ON THE ZEROS OF INFINITELY DIVISIBLE DENSITIES

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1.

Introduction. Making use of a representation theorem for infinitely divisible (inf div) distributions on the nonnegative integers, which is implicit in [3], and its continuous analogue, which is implicit in [5], some properties are proved regarding the zeros of inf div probability density functions (pdf's) on [0, \( \infty \)), both in the discrete and in the continuous case.

2.

Representation theorems.

Theorem 1. A probability distribution \( \{p_n\} \) on the nonnegative integers, with \( p_0 > 0 \), is inf div if and only if

\[
np_n = \sum_{j=0}^{n-1} j p_{j-1},
\]

where the \( q \)'s satisfy

\[
q_j \geq 0 \quad (j = 0, 1, 2, \cdots); \quad \sum_{j=1}^{\infty} j^{-1} q_j < \infty.
\]

Proof. From Feller [1] (page 270 seq.) one easily obtains, that \( \{p_n\} \) is inf div if and only if its generating function (pgf) \( P(z) \) is of the form

\[
P(z) = \exp \{-\lambda (1 - R(z))\} \quad (|z| \leq 1),
\]

where \( \lambda > 0 \) and \( R(z) \) is the pgf of some distribution \( \{r_n\} \) on the nonnegative integers. Equivalently we have, taking logarithmic derivatives,

\[
P'(z) = P(z)Q(z) \quad (|z| < 1),
\]

where \( Q(z) = \lambda R'(z) \).

Again equivalently,

\[
np_n = \sum_{j=0}^{n-1} j p_{j-1},
\]

where \( q_n = \lambda (n+1)r_{n+1} \), with \( \sum_{n=1}^{\infty} (n+1)^{-1} q_n = \lambda (1 - r_0) \).

In the same way for general distributions on \([0, \infty)\) we have

Theorem 2. A distribution function (df) \( F(x) \) on \([0, \infty)\) is inf div if and only if it satisfies

\[
\int_0^x u dF(u) = \int_0^x F(x-u) dP(u),
\]

where \( P \) is non-decreasing, and

\[
\int_1^{\infty} x^{-1} dP(x) < \infty.
\]

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\(^1\) Now at Twente Institute of Technology, Enschede, Netherlands.
Proof. According to Feller [2] the Laplace transform \( \tilde{F}(\tau) \) of a df on \([0, \infty)\) is inf div if and only if
\[
\tilde{F}(\tau) = \exp \left\{ -\int_0^\infty x^{-1}(1-e^{-\tau x}) \, dP(x) \right\},
\]
where \( P \) is non-decreasing and satisfies (4). Taking logarithmic derivatives and using the convolution theorem yields (3), as we have
\[
-d\tilde{F}(\tau)/d\tau = \int_0^\infty xe^{-\tau x} \, dF(x).
\]

**Corollary.** The pdf \( f(x) \) of a distribution on \([0, \infty)\) is inf div if and only if
\[
x f(x) = \int_0^x f(x-u) \, dP(u),
\]
where \( P \) is non-decreasing and satisfies (4).

Proof. This follows by writing \( F(u) = \int_0^u f(t) \, dt \) and changing the order of integration in (3).


**Lemma 1.** If \( \{p_n\} \) is an inf div distribution on the nonnegative integers, with \( p_0 > 0 \), then
\[
\begin{align*}
p_a &> 0 \\
q_{b-1} &> 0
\end{align*}
\rightarrow p_{a+b} > 0.
\]

Proof. \((a+b)p_{a+b} \geq p_a q_{b-1} > 0\), hence \( p_{a+b} > 0 \).

**Theorem 3.** If \( \{p_n\} \) is an inf div distribution on the nonnegative integers, with \( p_0 > 0 \), then
\[
\begin{align*}
p_a &> 0 \\
p_b &> 0
\end{align*}
\rightarrow p_{a+b} > 0.
\]

Proof. As \( bp_b = \sum_{j=0}^{b-1} p_j q_{b-j-1} \), there is a \( j_0 \), with \( 0 \leq j_0 < b \), such that \( p_{j_0} > 0 \) and \( q_{b-j_0-1} > 0 \). It follows by Lemma 1 that \( p_{a+b-j_0} > 0 \). There are two possibilities.

Case 1. \( q_{j_0-1} > 0 \) and hence by (6), \( p_{a+b} > 0 \).

Case 2. \( q_{j_0-1} = 0 \). Then, as \( p_{j_0} > 0 \), there is a \( j_1 \), with \( 0 \leq j_1 < j_0 \), such that \( p_{j_1} > 0 \) and \( q_{j_0-j_1-1} > 0 \). It follows that \( p_{a+b-j_1} > 0 \). Again there are two cases.

Case 2.1. \( q_{j_1-1} > 0 \) and hence by (6) \( p_{a+b} > 0 \).

Case 2.2. \( q_{j_1-1} = 0 \). Then, as \( p_{j_1} > 0 \), there is a \( j_2 \), with \( 0 \leq j_2 < j_1 \), such that \( p_{j_2} > 0 \) and \( q_{j_1-j_2-1} > 0 \). It follows that \( p_{a+b-j_2} > 0 \).

Proceeding in this way, in a finite number of steps we reach the situation that \( p_{a+b-j_m} > 0 \), and \( q_{j_m-1} > 0 \) or \( j_m = 0 \). Hence \( p_{a+b} > 0 \).

**Corollary.** If \( \{p_n\} \) is an inf div distribution on the nonnegative integers, \( p_0 > 0 \), then \( p_1 > 0 \rightarrow p_k > 0 \) \((k = 0, 1, 2, \ldots)\).
Remark. Theorem 3 can also be proved by direct application of the definition of infinite divisibility, without use of Theorem 1.

4. Zeros of densities.

Theorem 4. If \( f(x) \) is a continuous and \( \text{inf div} \) density on \( (0, \infty) \), then

\[
f(x_0) = 0 \rightarrow \{ f(x) = 0 \ (x \leq x_0) \}.
\]

Proof. It is no restriction (this can be achieved by a shift) to assume that for every \( \delta > 0 \) there is an \( x_1 < \delta \) such that \( f(x_1) > 0 \). We now have to prove that \( f(x) \) has no zeros in \( (0, \infty) \). Suppose, therefore, that \( f(x_1) > 0 \) and \( f(x_0') = 0 \) with \( x_0' > x_1 \). Then by the continuity of \( f(x) \) there is a smallest number \( x_0 \) satisfying \( x_0 > x_1 \) and \( f(x_0) = 0 \). By (5) we have

\[
0 = x_0 f(x_0) = \int_0^{x_0} f(x_0 - u) \, dP(u).
\]

As \( f(x) > 0 \) for all \( x \) with \( x_1 \leq x < x_0 \), it follows that \( \int_0^{x_0 - x_1} dP(u) = 0 \), and hence that \( \int_{x_0 - x_1}^{x_0} f(x - u) \, dP(u) = 0 \) for all \( x < x_0 - x_1 \). Therefore, by (5), \( xf(x) = f(x)P(0) \) for all \( x < x_0 - x_1 \). It follows from the continuity of \( f \) that \( f(x) = 0 \) for all \( x < x_0 - x_1 \). As this contradicts our assumption, it follows that \( x_0 \) does not exist and that \( f(x) \neq 0 \) for \( x > 0 \). This proves the theorem.

Corollary. An \( \text{inf div pdf} \) on \( (0, \infty) \), which is continuous on \( (0, \infty) \) and positive on \( (0, \delta) \) for some \( \delta > 0 \), has no zeros on the positive half-line.

It does not seem easy to extend the argument of Theorem 4 to pdf’s on \( (-\infty, \infty) \): if \( \phi(t) \) is the characteristic function (ch.f.) of a pdf \( f(x) \), having a representation of the form

\[
\phi(t) = \exp \int_{-\infty}^{\infty} (e^{itx} - 1) \, d\theta(x),
\]

where \( \theta \) is non-decreasing, then the analogue of (5) becomes (if differentiation is possible)

\[
xf(x) = \int_{-\infty}^{\infty} f(x-u)u \, d\theta(u),
\]

where however \( u \, d\theta(u) \) is not a measure. Theorem 4 provides a generalization of the Corollary to Theorem in [4], if \( \phi \) is the ch.f. of a pdf on \( (0, \infty) \) and if \( \phi \) is not integrable.

5. Examples. Examples of pdf’s which are not \( \text{inf div} \) by the Corollary to Theorem 4 are

1. \( f(x) = 6(e^{-x} - 2e^{-2x})^2 \) (cf. [6]).

2. \( f_\alpha(x) = \frac{1}{24} \exp \left(-x^2\right)(1-x \sin x^2) \) for \( \alpha = 1 \).

\( f_\alpha(x) \) is inf div (see [5]). It follows from the closure property of inf div distributions, that \( f_\alpha \) cannot be inf div for all \( \alpha \), with \( 0 \leq \alpha < 1 \), as this would imply that \( f_1(x) \) is inf div. The pdf \( f_\alpha \) has the same moments for all \( \alpha \) (cf. [2], page 224).
From the representation theorems it easily follows that Const. \( \{q^n p_n\} \) is inf div if \( \{p_n\} \) is inf div. In the same way if \( f(x) \) is inf div, then Const. \( e^{-\lambda x} f(x) \) is inf div.

REFERENCES


