In indirect adaptive-control algorithms estimates of the system parameters are made on the basis of the observed behavior of the (adaptively controlled) system. The controls that are applied to the system are based on the estimates and the external signals. Hence identification and control of the system take place simultaneously. As a result the identification part of the adaptive controller receives information about the closed-loop system rather than the open-loop system. This phenomenon is known as closed-loop identification. In the context of adaptive control it was first studied in [1]. There the adaptive control of a finite-state Markov chain was considered. It was proved that the sequence of estimates converged with positive probability to the wrong...
parameter value, due to the identification in closed loop and the resulting lack of excitation.

In the present paper we will deal with linear, time-invariant, finite-dimensional single-input, single-output systems, described in discrete time. The underlying control objective is the minimization of a quadratic cost functional on the input and the output, known as linear-quadratic control, or just LQ control.

Our main result is that for a broad class of adaptive control algorithms, closed-loop identification will most likely lead to suboptimal behavior of the controlled system. This rather vague statement will be made precise in the technical part of the paper. The result is of the same nature as that in [1]. For the first-order case it has been obtained by [6], and for the higher-order case partial results can be found in [7]. In [7] it was assumed that the state of the system was accessible for measurement. This assumption is now relaxed. The results presented here can also be found in [8].

The paper is organized as follows. In Section 2 we have collected the preliminaries that we will use. The problem statement is given and two subsets of the parameter-space are introduced. The main result will be stated in terms of these subsets. Section 3 contains the main theorem. The proof of this theorem is rather technical and is therefore divided into several parts. The proofs of these intermediate results are given in the Appendix.

2. PROBLEM STATEMENT

In this section we will provide the ingredients of which our main result is composed. We will first define the class of systems that we consider; we will then formulate a class of LQ-control problems, followed by their nonadaptive solutions. We will then briefly describe a class of adaptive-control algorithms. Finally we will define the subsets advertised in the introduction.

We consider systems of the form

\[ x(k+1) = Ax(k) + bu(k), \quad (2.1.a) \]
\[ y(k) = cx(k), \quad (2.1.b) \]

where \((A, b, c) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{1 \times n}\) is a minimal triple. The set of systems of the form (2.1) is denoted by

\[ E := \{(A, b, c) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{1 \times n} | (A, b, c) \text{ minimal}\}. \quad (2.2) \]
Control objective (nonadaptive): Given a system of the form (2.1), find a causal controller such that the following expression is minimized:

\[ J = \sum_{k=0}^{\infty} \left[ y(k)^2 + ru(k)^2 \right], \quad r > 0. \]  
(2.3)

The solution of this problem is well known (see [4]) and is given by

\[ u(k) = f(A, b, c)x(k), \]  
(2.4)

where:

\[ f(A, b, c) = -(b^TKb + r)^{-1}b^TKA, \]  
(2.5)

and \( K \) is the unique symmetric positive definite solution of the algebraic Riccati equation

\[ K - \Lambda^TK\Lambda + \Lambda^TKb(b^TKb + r)^{-1}b^TK\Lambda - c^Tc = 0. \]  
(2.6)

Moreover, the optimal value of \( J \) is given by

\[ x(0)^TKx(0), \]  
(2.7)

where \( x(0) \) is the initial state of the system.

In the sequel it will be a standing assumption that the system to be controlled is unknown and can be represented by an element of \( E \), which we denote by \((A_0, b_0, c_0)\). Without loss of generality we will assume that \((A_0, b_0, c_0)\) is in standard observable form:

\[ A_0 := \begin{bmatrix} a_0 & 1 & \cdots & 0 \\ a_1 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ a_{n-1} & 0 & \cdots & 0 \end{bmatrix}, \quad b_0 := \begin{bmatrix} b_0^0 \\ \vdots \\ 1 \\ b_{n-1}^0 \end{bmatrix}, \quad c_0 := [1 \ 0 \ \cdots \ 0]. \]  
(2.8)
The adaptive-control algorithms which we consider are recursive, based on certainty equivalence, and driven by the prediction error. The last two properties imply that given an estimate \((\hat{A}, \hat{b}, \hat{c}, \hat{x}_k)\) of \((A_0, b_0, c_0)\) and \(x_k\) we will apply to the system \((A_0, b_0, c_0)\):

\[
\hat{u}_k = f(\hat{A}, \hat{b}, \hat{c}) \hat{x} \quad \text{(certainty equivalence),} \tag{2.9}
\]

where \(f\) is defined by (2.5), (2.6). The predicted output will then be

\[
\hat{y}_{k+1} = \hat{c} [\hat{A} + \hat{b} f(\hat{A}, \hat{b}, \hat{c})] \hat{x}, \tag{2.10}
\]

whereas the actual output will be

\[
y = c_0 [A_0 x + b_0 f(\hat{A}, \hat{b}, \hat{c}) \hat{x}]. \tag{2.11}
\]

By saying that the output is driven by the prediction error we mean that the next estimate of \((A_0, b_0, c_0)\) equals \((\hat{A}, \hat{b}, \hat{c})\) if \(y - \hat{y} = 0\).

We will use the following sets.

\[
E_{ob} := \{(A, b) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times 1} | (A, b, c_0) \text{ standard observable,} \}
\]

\[
\quad (A, b) \text{ reachable, } A \text{ nonsingular} \} \tag{2.12}
\]

\[
E_{us} := \{(A, b) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times 1} | (A, b, c_0) \text{ minimal, } A \text{ nonsingular} \}. \tag{2.13}
\]

**Remark.** We restrict ourselves to nonsingular \(A\)-matrices for technical reasons only.

Define

\[
\phi(k) := [y(k), \ldots, y(k - n + 1), u(k - 1), \ldots, u(k - n + 1)]^T. \tag{2.14}
\]
As pointed out in [3] and [8], $\phi(k)$ is the state of a $2n-1$-dimensional realization of (2.1).

Define $M \in \mathbb{R}^{n \times (2n-1)}$ by

$$M := \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ 0 & a_1 & a_2 & \cdots & \cdots & a_{n-1} & b_1 & b_2 & \cdots & b_{n-1} \\ \cdot & a_2 & a_3 & \cdots & 0 & b_2 & b_3 & \cdots & \cdots & \cdots \\ \cdot & \cdot & a_4 & \cdots & \cdots & \cdots & b_4 & \cdots & \cdots & \cdots \\ \cdot & \cdot & \cdot & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdot & \cdot & a_{n-1} & \cdots & \cdots & b_{n-1} & \cdots & \cdots & \cdots & \cdots \\ 0 & a_{n-1} & 0 & \cdots & \cdots & 0 & b_{n-1} & 0 & \cdots & 0 \end{bmatrix}$$

(2.15)

It can easily be shown that for all $k$ the state $x(k)$ of (2.1) is related with $\phi(k)$ by

$$x(k) = M\phi(k).$$

(2.16)

Throughout the paper let the initial state $\phi(0) \in \mathbb{R}^{2n-1}$ be fixed. For any $(A, b) \in E_{ab}$, define $z(0) \in \mathbb{R}^n$ by

$$z(0) := M\phi(0),$$

(2.17)

and define $x(0) \in \mathbb{R}^n$ by

$$x(0) := M_0\phi(k),$$

(2.18)

where $M_0$ is obtained by adding the appropriate superscripts to the entries of $M$. Finally define

$$x(k+1) = A_0x(k) + b_0u(k), \quad z(k+1) = Az(k) + bu(k),$$

(2.19.a)

$$y(k) = c_0x(k), \quad \dot{y}(k) = c_0z(k),$$

(2.19.b)

where

$$u(k) := f(A, b, c_0)z(k).$$

(2.19.c)
The above recursions should be interpreted as follows. The first system represents the true system, and the second system is an estimate of the true system. By (2.16), (2.17), \( z(0) \) is an estimate of \( x(0) \) which is compatible with the estimate \( (A, b) \). Note that since we assume the standard observable form, \( c_0 \) need not to be estimated. According to the certainty-equivalence principle the input to both systems is defined by (2.19.c). The prediction error is given by \( y(k) - \hat{y}(k) \).

We now define two subsets of \( E_{ob} \):

**Definition 2.1.** The sets \( G \) and \( H \) contained in \( E_{ob} \) are defined as

\[
G := \{ (A, b) \in E_{ob} \mid \text{for all } k: \hat{y}(k) = y(k) \}, \tag{2.20}
\]

\[
H := \{ (A, b) \in E_{ob} \mid \text{for all } k: f(A, b, c_0)z(k) = f(A_0, b_0, c_0)x(k) \}, \tag{2.21}
\]

where the sequences \( \{x(k)\}, \{z(k)\}, \{y(k)\}, \) and \( \{\hat{y}(k)\} \) are defined as in (2.19).

**Interpretation.**

(i) The set \( G \) can be seen as the set of those estimates \( (A, b) \) that are invariant under any algorithm of the considered type. For choose \( (A, b) \in G \). Since at every time instant the prediction error is zero, this estimate will never be changed, because the identification part of any algorithm is driven by the prediction error.

(ii) The set \( H \) can be viewed as the set of those parameters that generate the optimal controls.

The relevance of the sets \( G \) and \( H \) lies in the fact that if an algorithm produces a sequence of estimates that converges, then the limit will be an invariant point of the algorithm and hence will be an element of \( G \). That implies that \( G \) contains the set of possible limits. Whether or not an element of \( G \) is attractive, however, depends on the particular algorithm that generates the sequence of estimates. In [6] it was shown, using the ODE method, that a significant part of \( G \) can be attractive (see also Example 3.12). On the other hand, if we want the system to be controlled optimally according to the cost criterion, the limit should be an element of \( H \). \( H \) can therefore be seen as the set of desirable limits. Indeed, if the sequence of estimates converges
to an element of \( H \), then the corresponding sequence of controllers converges to the optimal one, since \( f \) is \( C^\infty \) on \( E \) (Corollary 4.7). Combining the properties of \( G \) and \( H \), we conclude that a limit of the sequence of estimates necessarily is an element of \( G \), whereas we also want it to be an element of \( H \). The question that now arises is: how is \( G \cap H \) related to \( G \)?

In general \( G \) will consist of an infinite number of pairs \( (A, b) \). Thus it is not at all obvious that an element of \( H \) will also be an element of \( G \). All that we can say at this stage is that \( (A_0, b_0) \) belongs to both \( G \) and \( H \). The phenomenon that \( G \) is larger than just \( \{(A_0, b_0)\} \) is due to the fact that identification takes place in closed loop: Information is obtained only about the closed-loop behavior of the system. It is very likely that there are many parameter values that give rise to the same closed-loop behavior.

3. \( G \) AND \( H \) FOR LQ CONTROL: CONFLICT BETWEEN IDENTIFICATION AND CONTROL

In this section we will investigate the relation between \( G \cap H \) and \( G \). It will turn out that the desirable property

\[ G \subset H \]  

(3.1)

does not hold. In fact we will show that \( G \cap H \) is a negligible subset of \( G \). The results in this section are refinements of those obtained in [7]. Theorem 3.11(i) was also proven in [6] for the first-order case.

We will first state the main result of this section:

**Theorem 3.1.** \( G \cap H \) is a nowhere dense subset of \( G \).

Theorem 3.1 can be rephrased by saying that \( G \cap H \) is a negligible subset of \( G \). This is of course not a mathematical statement. Intuitively it means that within the set of invariant points of an adaptive algorithm only a negligible part consists of points that correspond to the desired (optimal) control law. This is in contrast to the pole assignment problem, where every invariant point corresponds to the desired control law [8, 9]. In this sense adaptive LQ control is more difficult. In pole assignment the only concern is convergence of the parameter estimates; every limit point will be invariant and will hence produce the right controls. In LQ control we have to prevent the estimates from converging to suboptimal invariant points. That means that we have to develop an algorithm for which those invariant points can never be attractive.
The proof of Theorem 3.1 will be divided into several steps, some of which are interesting on their own merit. It is difficult to get a direct grip on the sets $G$ and $G \cap H$; therefore we will introduce two other sets, $G_0$ and $H_0$, which are easier to analyze and which are closely related to $G$ and $H$. In order to relate $G$ and $H$ with $G_0$ and $H_0$ we will also define a subset $\tilde{G}$ of $G$ and a subset of $G_0$ of $G$.

**Definition 3.2.**

$$C_0 := \{(A, b) \in E_{ns} | A_0 + b_0 f(A, b, c_0) = A + b f(A, b, c_0)\}, \quad (3.2)$$

$$H_0 := \{(A, b) \in E_{ns} | f(A, b, c_0) - f(A_0, b_0, c_0)\}. \quad (3.3)$$

**Definition 3.3.** For every $(A, b) \in E_{ob}$, define

$$V(A, b) := \text{span} \{x(k)\}_{k \in \mathbb{N}}, \quad (3.4)$$

where $\{x(k)\}$ and $\{z(k)\}$ are defined by (2.19).

**Definition 3.4.**

$$\tilde{G}_0 := \{(A, b) \in G_0 | (A_0 + b_0 f(A, b, c_0), x(0)) \text{ is reachable}\}, \quad (3.5)$$

$$\tilde{G} := \{(A, b) \in G | V(A, b) = \mathbb{R}^n\}. \quad (3.6)$$

**Theorem 3.5.** $\tilde{G}$ and $\tilde{G}_0$ are $C^\infty$ diffeomorphic.

**Proof.** For the proof of this statement we use the following theorem, which can be viewed as an extension of classical realization theory. We find this theorem interesting enough to give it here rather than in the appendix. Its proof and the proof Theorem 3.5 are given in the appendix, though.

**Theorem 3.6.** Let $\{(u(k), y(k))\}_{k \in \mathbb{N}}$ be a sequence in $\mathbb{R}^2$, and suppose there exist $(A_1, b_1, c_1), (A_2, b_2, c_2)$, minimal triples of order $n$, and
sequences \( \{x(k)^{(1)}, x(k)^{(2)}\} \) in \( \mathbb{R}^n \), such that for all \( k \)

\[
\begin{align*}
x(k + 1)^{(1)} &= A_1 x(k)^{(1)} + b_1 u(k), & x(k + 1)^{(2)} &= A_2 x(k)^{(2)} + b_2 u(k), \\
y(k) &= c_1 x(k)^{(1)}, & y(k) &= c_2 x(k)^{(2)}.
\end{align*}
\]

(3.7.a) (3.7.b)

Define \( X_i = \text{span} \{ x(k)^{(i)} \}_{k \in \mathbb{N}} \), and \( d_i = \dim(X_i) \), \( i = 1, 2 \).

(i) If \( d_1 < n \), then there exists a nonsingular matrix \( S \), such that \( S x(k)^{(1)} = x(k)^{(2)} \).

(ii) \( d_1 = d_2 \).

(iii) If there exists \( g_1 \) such that \( u(k) = g_1 x(k)^{(1)} \), then there exists a nonsingular matrix \( S \) such that \( S x(k)^{(1)} = x(k)^{(2)} \).

Remark. Note that the statement would have followed from classical realization theory if (3.7) were true for any input-output sequence. Note also that we do not claim that the system matrices are related by the transformation matrix.

Theorem 3.7.

(i) \( \tilde{G}_0 \) is open and dense in \( G_0 \).

(ii) \( \tilde{G} \) is open and dense in \( G \).

Proof. See the appendix.

Theorem 3.8. \( G_0 \) is an embedded analytic manifold of dimension \( n \).

Proof. See the appendix.

Lemma 3.9. For all \( (A, b) \in C \cap H, V(A, b) = V(A_0, b_0) \).

Proof. This is immediate from the fact that \( (A, b) \in H \) implies that for all \( k \), \( f(A, b, c_0) z(k) = f(A_0, b_0, c_0) x(k) \).

\( \blacksquare \)
Definition 3.10. \( V_0 \coloneqq V(A_0, b_0) \), the linear span of the optimal state trajectory.

Theorem 3.11.

(i) If \( \dim(V_0) = n \), then \( G \cap H = \{(A_0, b_0)\} \).

(ii) If \( \dim(V_0) < n \), then \( G \cap H \) is contained in \( G \setminus \tilde{G} \).

Proof. See the appendix.

The proof of Theorem 3.1 follows directly from Theorem 3.11.

Comment. Let us now discuss some of the consequences of Theorem 3.11. First of all it is the mathematical formalization of the statement that \( G \cap H \) is a negligible subset of \( G \). For suppose \( \phi(0) \) is such that \( \dim(V_0) = n \). Then from Theorem 3.11(i) we know that \( G \cap H = \{(A_0, b_0)\} \), a singleton. Now \( G \) contains an open and dense subset that is diffeomorphic to an open and dense subset of an \( n \)-dimensional manifold (by Theorems 3.5, 3.7, and 3.8). In that sense \( G \cap H \), being a singleton, is a negligible subset of \( G \). In the other case, where \( \phi(0) \) is such that \( \dim(V_0) < n \), \( G \cap H \) is contained in \( G \setminus \tilde{G} \). In other words, \( G \cap H \) is contained in the boundary of a set that is diffeomorphic to an open and dense subset of an \( n \)-dimensional manifold. Since the boundary of an \( n \)-dimensional manifold has a strictly smaller dimension, again \( G \cap H \) is a negligible subset of \( G \).

Now suppose that an algorithm of the considered type is used. Then almost every invariant point of the algorithm will result in suboptimal behavior. This means that almost every invariant point must not be attractive, i.e. must not be a possible limit of the algorithm. This seems to be very difficult, if not impossible, to achieve.

One possibility to ensure the convergence of the sequence of estimates to the true parameter is to inject external excitation into the system. A draw-back of this procedure is that the system will be excited persistently, thus influencing the asymptotic behavior negatively. In [8] another method is proposed. There a closed-loop excitation signal in combination with a probing signal driven by the prediction error is used. The advantage of this approach is twofold. Firstly, the excitation is in closed loop, which means that it is proportional to the signals of the system. This implies that if the system stabilizes, the output of the system vanishes asymptotically, which is not possible if the excitation is persistent. Secondly, since extra excitation is used when the (normalized) prediction error is large, the identification task of the input is emphasized as long as the parameter estimates are far away from the true parameter, which improves the transient behavior of the algorithm. As
the prediction error decreases, the extra excitation damps out and the control task of the input becomes more prominent.

We conclude this section with a simple example.

**Example 3.12.** In Figure 1, we have depicted the sets \( G \) and \( H \) for a first-order system. The parameter values were \((a_0, b_0) = (1, 1), \ r = 2\). The upper graph shows the branches of \( G \) and \( H \) in the right half plane; the lower graph shows the branches in the left half plane. The picture illustrates that \( G \cap H = \{a_0, b_0\} \), as was already predicted by Theorem 3.11(i).

In [6] it was shown that the elements of the upper branch of \( G \) are indeed all attractive for a specific algorithm, and also simulations have suggested that \( G \) contains at least an open set consisting of attractive points.

![Diagram of G and H for a first-order system](image)

**Fig. 1.** \( G \) and \( H \) for a first-order system.
4. APPENDIX

Throughout the appendix we will use the inner product on \( \mathbb{R}^{p_1 \times q_1} \times \mathbb{R}^{p_2 \times q_2} \) defined by 
\[
[(M_1, N_1), (M_2, N_2)] := \text{Tr}(M_1 N_1^T) + \text{Tr}(M_2 N_2^T),
\]
where \( M_i, N_i \in \mathbb{R}^{p_i \times q_i}, i = 1, 2 \), and \( \text{Tr} \) denotes the trace of a matrix. If we refer to the adjoint of a linear map that acts on a space of matrices, then it is understood to be the adjoint with respect to this inner product, unless otherwise stated.

We will use the following lemma:

**Lemma 4.1.** Let \( M, N \in \mathbb{R}^{p \times p} \); let \( \Lambda : \mathbb{R}^{p \times p} \to \mathbb{R}^{p \times p} \) be defined by 
\[
\Lambda(X) = X - M^T X N.
\]
Then 
\[
\text{Spec}(\Lambda) = 1 - \text{Spec}(M) \times \text{Spec}(N) = \{ 1 - \lambda \mu | \lambda \in \text{Spec}(M), \mu \in \text{Spec}(N) \}.
\]

**Proof.** See [5].

For the proof of Theorem 3.6 we will use the following:

**Lemma 4.2.** Let \( (A, b) \) be reachable, and \( x(0) \in \mathbb{R}^n \). Let \( \{ u(k) \} \) be a sequence of real numbers. Define 
\[
x(k + 1) = Ax(k) + bu(k), \quad k = 0, 1, 2, \ldots \quad (4.1)
\]
Define \( X := \text{span}\{ x(k) \}_{k \in \mathbb{N}} \) and \( d := \text{dim}(X) \). If \( d < n \), then there exists an \( g \in \mathbb{R}^{1 \times n} \) such that for all \( k \)
\[
u(k) = gx(k). \quad (4.2)
\]

**Proof.** Suppose \( (A, b) \) is in standard controllable form, i.e.
\[
A = \begin{bmatrix}
0 & 1 & \cdots & & 0 \\
\cdot & 0 & \cdots & & \cdot \\
\cdot & 0 & \cdots & & \cdot \\
\cdot & \cdot & \cdots & & 0 \\
0 & 0 & \cdots & & 1 \\
a_1 & a_2 & \cdots & & a_n
\end{bmatrix}, \quad b = \begin{bmatrix}
0 \\
\cdot \\
\cdot \\
\cdot \\
0 \\
1
\end{bmatrix}. \quad (4.3)
\]
Define \( a \in \mathbb{R}^{1 \times n} \) by \( a := (a_1, \ldots, a_n) \). Define

\[
\tilde{A} := A - ba,
\]

\[
\tilde{u} := ax(k) + u(k).
\]

Then

\[
x(k + 1) = \tilde{A}x(k) + b\tilde{u}(k).
\]

Suppose \( x(0) = [x_1(0), \ldots, x_n(0)]^T \); define \( H \in \mathbb{R}^{n \times N} \) by

\[
H := [x(0), x(1), x(2), x(3), \ldots].
\]

Then

\[
H := \begin{bmatrix}
x_1(0) & x_2(0) & x_3(0) & \cdots & x_n(0) & \tilde{u}(0) & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \tilde{u}(0) & \cdots \\
\vdots & x_{n-1}(0) & x_n(0) & \cdots & \tilde{u}(1) & \cdots \\
x_n(0) & \tilde{u}(0) & \cdots & \cdots & \cdots & \cdots \\
x_n(0) & \tilde{u}(0) & \tilde{u}(1) & \cdots & \tilde{u}(n-2) & \tilde{u}(n-1) & \cdots
\end{bmatrix}.
\]

Since \( d < n \), \( \text{rank}(H) < n \). Now \( H \) is a truncated Hankel matrix; hence its rank does not increase if we add the last row, shifted to the left, as the \( n + 1 \)th row. This shifted row is

\[
[\tilde{u}(0), \tilde{u}(1), \tilde{u}(2), \tilde{u}(3), \ldots].
\]

Since the rank of the increased matrix is equal to the original one, the last row is a linear combination of the first \( n \) rows. In other words, there exist \( \tilde{g}_1, \ldots, \tilde{g}_n \in \mathbb{R} \) such that

\[
r_{n+1} = \tilde{g}_1 r_1 + \cdots + \tilde{g}_n r_n,
\]
where \( r_i \) denotes the \( i \)th row. Define \( \tilde{g} \in \mathbb{R}^{1 \times n} \) by \( \tilde{g} := [\tilde{g}_1, \ldots, \tilde{g}_n] \). Then for all \( k \)

\[
\tilde{u}(k) = \tilde{g}x(k).
\]  

(4.11)

Define \( g \) by \( g := \tilde{g} - a \). Finally,

\[
u(k) = \tilde{u}(k) - ax(k) = \tilde{g}x(k) - ax(k) = gx(k).
\]  

(4.12)

(4.13)

(4.14)

We will now prove Theorem 3.6:

Proof of Theorem 3.6. (i): Suppose \( d_1 < n \).

\[
y(k + i) = c_1 \left[ A_1^i x(k)^{(1)} + \sum_{j=0}^{i-1} A_1^j b_1 u(k + i - j - 1) \right].
\]  

(4.15)

Define

\[
W := \begin{bmatrix} c_1 \\ c_1 A_1 \\ \vdots \\ c_1 A_1^{n-1} \end{bmatrix}.
\]  

(4.16)

then

\[
Wx(k)^{(1)} = \begin{bmatrix} y(k) \\ y(k + 1) - c_1 b_1 u(k) \\ \vdots \\ y(k + n - 1) - c_1 A_1^{n-2} b_1 u(k) - \cdots - c_1 b_1 u(k + n - 2) \end{bmatrix},
\]  

(4.17)
from which we conclude that

\[
\begin{bmatrix}
  x(k)^{(1)} \\
  \vdots \\
  x(k)^{(1)} \\
  u(k) \\
  \vdots \\
  u(k + n - 2)
\end{bmatrix}
= 
\begin{bmatrix}
  1 & 0 & \cdots & 0 & \cdots & 0 \\
  0 & -c_1b_1 & \cdots & 0 \\
  \vdots & -c_1A_1b_1 & \cdots & \vdots \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & & & & & 0 \\
  0 & & & & & 1
\end{bmatrix}
\begin{bmatrix}
  y(k) \\
  \vdots \\
  y(k + n - 1) \\
  u(k) \\
  \vdots \\
  u(k + n - 2)
\end{bmatrix}
\]

from which we derive

\[
\begin{bmatrix}
  x_1^{(1)}(k) \\
  \vdots \\
  x_n^{(1)}(k) \\
  u(k) \\
  \vdots \\
  u(k + n - 2)
\end{bmatrix}
= T_1
\begin{bmatrix}
  y(k) \\
  \vdots \\
  y(k + n - 1) \\
  u(k) \\
  \vdots \\
  u(k + n - 2)
\end{bmatrix}
\]

with \( T_1 \) nonsingular. In the same way one derives that

\[
\begin{bmatrix}
  x_1^{(2)}(k) \\
  \vdots \\
  x_n^{(2)}(k) \\
  u(k) \\
  \vdots \\
  u(k + n - 2)
\end{bmatrix}
= T_2
\begin{bmatrix}
  y(k) \\
  \vdots \\
  y(k + n - 1) \\
  u(k) \\
  \vdots \\
  u(k + n - 2)
\end{bmatrix}
\]

(4.20)
Hence

\[
\begin{bmatrix}
  x(k)^{(1)} \\
  u(k) \\
  \vdots \\
  u(k+n-2)
\end{bmatrix} = R
\begin{bmatrix}
  x(k)^{(2)} \\
  u(k) \\
  \vdots \\
  u(k+n-2)
\end{bmatrix},
\]

(4.21)

where \( R = T_1 T_2^{-1} \). Now since \( u(k+i) = b_2^T [x(k+i+1)^{(2)} - A_2 x(k+i)^{(2)}] \), there exist matrices \( M_1^{(2)}, \ldots, M_n^{(2)} \in \mathbb{R}^{n \times n} \), such that for all \( k \)

\[
x(k)^{(1)} = M_1^{(2)} x(k)^{(2)} + \cdots + M_n^{(2)} x(k+n-1)^{(2)},
\]

(4.22)

and similarly

\[
x(k)^{(2)} = M_1^{(1)} x(k)^{(1)} + \cdots + M_n^{(1)} x_{k+n-1}.
\]

(4.23)

Since by assumption \( d_1 < n \), we conclude from Lemma 4.2 that there exists \( g_1 \) such that \( u(k) = g_1 x(k)^{(1)} \), hence \( x(k+1)^{(1)} = (A_1 + b_1 g_1) x(k)^{(1)} \). Together with (4.23) this gives that there exists a matrix \( N_1 \) such that for all \( k \)

\[
x(k)^{(2)} = N_1 x(k)^{(1)}.
\]

(4.24)

Denote by \( X_2 \) the linear span of \( \{ x(k)^{(2)} \} \in \mathbb{N} \), and by \( d_2 \) its dimension. From (4.24) it follows that \( d_2 \leq d_1 < n \); hence by Lemma 4.2 there exists \( g_2 \) such that \( u(k) = g_2 x(k)^{(2)} \) for all \( k \). As above, we conclude that there exists a matrix \( N_2 \) such that for all \( k \)

\[
x(k)^{(1)} = N_2 x(k)^{(2)}.
\]

(4.25)

Finally (4.24) together with (4.25) gives the statement.

(ii): This follows immediately from part (i).

(iii): If \( d_1 < n \), then the statement follows from part (i). Assume that \( d_1 = n \), and suppose \( u(k) = g_1 x(k)^{(1)} \); then just as in the proof of part (i), (4.23), we conclude that

\[
x(k)^{(2)} = N_1 x(k)^{(1)}.
\]

(4.26)

Since \( d_1 = d_2 = n \), it follows that \( N_1 \) is nonsingular.

\[ \square \]

**Lemma 4.3.** For every minimal triples \( (A, b, c) \) one has \( \text{Ker}[A + bf(A, b, c)] = \text{Ker} A \).
Proof. Suppose \( x_0 \in \text{Ker}[A + bf(A, b, c)] \); then \( x_k = 0 \) and \( u_k = 0 \) for all \( k \geq 1 \). Hence

\[
x_0^T K x_0 = x_0^T c^T c x_0 + u_0^T r u_0
\]

by (2.3) and (2.7)

\[
= x_0^T \left[ c^T c + f(A, b, c)^T r f(A, b, c) \right] x_0
\]

by (2.4)

\[
= x_0^T \left[ K - A^T K \left( A + bf(A, b, c) \right) + f(A, b, c)^T r f(A, b, c) \right] x_0
\]

by (2.5) and (2.6)

This implies that \( x_0^T f(A, b, c)^T r f(A, b, c) x_0 = 0 \), and thus that \( f(A, b, c) x_0 = 0 \). Together with \( (A + bf(A, b, c)) x_0 = 0 \), this gives \( Ax_0 = 0 \).

Suppose on the other hand that \( Ax_0 = 0 \); then also \( f(A, b, c) x_0 = 0 \) [by (2.5)] and thus \( [A + bf(A, b, c)] x_0 = 0 \).

Corollary 4.4. For all \((A, b, c) \in E_{NS}\), \( A + bf(A, b, c) \) is nonsingular.

Proof. This follows from Lemma 4.3 and from the fact that by definition of \( E_{NS}\), \((A, b, c) \in E_{NS}\), implies that \( A \) is nonsingular.

Lemma 4.5. For all \((\bar{A}, \bar{b}, c_0) \in C_0\) there exists an \( \bar{\epsilon} > 0 \) such that \( \forall \bar{f} \) with \( \| \bar{f} - f(\bar{A}, \bar{b}, c_0) \| < \epsilon \), there exists \((\bar{A}, \bar{b}, c_0) \in E\) such that:

(i) \( f(\bar{A}, \bar{b}, c_0) \equiv \bar{f} \),
(ii) \( A_0 + b_0 f(\bar{A}, \bar{b}, c_0) = \bar{A} + b_0 f(\bar{A}, \bar{b}, c_0) \),
(iii) \((\bar{A}, \bar{b})\) depends continuously on \( \bar{f} \).

Proof. Choose \((\bar{A}, \bar{b}, c_0) \in C_0\). We will prove that the map \( f \), subject to the constraint that \((A, b, c_0) \in C_0\), is locally surjective. To this end it is enough to prove that, locally, \((A, b)\) can be written as a continuous function of \( \bar{f} \). Define

\[
L : \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{1 \times (n + 1)} \to \mathbb{R}^{n \times n} \times \mathbb{R}^{1 \times n} \mathbb{R}^{1 \times (n + 1)}
\]

by

\[
L(A, b, K, \bar{f}) := (L_1(A, b, K, \bar{f}), L_2(A, b, K, \bar{f}), L_3(A, b, K, \bar{f})),
\]

(4.27)
where

\[ L_1(A, b, K, \tilde{f}) := A_0 + b_0 \tilde{f} - \Lambda - b_0 f, \]  

\[ L_2(A, b, K, \tilde{f}) := b^T K b \tilde{f} + r \tilde{f} + b^T K A, \]  

\[ L_3(A, b, K, \tilde{f}) := K - A^T K A + A^T K b (b^T K b + r)^{-1} b^T K A - c_0^T c_0. \]

By definition of \( L \) it follows that \( L(\tilde{A}, \tilde{b}, \tilde{K}, \tilde{f}) = (0, 0, 0) \), where \( \tilde{K} \) is the positive definite solution of the algebraic Riccati equation and \( \tilde{f} = f(\tilde{A}, \tilde{b}, c_0) \). We will now calculate the derivative of \( L \) with respect to \((A, b, K)\) evaluated in \((\tilde{A}, \tilde{K}, \tilde{b}, \tilde{f})\):

\[ \Lambda_1(\Delta A, \Delta b, \Delta K) = - \Delta A - \Delta b \tilde{f}, \]  

\[ \Lambda_2(\Delta A, \Delta b, \Delta K) = \Delta b^T \tilde{K} \tilde{b} \tilde{f} + \tilde{b}^T \Delta K \tilde{b} \tilde{f} + \tilde{b}^T \tilde{K} \Delta b \tilde{f} \]  

\[ + \Delta b^T \tilde{K} \tilde{A} + \tilde{b}^T \Delta K \tilde{A} + \tilde{b}^T \tilde{K} \Delta A, \]  

\[ \Lambda_3(\Delta A, \Delta b, \Delta K) = \Delta K - (\tilde{A} + \tilde{b} \tilde{f})^T \Delta K (\tilde{A} + \tilde{b} \tilde{f}) \]  

\[ - \Delta A^T \tilde{K} (\tilde{A} + \tilde{b} \tilde{f}) - (\tilde{A} + \tilde{b} \tilde{f})^T \tilde{K} \Delta A \]  

\[ - \tilde{A}^T \tilde{K} \Delta b \tilde{f} - \tilde{f}^T \Delta b^T \tilde{K} \tilde{A} - \tilde{f}^T (\Delta b^T \tilde{K} \tilde{b} + \tilde{b}^T \tilde{K} \Delta b) \tilde{f}. \]  

To show that \( \Lambda \) has full rank it is sufficient to show that it is injective: Put

\[ E_1: \Lambda_1(\Delta A, \Delta b, \Delta K) = 0, \]  

\[ E_2: \Lambda_2(\Delta A, \Delta b, \Delta K) = 0, \]  

\[ E_3: \Lambda_3(\Delta A, \Delta b, \Delta K) = 0. \]

\[ E_3 + (\tilde{A} + \tilde{b} \tilde{f})^T \tilde{K} E_1 + E_1^T \tilde{K} (\tilde{A} + \tilde{b} \tilde{f}) \]  

gives

\[ \Delta K - (\tilde{A} + \tilde{b} \tilde{f})^T \Delta K (\tilde{A} + \tilde{b} \tilde{f}) = 0. \]

By Lemma 4.1 and the strict stability of \( \tilde{A} + \tilde{b} \tilde{f} \) it follows that \( \Delta K = 0 \).
Substituting this in $E_2$ gives

$$E_2' := \Delta b^T \tilde{\mathbf{K}} \tilde{\mathbf{f}} + \tilde{\mathbf{b}}^T \tilde{\mathbf{K}} \Delta \mathbf{f} + \Delta b^T \tilde{\mathbf{K}} \tilde{\mathbf{A}} + \tilde{\mathbf{b}}^T \tilde{\mathbf{K}} \Delta \mathbf{A} = 0. \quad (4.38)$$

$E_2' - b^T \tilde{\mathbf{K}} E_1$ gives

$$\Delta b^T \tilde{\mathbf{K}} (\tilde{\mathbf{A}} + \tilde{\mathbf{b}} \tilde{\mathbf{f}}) = 0. \quad (4.39)$$

By Corollary 4.4 $\tilde{\mathbf{A}} + \tilde{\mathbf{b}} \tilde{\mathbf{f}}$ is nonsingular, and hence $\Delta \mathbf{b} = 0$. Finally, substituting this in $E_1$ gives $\Delta \mathbf{A} = 0$.

Now, the implicit-function theorem yields the existence of an open neighborhood of $\tilde{\mathbf{f}}$ and a $C^\omega$ function defined on that open set to $(\mathbf{A}, b, K)$. This completes the proof.

**Proof of Theorem 3.5.** Define

$$\phi : E_{ns} \rightarrow E_{ob}$$

by

$$\phi(\mathbf{A}, b) = (\mathbf{SAS}^{-1}, \mathbf{Sb}), \quad (4.40)$$

where $\mathbf{S} \in \text{Gl}(n)$ is the unique nonsingular matrix which transforms $(\mathbf{A}, c_0)$ into standard observable form. Since $\mathbf{S}$ depends $C^\omega$ on $\mathbf{A}$, it follows that $\phi$ is $C^\omega$.

Now, let $(\mathbf{A}, b) \in \tilde{\mathcal{G}}_0$. Define $x(0) := M_0 \phi(0)$, and $x(k)$ by

$$x(k + 1) = [\mathbf{A}_0 + b_0 f(\mathbf{A}, b, c_0)] x(k). \quad (4.41)$$

Define

$$z(k) := \mathbf{S} x(k). \quad (4.42)$$

Then

$$z(k + 1) = \mathbf{S} x(k + 1) = \mathbf{S} [\mathbf{A}_0 + b_0 f(\mathbf{A}, b, c_0)] x(k) \quad (4.43)$$

$$= \mathbf{S} [\mathbf{A} + b f(\mathbf{A}, b, c_0)] x(k) \quad (4.44)$$

$$= \mathbf{S} [\mathbf{A} + b f(\mathbf{A}, b, c_0)] \mathbf{S}^{-1} z(k) \quad (4.45)$$

$$= [\mathbf{SAS}^{-1} + \mathbf{Sb} f(\mathbf{SAS}^{-1}, \mathbf{Sb}, c_0)] z(k). \quad (4.46)$$
From the standard observable form and the recursion for $z(k)$ it follows that $z(k) = M\phi(k)$, where $M$ is derived from $(SAS^{-1}, Sb)$ as in (2.15). In particular it follows that $z(0) = M\phi(0)$. Finally, $y(k) = c_0x(k) = c_0S^{-1}z(k) = c_0z(k) = \hat{y}(k)$. We conclude that $\phi(A, b) \in G$. Moreover since by definition of $\tilde{G}_0$, span{ $x(k)$} $= \mathbb{R}^n$, it follows that $\phi(A, b) \in \tilde{G}$.

Define $\psi: \tilde{G} \rightarrow \tilde{G}_0$ as follows: Choose $(A, b) \in \tilde{G}$. By Theorem 3.6 there exists $T \in \text{Gl}(n)$ such that for all $k$ one has $x(k) = Tz(k)$, and since $V(A, b) = \mathbb{R}^n$, this $T$ is unique. Moreover, from the proof of Theorem 3.6 it follows easily that $T$ depends $C^\omega$ on $(A, b)$. Define

$$\psi(A, b) := (TAT^{-1}, Tb). \quad (4.47)$$

Since $y(k) = \hat{y}(k)$, it follows from $x(k) = Tz(k)$ and $V(A, b) = \mathbb{R}^n$ that $c_0T^{-1} = c_0$. Now

$$x(k + 1) = A_0x(k) + b_0f(A, b, c_0)z(k) \quad (4.48)$$

$$= \left[ A_0 + b_0f(TAT^{-1}, Tb, c_0T^{-1}) \right] x(k) \quad (4.49)$$

and also

$$x(k + 1) = Tz(k + 1) = T \left[ A + bf(A, b, c_0) \right] z(k) \quad (4.50)$$

$$= T \left[ A + bf(A, b, c_0) \right] T^{-1}x(k) \quad (4.51)$$

$$= \left[ TAT^{-1} + Tbf(TAT^{-1}, Tb, c_0) \right] x(k). \quad (4.52)$$

Since $V(A, b) = \mathbb{R}^n$, it follows that

$$A_0 + b_0f(TAT^{-1}, Tb, c_0) = TAT^{-1} + Tbf(TAT^{-1}, Tb, c_0); \quad (4.53)$$

hence $\psi(A, b) \in \tilde{G}_0$.

Finally, from the uniqueness of the matrices $S$ and $T$ one can easily check that

$$\psi \cdot \phi = \text{id}_{\tilde{G}}, \quad (4.54)$$

$$\phi \cdot \psi = \text{id}_{\tilde{G}_0}. \quad (4.55)$$

This finishes the proof. $\blacksquare$
Proof of Theorem 3.7. (i): Choose \((\bar{A}, \bar{b}) \in G_0\), and suppose that \((A_0 + b_0, f(\bar{A}, \bar{b}, c_0), x(0))\) is nonreachable. Choose an open neighborhood \(W\) of \((A, b)\) in \(G_0\). By Lemma 4.5 there exists an open neighborhood \(V\) of \(f(\bar{A}, \bar{b}, c_0)\) such that for every \(\tilde{f} \in V\), the unique pair \((A, b) \in G_0\) with \(f(A, b, c_0) = \tilde{f}\) has the property that \((A, b) \in W\). Choose \(\tilde{f} \in V\) such that \((A_0 + b_0, \tilde{f}, x(0))\) is reachable, and it follows that \(G_0\) is dense in \(G_0\). Since \(\tilde{G}\) is the complement of the zero set of a continuous function, it follows that \(\tilde{G}_0\) is also open in \(G_0\).

(ii): Choose \((A, b) \in G\), and suppose that \(V(A, b) \neq \mathbb{R}^n\). Choose an open neighborhood \(W\) of \((A, b)\) in \(G\). From the proof of Theorem 3.5 it follows that there exists a nonsingular matrix \(S\) such that \((SAS^{-1}, Sb) \in G_0\). The function \(\phi\) as defined by (4.40) is continuous, and hence there exists an open neighborhood \(V\) of \((SAS^{-1}, Sb)\) in \(G_0\) such that \(\phi(V) \subset W\). By part (i) we know that \(W \cap \tilde{G}_0 \neq \emptyset\). It is not difficult to check that this implies that \(W \cap \tilde{G} \neq \emptyset\), which shows that \(\tilde{G}\) is dense in \(G\). Also, since \(\tilde{G}\) is the complement of the zero set of a collection of polynomials, \(\tilde{G}\) is open in \(G\). ■

Lemma 4.6. There exists a \(C^\infty\) function \(K: E \to P\) such that \(K(A, b, c)\) satisfies (2.6) for all \((A, b, c) \in E\).

Proof. A proof for the continuous-time case can be found in [2]; for the discrete-time case, the reader is referred to [7, 8]. ■

Corollary 4.7. \(f\) is a \(C^\infty\) function on \(E\).

Proof. This is immediate from the fact that \(f\) is a \(C^\infty\) function of \((A, b, c, K)\) and Lemma 4.6. ■

Proof of Theorem 3.8. By Theorem 3.7, \(\tilde{G}_0\) is nonempty. Define \(\tilde{G}'_0 \subset \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{n(n+1)/2}\) by

\[
\tilde{G}'_0 := \{(A, b, K) | (A, b) \in \tilde{G}_0, K = K(A, b) \in P\}.
\]

Define \(L: E_{ns} \times P \to \mathbb{R}^{n(n+1)/2} \times \mathbb{R}^{n \times n}\) by

\[
L(A, b, K) = (L_1(A, b, K), L_2(A, b, K)),
\]

(4.56)
where

\[ L_1(A, b, K) = K - A^T K A + A^T K b (b^T K b + r)^{-1} b^T K A - c_0 r c_0, \]  
\( \text{(4.57)} \)

\[ L_2(A, b, K) = \begin{bmatrix} A - b (b^T K b + r)^{-1} b^T K A \end{bmatrix} - \begin{bmatrix} A_0 - b_0 (b^T K b + r)^{-1} b^T K A \end{bmatrix}. \]  
\( \text{(4.58)} \)

Note that \((A, b, K) \in \mathcal{C}'_0 \) if and only if \( L(A, b, K) = (0, 0) \), and that \( L \) is \( C^\infty \).

Fix a triple \((\tilde{A}, \tilde{b}, \tilde{K}) \in \mathcal{C}'_0 \), and let \( \tilde{f} = f(\tilde{A}, \tilde{b}, c_0) \). We will show that the derivative of \( L \) with respect to \((A, b, K)\), evaluated in \((\tilde{A}, \tilde{b}, \tilde{K})\), has full rank. The derivative of \( L \) evaluated in \((\tilde{A}, \tilde{b}, \tilde{K})\) is a linear map \( \Lambda \) given by

\[ \Lambda_1(\Delta A, \Delta b, \Delta K) = \Delta K - (\tilde{A} + \tilde{b} \tilde{f})^T \Delta K (\tilde{A} + \tilde{b} \tilde{f}) - \Delta A^T \tilde{K} (\tilde{A} + \tilde{b} \tilde{f}) \]  
\[ + \tilde{b}^T \tilde{K} \Delta A + \tilde{b} \tilde{f}^T \Delta b^T \tilde{K} \tilde{A} \]  
\[ - \tilde{f}^T (\Delta b^T \tilde{K} \tilde{b} + \tilde{b}^T \tilde{K} \Delta b) \tilde{f}, \]  
\( \text{(4.59)} \)

\[ \Lambda_2(\Delta A, \Delta b, \Delta K) = \Delta A - \tilde{b} (\tilde{b}^T \tilde{K} \tilde{b} + r)^{-1} (\Delta b^T \tilde{K} \tilde{A} + \tilde{b}^T \Delta K \tilde{A} + \tilde{b}^T \tilde{K} \Delta A) \]  
\[ - \Delta b (\tilde{b}^T \tilde{K} \tilde{b} + r)^{-1} \tilde{b}^T \tilde{K} \tilde{A} \]  
\[ - \tilde{b} (\tilde{b}^T \tilde{K} \tilde{b} + r)^{-1} (\Delta b^T \tilde{K} \tilde{b} + \tilde{b}^T \Delta K \tilde{b} + \tilde{b}^T \tilde{K} \Delta b) \tilde{f} \]  
\[ + b_0 (\tilde{b}^T \tilde{K} \tilde{b} + r)^{-1} (\Delta b^T \tilde{K} \tilde{A} + \tilde{b}^T \Delta K \tilde{A} + \tilde{b}^T \tilde{K} \Delta A) \]  
\[ + b_0 (\tilde{b}^T \tilde{K} \tilde{b} + r)^{-1} (\Delta b^T \tilde{K} \tilde{b} + \tilde{b}^T \Delta K \tilde{b} + \tilde{b}^T \tilde{K} \Delta b) \tilde{f}. \]  
\( \text{(4.60)} \)

Let \((M, N) \in \mathbb{R}^{(n+1)/2} \times \mathbb{R}^{n \times n} \). We will calculate the inner product of
$\Lambda(\Delta A, \Delta b, \Delta K)$ with $(M, N)$ in order to establish a formula for its adjoint:

\[
\Lambda(\Delta A, \Delta b, \Delta K) \cong (M, N)\]

\[
\begin{align*}
\Lambda(\Delta A, \Delta b, \Delta K) & \cong (M, N) \\
& = \text{Tr}(\Lambda_1(\Delta A, \Delta b, \Delta K)M) + \text{Tr}(\Lambda_2(\Delta A, \Delta b, \Delta K)N^T) \\
& = \text{Tr}\left(\Delta K \left[ M - (\bar{A} + \bar{b}\bar{f})M(\bar{A} + \bar{b}\bar{f})^T ight. \right. \\
& \left. \quad - \bar{A}N^Tb(\bar{b}^T\bar{K}\bar{b} + r)^{-1}\bar{b}^T + \bar{A}N^Tb_0(\bar{b}^T\bar{K}\bar{b} + r)^{-1}\bar{b}^T \right) \right] \\
& \left. + \text{Tr}\left(\Delta A \left[ -2M(\bar{A} + \bar{b}\bar{f})^TK + N^T \right. \right. \right. \\
& \left. \quad - N^Tb(\bar{b}^T\bar{K}\bar{b} + r)^{-1}\bar{b}^T + N^Tb_0(\bar{b}^T\bar{K}\bar{b} + r)^{-1}\bar{b}^T \right) \right] \\
& \left. + \text{Tr}\left(\Delta b \left[ -2\bar{f}M\bar{A}^TK - 2\bar{f}M\bar{f}^T\bar{b}^TK - (\bar{b}^T\bar{K}\bar{b} + r)^{-1}\bar{b}^TN\bar{A}^TK \right. \right. \right. \\
& \left. \quad + \bar{f}N^T(\bar{b}^T\bar{K}\bar{b} + r)^{-1}\bar{b}^TN\bar{f}^T\bar{b}^TK \right. \\
& \left. \quad - \bar{f}N^Tb(\bar{b}^T\bar{K}\bar{b} + r)^{-1}\bar{b}^T + (\bar{b}^T\bar{K}\bar{b} + r)^{-1}b_0^TN\bar{A}^TK \right) \\
& \left. \quad + (\bar{b}^T\bar{K}\bar{b} + r)^{-1}b_0^TN\bar{f}^TbK + \bar{f}N^Tb_0(\bar{b}^T\bar{K}\bar{b} + r)^{-1}\bar{b}^TK \right] \right), \quad (4.62)
\end{align*}
\]

Hence the adjoint map of $\Lambda$ is given by $\Lambda^* = (\Lambda_1^*, \Lambda_2^*, \Lambda_3^*, \Lambda_4^*)$, where

\[
\begin{align*}
\Lambda_1^*(M, N) &= M - (\bar{A} + \bar{b}\bar{f})M(\bar{A} + \bar{b}\bar{f})^T \\
& \quad - \bar{b}(\bar{b}^T\bar{K}\bar{b} + r)^{-1}\bar{b}^TN\bar{A}^T - \bar{b}(\bar{b}^T\bar{K}\bar{b} + r)^{-1}\bar{b}^TN\bar{f}^T\bar{b}^T \\
& \quad + \bar{b}(\bar{b}^T\bar{K}\bar{b} + r)^{-1}b_0^TN\bar{A}^TK + \bar{b}(\bar{b}^T\bar{K}\bar{b} + r)^{-1}b_0^TN\bar{f}^T\bar{b}^T, \quad (4.63)
\end{align*}
\]

\[
\begin{align*}
\Lambda_2^*(M, N) &= -2\bar{K}(\bar{A} + \bar{b}\bar{f})M + N \\
& \quad - \bar{K}\bar{b}(\bar{b}^T\bar{K}\bar{b} + r)^{-1}\bar{b}^TN + \bar{K}\bar{b}(\bar{b}^T\bar{K}\bar{b} + r)^{-1}b_0^TN, \quad (4.64)
\end{align*}
\]

\[
\begin{align*}
\Lambda_3^*(M, N) &= -2\bar{K}(\bar{A} + \bar{b}\bar{f})M\bar{f}^T + N\bar{f}^T \\
& \quad - \bar{K}\bar{b}(\bar{b}^T\bar{K}\bar{b} + r)^{-1}\bar{b}^TN\bar{f}^T \\
& \quad - \bar{K}(\bar{A} + \bar{b}\bar{f})N^Tb(\bar{b}^T\bar{K}\bar{b} + r)^{-1} \\
& \quad + \bar{K}(\bar{A} + \bar{b}\bar{f})N^Tb_0(\bar{b}^T\bar{K}\bar{b} + r)^{-1}. \quad (4.65)
\end{align*}
\]
To show that $\Lambda^*$ is injective, we put $\Lambda^*(M, N) = 0$, which gives the following equations:

$$E_1: M - (\bar{A} + \bar{b} \tilde{f}) M (\bar{A} + \bar{b} \tilde{f})^T - \bar{b} (\bar{b}^T \bar{Kb} + r)^{-1} \bar{b}^T N \bar{A}^T$$

$$- \bar{b}(\bar{b}^T \bar{Kb} + r)^{-1} \bar{b}^T N \bar{f}^T \bar{b}^T + \bar{b}(\bar{b}^T \bar{Kb} + r)^{-1} \bar{b}^T N \bar{A}^T$$

$$+ \bar{b}(\bar{b}^T \bar{Kb} + r)^{-1} \bar{b}^T N \bar{f}^T \bar{b}^T = 0,$$

(4.66)

$$E_2: - 2 \bar{K}(\bar{A} + \bar{b} \tilde{f}) M + N - \bar{K} \bar{b}(\bar{b}^T \bar{Kb} + r)^{-1} \bar{b}^T N$$

$$+ \bar{K} \bar{b}(\bar{b}^T \bar{Kb} + r)^{-1} \bar{b}^T N = 0,$$

(4.67)

$$E_3: - 2 \bar{K}(\bar{A} + \bar{b} \tilde{f}) M \bar{f}^T + N \bar{f}^T - \bar{K} \bar{b}(\bar{b}^T \bar{Kb} + r)^{-1} \bar{b}^T N \bar{f}^T$$

$$- \bar{K}(\bar{A} + \bar{b} \tilde{f}) N^T b(\bar{b}^T \bar{Kb} + r)^{-1} + \bar{K}(\bar{A} + \bar{b} \tilde{f}) N^T b_0(\bar{b}^T \bar{Kb} + r)^{-1} = 0.$$  

(4.68)

$$E_3 - E_2 \bar{f}^T$$ gives

$$- \bar{K}(\bar{A} + \bar{b} \tilde{f}) N^T b(\bar{b}^T \bar{Kb} + r)^{-1} + \bar{K}(\bar{A} + \bar{b} \tilde{f}) N^T b_0(\bar{b}^T \bar{Kb} + r)^{-1} = 0.$$  

(4.69)

Substituting (4.69) in $E_1$ gives

$$M - (\bar{A} + \bar{b} \tilde{f}) M (\bar{A} + \bar{b} \tilde{f})^T = 0.$$  

(4.70)

By Lemma 4.1 and the asymptotic stability of $\bar{A} + \bar{b} \tilde{f}$ we conclude that $M = 0$. Since $\bar{K}(\bar{A} + \bar{b} \tilde{f})$ is nonsingular, (4.69) implies that $- N^T \bar{b}(\bar{b}^T \bar{Kb} + r)^{-1} + N^T b_0(\bar{b}^T \bar{Kb} + r)^{-1} = 0$. Substituting this and $M = 0$ in $E_2$ gives

$$N = 0.$$  

(4.71)

This shows that $\tilde{G}_0'$ is an $n$-dimensional manifold in $\mathbb{R}^{n \times n} \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{n(n+1)/2}$. Since $\tilde{K}$ depends $C^\omega$ on $(\bar{A}, \bar{b})$, it follows that $\tilde{G}_0'$ is an $n$-dimensional $C^\omega$ manifold in $\mathbb{R}^{n \times n} \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{1 \times n}$. This completes the proof. 

$\blacksquare$
Lemma 4.8. Let \((A, b) \in G_0\), denote the solution of (2.6) by \(K\), and let \(K_0\) be the solution of (2.6) with \((A, b, c_0)\) replaced by \((A_0, b_0, c_0)\). Then \(K \geq K_0\).

Proof. Let \(x_0 \in \mathbb{R}^n\). The optimal cost for the system \((A, b, c_0)\) starting in \(x_0\) is \(x_0^TKx_0\); the optimal cost for \((A_0, b_0)\) is \(x_0^TK_0x_0\). The real cost incurred when the feedback \(f(A, b, c_0)\) is applied to the system \((A_0, b_0)\) is equal to the optimal cost of the system \((A, b, c_0)\), since \((A, b) \in G_0\) and hence both the state and the input trajectories of \(A + bf(A, b, c_0)\) and \(A_0 + b_0f(A, b, c_0)\) are equal. However, for \((A_0, b_0)\), \(f(A, b, c_0)\) can do no better than \(f(A_0, b_0, c_0)\). Hence \(x_0^TKx_0 \geq x_0^TK_0x_0\). Since \(x_0\) was arbitrary, it follows that \(K \geq K_0\). \(\blacksquare\)

Corollary 4.9. If \((A, b) \in G_0\) and \(f(A, b, c_0) = f(A_0, b_0, c_0)\), then \(K = K_0\).

Proof. Since \((A, b) \in G_0\), we have \(A + bf(A, b, c_0) = A_0 + b_0f(A_0, b_0, c_0)\), which by Lemma 4.8 implies that \(K \geq K_0\). On the other hand, since \(f(A, b, c_0) = f(A_0, b_0, c_0)\), we also have \(A_0 + b_0f(A_0, b_0, c_0) = A + bf(A_0, b_0, c_0)\). We can apply Lemma 4.8 once again, now with \((A_0, b_0, c_0)\) and \((A, b, c_0)\) interchanged, showing that \(K_0 \geq K\). \(\blacksquare\)

Proof of Theorem 3.11. Choose \((A, b) \in G \cap H\). Define \((\bar{A}, \bar{b}) \in E_{ns}\) by

\[
(\bar{A}, \bar{b}) := \psi(A, b) \tag{4.72}
\]

with \(\psi\) defined as in (4.47).

Then \((\bar{A}, \bar{b}) \in C_0\), and also, since \((A, b) \in H\), \(f(\bar{A}, \bar{b}, c_0) = f(A_0, b_0, c_0)\). Hence by Corollary 4.9, \(\bar{K} = K_0\). Now

\[
(\bar{A}, \bar{b}) \in C_0 \quad \Rightarrow \quad \bar{A} = A_0 + (b_0 - \bar{b})\bar{f} \tag{4.73}
\]

\[
= A_0 + (b_0 - \bar{b})f_0, \tag{4.74}
\]

\[
f(\bar{A}, \bar{b}, c_0) = f(A_0, b_0, c_0) \quad \Rightarrow \quad (\bar{b}^TK_0\bar{b} + r)^{-1}\bar{b}^TK_0\bar{A} = -f_0. \tag{4.75}
\]

Substituting (4.74) in (4.75) gives

\[
\bar{b}^TK_0(\bar{A}_0 + (b_0 - \bar{b})f_0) = -(\bar{b}^TK_0\bar{b} + r)f_0, \tag{4.76}
\]
which implies
\[ b^T K_0 (A_0 + b_0 f_0) = -rf_0. \] (4.77)

Now, since \( K_0 \) and \( A_0 + b_0 f_0 \) are nonsingular, and \( \bar{b} = b_0 \) is by construction a solution of (4.77), it follows that \( \bar{b} = b_0 \). Substituting this in (4.73) gives \( \bar{A} = A_0 \). Now \( \bar{A} = SAS^{-1} \) for some \( S \in \text{Gl}(n) \). Hence \((A, c_0)\) and \((SAS^{-1}, c_0)\) are in standard observable form. This implies that \( S = I \). This completes the proof of Theorem 3.11(i).

The proof of part (ii) is immediate.

REFERENCES


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