ANALYTICAL RENORMALIZATION RESULTS FOR THE CROSS-OVER BEHAVIOR OF PERIOD DOUBLING, FROM CONSERVATIVE TO DISSIPATIVE SYSTEMS

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Extended abstract

It has been shown that there is a universal scaling function describing the cross-over of the effective Feigenbaum convergence rate $\delta$ from its conservative value ($\delta = 8.721097\ldots$) to its dissipative value ($\delta = 4.669201\ldots$), as a function of the "effective dissipation". Using renormalization theory I obtain an explicit analytical expression for this cross-over function and show that it's not monotonic but has a minimum, just before it reaches its asymptotic dissipative value. I also derive an analytical expression for the (period-doubling) bifurcation values in a particular map (the Hénon map), at all values of the Jacobian.

1. Introduction

Many systems display infinite series of period-doubling bifurcations [1]. Using renormalization theory I derive explicit expressions for the dependence, of the parameter values at bifurcation and their "effective" rate of convergence $\delta$ (i.e. at the $n$th bifurcation), on $B$ and $n$, for planar maps of constant Jacobian $B$.

Zisook has shown that there is a universal scaling function describing the cross-over of the effective rate of convergence $\delta$ from its conservative value ($\delta = 8.721097\ldots$) to its dissipative value ($\delta = 4.669201\ldots$), as a function of the "effective dissipation", i.e. of $B_e = B^{2^n}$. Using a technique of Ghendrih and renormalization theory, I obtain an analytical expression for this scaling function and show that it's not monotonic but has a minimum at $B_{e, \text{min}} \approx 3 \times 10^{-6}$. The true existence of this minimum has been demonstrated numerically for the Hénon map [3]. Finally, an expression for the bifurcation values of a particular map, the Hénon map, is derived. A more extended version of this work, including results on the orbit-scaling factor $\alpha$, are presented in ref. 4.

2. The universal rate of convergence $\delta$

The behavior of almost every planar mapping, of constant Jacobian $B$, can be approximated locally by a quadratic mapping that can be brought into the standard form [5]

$$y_{t+1} + B y_{t-1} = 2C y_t + 2 y_t^2, \quad t = 0, 1, 2, \ldots$$

the two-dimensional Hénon map. The parameter values $C_n$ at which a period $2^n$ is born are determined in first-order renormalization theory by

$$C_{n-1}(B^2) = -2C_n^2(B) + 2(1 + B)C_n(B) + 2B^2 + 3B + 2, \quad C_0(B) = \frac{1}{2} + \frac{1}{2}B.$$

(2)

The effective rate of convergence is defined as

$$\delta_n(B) = \frac{C_{n-1}(B) - C_n(B)}{C_n(B) - C_{n+1}(B)}.$$

(3)

Using (2) it can be shown that for large $n$ the effective rate of convergence $\delta_n(B)$ depends only on $B_e = B^{2^n}$ [2] and that this universal function
can be approximated by

\[ \delta_n(B) = \delta(B_e) \]

\[ = \left\{ \left[ 6 + 8\sqrt{B_e} + 6\sqrt{B_e} \right]^{1/2} - 2 - 2\sqrt{B_e} \right\} \]

\[ / \left\{ \left[ 4 + 8\sqrt{B_e} + 4\sqrt{B_e} + \left[ 6 + 8\sqrt{B_e} + 6\sqrt{B_e} \right]^{1/2} \right]^{1/2} \]

\[ - \left[ 6 + 8\sqrt{B_e} + 6\sqrt{B_e} \right]^{1/2} \} \]

which I derive in ref. 4, cf. also ref. 6. The function \( \delta(B_e) \) is not monotonic but has a minimum at \( B_e = 3 \times 10^{-6} \). This can be seen from fig. 1, where \( \delta(B_e) \) is plotted at small \( B_e \). This behavior agrees with numerical simulation [3].

3. The bifurcation values \( C_n(B) \) for the Hénon map

Defining the distance to the fixed point,

\[ D_n(B) = C_n(B) - C_\infty(B), \]  

the linearization of eq. (2) about its fixed point is

\[ D_n(B) = k(B) D_{n-1}(B^2), \]  

where

\[ k(B) = \left( -4C_\infty(B) + 2 + 2B \right)^{-1}. \]  

Iterating eq. (6) we get

\[ D_n(B) = k(B) k(B^2) \cdots k(B^{2^{n-2}}) D_1(B^{2^{n-1}}). \]  

The product in eq. (8) can be simplified introducing a function \( f(B) \) such that

\[ k(B) = f(B) / f(B^2). \]  

Eq. (8) then becomes

\[ D_n(B) = f(B) / f(B^{2n-1}) D_1(B^{2^{n-1}}). \]  

Since \( D_1 \) can be found (as a Taylor series) using eq. (2), it remains to determine \( f(B) \). In order to do this we split off two factors that are non-analytic at \( B = 1 \) and \( B = 0 \), respectively. The remaining factor is analytic and is found recursively in a Taylor series using (9), (7) and (2). The \( C_n(B) \) are then found for all \( n \) and \( B \), using eq. (5), cf. ref. 4.

References