ANALYTICAL DESCRIPTION OF THE BASIN AND THE TRANSIENTS OF A POINT ATTRACTOR OF THE HÉNON MAPPING

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Extended abstract

Explicit expressions for closed curves $C_n$ about the attractor that enclose a part of its basin are obtained. If $n \to \infty$ the interior of $C_n$ covers the complete basin. A sequence of invertible polynomial transformations is given, that converges rapidly towards the normal transformation, that transforms the nonlinear motion in the basin to a linear one. Using these transformations we obtain some practical tools to describe one aspect of the transient behavior in the basin.

1. Results

Consider a quadratic invertible mapping of the real plane, $x \to x'$, with

$$x' = Hx, \quad Hx = Px + Q(x),$$

with constant positive Jacobian $b$, smaller than unity. $Q(x)$ is a homogeneous quadratic expression in the components $x_1$ and $x_2$. The matrix $P$ is assumed to have two different complex eigenvalues, such that the origin is a spiral attractor.

We obtain a sequence of polynomial expressions, representing closed curves $C_n$ about the origin, which enclose a part of its basin of attraction. For each enclosed domain we shall obtain a polynomial Lyapunov function $L_n(x)$, which controls the rate of convergence,

$$L_n(Hx) \leq \theta_n(L_n(x))L_n(x),$$

$$\theta_n(L_n) \equiv b^{1/2} + b^{1/2}qL_n,$$

where $q$ is some positive constant representing the 'strength' of the nonlinear term (cf. sec. 2). At sufficiently large $n$, each $x$ in the basin is enclosed by a contour $C_n$. Fig. 1 shows some contours for the Hénon mapping [1].

To describe the transient behavior we introduce a functional $n_x(x)$ that represents the number of steps necessary to map $x$ into a well defined $\epsilon$-neighborhood of the origin: below a Lyapunov function $L(x)$ is defined, for which

$$L(Hx) = b^{1/2}L(x)$$

for each $x$ in the basin. Eq. (3) shows that a level line $L(x) = L_0$ is mapped onto the level line $L(x) = b^{1/2}L_0$. Thus the number of steps, necessary to map $x$ into a small region given by $L(x) \leq \epsilon$, depends on $L$ only and is

$$n_x(x) \equiv \text{INT} \left[ \frac{(\ln L(x) - \ln \epsilon) / \ln b^{1/2}}{L_0} \right],$$

i.e. the integer part of the expression in brackets.

The irregular shape of the area enclosed by a level line (cf. fig. 2) indicates irregular transient behavior. One aspect of this behavior is formulated more precisely in terms of the gradient of $L$: in a (sufficiently) small neighborhood of some $x_0$ the average value of $n_x(x)$ equals $n_x(x_0)$ and one readily shows that its root mean square is proportional to the length of $(\nabla L/L)_{x=x_0}$ (cf. fig. 3). Both for the level lines and the gradient we have explicit approximate expressions.

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2. Method: transformation to normal form

To obtain the results above a sequence of invertible coordinate transformations \( x_n(u) \), with inverse \( u_n(x) \),

\[
\begin{aligned}
x_n(u) &\equiv H^{-n}(P^n u), \\
u_n(x) &\equiv P^{-n}H^n(x),
\end{aligned}
\]

is introduced. It can be proved [2] that the sequence \( \{x_n(u)\} \) converges to an analytic function \( x(u) \), that is defined for \( u \in \mathbb{R}^2 \) and whose range is the basin of attraction of the origin. Furthermore, \( x(u) \) transforms the mapping \( x' = H(x) \) restricted to this basin, to the linear mapping \( u' = Pu \). The existence of this function and its inverse \( u(x) \) is guaranteed by an easy extension of a theorem of Poincaré [3].

With these transformations the functionals \( L_n \) and \( L \) are defined,

\[
\begin{aligned}
L_n(x) &\equiv (u_n(x), Au_n(x))^{1/2}, \\
L(x) &\equiv (u(x), Au(x))^{1/2},
\end{aligned}
\]

where \( A \) is a real symmetric positive matrix such that \( (Pu, APu) = b(u, Au) \). Such an \( A \) exists since
\( \dot{P} \) has complex eigenvalues. One readily proves (2) and (3), with \( q \) being the smallest positive number such that \( \langle Q(x), AQ(x) \rangle \leq q^2 \langle x, Ax \rangle \).

Furthermore, observe that a level line \( L_n(x) = L_0 \) is the image in the \( x \) plane of an ellipse \( (u, Au) = L_0 \) in the \( u \) plane by the mapping \( x_n(u) \). Such an ellipse is parametrized by \( u = L_0^{1/2} (a_1 \cos \phi + a_2 \sin \phi), \) \( a_1 \) and \( a_2 \) being eigenvectors of \( A \) of appropriate length. Consequently we have explicit expressions for the level lines of \( L_n \). Since \( L_n(x) \) has only one stationary point, which is a minimum at the origin, it is clear from (2) that curves \( C_n \) for which \( \theta_n = 1 \) enclose a domain in the basin of attraction. It can be proved that a level line \( L_n(x) = L_0 \) converges to a level line \( L(x) = L_0 \) [2], and that \( \nabla L_n(x) \) along a level line converges to \( \nabla L(x) \).

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References

[2] T.P. Valkering, to be published. More details of the present work and proofs will be presented in this paper.