ON MAXIMUM CRITICALLY $h$-CONNECTED GRAPHS

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Let $h$ be an integer with $h \geq 2$. A graph $G$ is called critically $h$-connected or $h$-critical if $G$ is $h$-connected while, for every vertex $v$ of $G$, the graph $G - v$ is not $h$-connected. $\mathcal{C}$ denotes the class of all $h$-critical graphs and $\mathcal{A}$ the class of all graphs of $\mathcal{C}$ in which every vertex is adjacent to a vertex of degree $h$. $\mathcal{C}$ and $\mathcal{A}$ are the classes of maximum graphs in $\mathcal{C}$ and $\mathcal{A}$, respectively. Entringer's characterization of $\mathcal{C}$ for $h = 2$ shows that $\mathcal{C} \neq \mathcal{A}$ in case $h = 2$. Here $\mathcal{A}$ is determined for each $h \geq 2$. Then it is shown that $\mathcal{C} = \mathcal{A}$ for $h = 3$ and it is conjectured that $\mathcal{C} = \mathcal{A}$ for each $h \geq 3$.

Terminology

We use [2] for basic terminology and notations, but speak of vertices and edges instead of points and lines. Accordingly we denote the edge set of a graph $G$ by $E(G)$.

If $G$ is a connected graph, then by a cut of $G$ we mean a set of vertices of $G$ whose deletion results in a disconnected graph. If $T_1$ and $T_2$ are cuts of $G$, then $T_1$ interferes with $T_2$ if at least two components of $G - T_1$ contain vertices of $T_2$. An $h$-cut is a cut of $h$ elements. A vertex $v$ of $G$ is critical if $\kappa(G - v) < \kappa(G)$. $G$ is called critically $h$-connected, or briefly $h$-critical, if $\kappa(G) = h$ and every vertex of $G$ is critical.

If $\mathcal{G}$ is a class of graphs, then the elements of $\mathcal{G}$ are called $\mathcal{G}$-graphs. The set of graphs in $\mathcal{G}$ with $n$ vertices is denoted by $\mathcal{G}_n$. $G$ is a maximum $\mathcal{G}$-graph if no $\mathcal{G}$-graph with $|V(G)|$ vertices has more edges than $G$. The set of maximum $\mathcal{G}$-graphs is denoted by $\mathcal{G}_n$. $\mu_{\mathcal{G}}(n)$ is the number of edges of graphs in $\mathcal{G}_n$.

Let $h$ be a fixed integer with $h \geq 2$. By $\mathcal{C}$ we denote the set of all $h$-critical graphs. $\mathcal{A}$ is the subset of $\mathcal{C}$ consisting of all $h$-connected graphs in which every vertex is adjacent to a vertex of degree $h$. The set $\mathcal{B}$ is defined by $\mathcal{B} = \mathcal{C} - \mathcal{A}$. For a graph $G$, $M(G) = \{v \in V(G) \mid \deg_G v = h\}$, $K(G) = V(G) - M(G)$, $\rho(G) = \sum_{e \in M(G)} \deg_{M(G)} e$ and $B(G)$ is the set of edges of $G$ with one end in $K(G)$ and the other in $M(G)$. For the sake of notational simplicity we have chosen not to express the fact that the above notions depend on $h$; unless $h$ is specified, propositions involving the relevant notions will hold for each $h \geq 2$.

We use $[x]$ to denote the greatest integer less than or equal to $x$. 0012-365X/84/$3.00 \textcopyright 1984, Elsevier Science Publishers B.V. (North-Holland)
1. Introduction

Entringer [1] characterized $\tilde{\mathcal{G}}_n$-graphs for $h = 2$ and $n \geq 3$. In the proof of his characterization, which is by induction on $n$, he uses an upper bound for $\mu_{\tilde{\mathcal{G}}}(n)$ [1, Lemma 2]. It appears that, for $h = 2$, there are infinitely many $\tilde{\mathcal{G}}$-graphs which are not $\mathcal{H}$-graphs.

Here we first determine $\mathcal{A}$ for each $h \geq 2$ (Section 2). Then for $h = 3$ it is proved, also by induction on $n$, that $\mu_{\mathcal{A}}(n) < \mu_{\tilde{\mathcal{G}}}(n)$ for all $n$, so that, in consequence, $\hat{\mathcal{G}} = \mathcal{A}$ (Section 3). Finally it is conjectured that $\hat{\mathcal{G}} = \mathcal{A}$ for each $h \geq 3$ (Section 4).

2. Characterization of $\mathcal{A}$-graphs

Throughout this section $h$ will be a fixed integer with $h \geq 2$.

Noting that no $h$-connected graph with less than $h + 1$ vertices exists, we first determine $\mathcal{A}_n$ for $h + 1 \leq n \leq 2h$. Define, for $h + 1 \leq n \leq 2h$, the graph $H_n$ as follows:

(a) $V(H_n) = \{v_1, v_2, \ldots, v_n\}$;

(b) $N(v_1) = \{v_2, v_3, \ldots, v_{h+1}\}$;

(c) $N(v_2) = \{v_1, v_{n-h+2}, v_{n-h+3}, \ldots, v_n\}$;

(d) $\langle v_3, v_4, \ldots, v_n \rangle$ is complete.

Clearly, $H_n \in \mathcal{A}_n$.

Lemma 1. If $h + 1 \leq n \leq 2h$, then $\mu_{\mathcal{A}}(n) = \frac{1}{2}(n^2 - 5n + 4h + 4)$ and $\mathcal{A}_n = \{H_n\}$.

Proof. Let $G$ be an $\mathcal{A}_n$-graph with $h + 1 \leq n \leq 2h$. Then $G$ contains, by definition of $\mathcal{A}$, two adjacent vertices $v_1$ and $v_2$ with $\deg v_1 = \deg v_2 = h$. Hence

$$|E(G)| = 2h - 1 + |E(G - \{v_1, v_2\})|$$

$$\leq 2h - 1 + \binom{n-2}{2} = \frac{1}{2}(n^2 - 5n + 4h + 4). \quad (1)$$

Suppose equality holds in (1). Then $G - \{v_1, v_2\}$ is complete; furthermore, since $G$ is $h$-connected and every vertex of $G$ is adjacent to a vertex of degree $h$, one easily deduces that $N(v_1) \cup N(v_2) = V(G)$. These properties determine $G$ up to isomorphism: $G \cong H_n$. \hfill $\square$

We proceed by deriving (for $n \geq 2h + 1$) an upper bound for the number of edges of an $\mathcal{A}_n$-graph $G$ in case $|K(G)|$ has a prescribed value. Let the function $f_n$ be defined by

$$f_n(x) = \frac{1}{2}(x^2 - 2nx + (2h - 1)n).$$

Lemma 2. Let $G$ be an $\mathcal{A}_n$-graph with $|K(G)| = k$. Then $|E(G)| \leq f_n(k)$. Moreover, if $n \geq 2h + 1$ and $k \leq h - 1$, then $|E(G)| \leq f_n(h) - 1$ unless $h = 2$ and $n = 5$. 

On maximum critically $h$-connected graphs

**Proof.** Let $G$ be an $\mathcal{A}_n$-graph with $|K(G)| = k$. Then

$$
|E(G)| = |E(K(G))| + |E(M(G))| + |B(G)| \\
\leq \binom{k}{2} + \frac{1}{2} \rho(G) + \sum_{v \in M(G)} (h - \deg_{M(G)} v) \\
= \frac{1}{2} k(k-1) + h(n-k) - \frac{1}{2} \rho(G). \quad (2)
$$

Since $G \in \mathcal{A}$, $\rho(G) \geq |M(G)| = n-k$. Thus

$$
|E(G)| \leq \frac{1}{2} k(k-1) + h(n-k) - \frac{1}{2} (n-k) = f_n(k),
$$

proving the first part of the lemma.

Now let $n \geq 2h+1$ and assume first that $k \leq h-2$. Then

$$
|E(G)| = \frac{1}{2} \left( \sum_{v \in K(G)} \deg_G v + \sum_{v \in M(G)} \deg_G v \right) \\
\leq \frac{1}{2}(k(n-1) + h(n-k)) = \frac{1}{2}(n-1 \cdot h)k + hn \\
\leq \frac{1}{2}(n-1-h)(h-2) + hn = f_n(h) - \frac{1}{2}(n-h-2) \\
= f_n(h) - 1 \quad \text{unless } h = 2 \text{ and } n = 5.
$$

Assume next that $k = h-1$ (and $n \geq 2h+1$). Then, since $G$ is $h$-connected, $G - K(G)$ is connected, implying that $|E(M(G))| \geq |V(M(G))| = 1$, or, equivalently, $\rho(G) \geq 2(n-h)$. From (2) we deduce that

$$
|E(G)| \leq \frac{1}{2}(h-1)(h-2) + h(n-h+1) - (n-h) \\
= f_n(h) - \frac{1}{2}(n-h-2) \\
= f_n(h) - 1 \quad \text{unless } h = 2 \text{ and } n = 5. \quad \Box
$$

In the following lemma an upper bound for the cardinality of $|K(G)|$ in an $\mathcal{A}_n$-graph $G$ is obtained. Define

$$
k_n = \begin{cases} 
\left[ \frac{h-1}{h} \right] n & \text{if } n \not\equiv h \mod 2h, \\
\frac{h-1}{h} n - 1 & \text{if } n \equiv h \mod 2h.
\end{cases}
$$

**Lemma 3.** If $G$ is an $\mathcal{A}_n$-graph, then $|K(G)| \leq k_n$.

**Proof.** Let $G$ be an $\mathcal{A}_n$-graph. Every vertex of $K(G)$ has a neighbour in $M(G)$, so

$$
|B(G)| \Rightarrow |K(G)|. \quad (3)
$$

On the other hand, every vertex of $M(G)$ has at most $h-1$ neighbours in $K(G)$, since each vertex of $M(G)$ also has at least one neighbour in $M(G)$. Hence

$$
|B(G)| \leq (h-1)|M(G)| = (h-1)(n-|K(G)|). \quad (4)
$$
From (3) and (4) it follows that $|K(G)| \leq (h-1)(n-|K(G)|)$, or, equivalently,

$$|K(G)| \leq \frac{h-1}{h} n. \quad (5)$$

To complete the proof we show that the inequality (5) is strict if $n = h \mod 2h$. Assume that $n = 2hi + h$ and (5) holds with equality. Then $|M(G)| = 2i + 1$. Since (4) also holds with equality, the graph $\langle M(G) \rangle$ is regular of degree 1, implying that $|M(G)|$ is even, a contradiction. □

Lemmas 2 and 3 enable us to determine an upper bound for $\mu_{\mathcal{A}}(n)$ in case $n \geq 2h + 1$. Define

$$a(n) = [f_n(k_n)].$$

**Lemma 4.** Let $G$ be an $\mathcal{A}_n$-graph such that $n \geq 2h + 1$ and either $h \neq 2$ or $n \neq 5$. Then $|E(G)| \leq a(n)$. Moreover, unless $h = 3$ and $n = 7$, $|E(G)| = a(n)$ only if $|K(G)| = k_n$; if $h = 3$ and $n = 7$, then $|E(G)| = a(n)$ only if $|K(G)| \in \{k_n - 1, k_n\}$.

**Proof.** Let $G$ satisfy the conditions of the lemma. For $x > h$, $f_n(x)$ is an increasing function of $x$. Since $n \geq 2h + 1$, $k_n \geq h$. By using Lemmas 2 and 3 it follows that

$$|E(G)| \leq [f_n(k_n)]$$

and, if $k_n = h$,

$$|E(G)| = [f_n(k_n)] \text{ only if } |K(G)| = k_n.$$

Now assume $k_n \geq h + 1$. For every $x$ we have

$$f_n(x) - f_n(x - 1) = x - h - \frac{1}{2},$$

implying that $[f_n(k_n)] = [f_n(k_n - 1)]$ if and only if $k_n = h + 1$ and $f_n(k_n)$ is not integer-valued, i.e., if and only if $h = 3$ and $n = 7$, as is easily checked. The result follows. □

We finally show that $\mu_{\mathcal{A}}(n) = a(n)$ for all $n \geq h + 1$ and characterize $\mathcal{A}$. Let $T_7$, $T'_7$ be the graphs depicted in Fig. 1 and define a class $\mathcal{K}$ of graphs
by the assertion that a graph \( G \) with \( n \) vertices belongs to \( \mathcal{H} \) if and only if the following requirements are met:

(i) \( n \geq 2h + 1 \);

(ii) \( |K(G)| = k_n \) and \( \langle K(G) \rangle \) is complete;

(iii) if \( n - k_n \) is even, then \( \langle M(G) \rangle = \frac{1}{2}(n - k_n)P_2 \); if \( n - k_n \) is odd, then \( \langle M(G) \rangle = P_3 \cup \frac{1}{2}(n - k_n - 3)P_2 \);

(iv) every vertex of \( K(G) \) is incident with at least one edge of \( B(G) \);

(v) if \( v_1 \) and \( v_2 \) are the vertices of a component of \( \langle M(G) \rangle \) isomorphic to \( P_2 \), then \( |(N(v_1) \cup N(v_2)) \cap K(G)| \geq h \).

Note that \( \mathcal{H}_n = \emptyset \) if \( h = 2 \) and \( n = 5 \); if \( h \neq 2 \) or \( n \neq 5 \), then \( \mathcal{H}_n \neq \emptyset \) for all \( n \geq 2h + 1 \). In Fig. 2 an element of \( \mathcal{H}_{2h+1} \) is sketched for \( h = 3 \) and \( j \in \{1, 2, \ldots, 2h\} \); \( i \) is an arbitrary positive integer.

For \( h = 2 \), \( \mathcal{H}_n \)-graphs are unique up to isomorphism unless \( n \equiv 2 \mod 4 \) and \( n \neq 6 \). For \( h \geq 3 \), \( \mathcal{H}_n \)-graphs are unique up to isomorphism if and only if \( n \equiv 0 \mod 2h \) or \( n \equiv -(h - 1) \mod 2h \). For relevant values of \( h \) and \( n \), nonisomorphic \( \mathcal{H}_n \)-graphs can be obtained from one another by repeatedly applying the following operation: find two vertices \( u_1 \) and \( u_2 \) of degree greater than \( k_n \) such that \( u_1 \) has at least two neighbours of degree \( h \), \( v_1 \) and \( v_2 \) say; replace the edge \( u_1v_1 \) by the edge \( u_2v_1 \).

Define

\[ \mathcal{H}' = \mathcal{H} \cup \{H_n \mid h + 1 \leq n \leq 2h \}. \]

**Theorem 5.** \( \mu_n(n) = a(n) \) for \( n \geq h + 1 \) and

\[ \mathcal{A} = \begin{cases} \mathcal{H}' & \text{if } h \neq 2, 3, \\ \mathcal{H}' \cup \{C_3\} & \text{if } h = 2, \\ \mathcal{H}' \cup \{T_7, T_7'\} & \text{if } h = 3. \end{cases} \]

**Proof.** For \( h + 1 \leq n \leq 2h \) we are through by Lemma 1 and the observation that

\[ f_n(k_n) = f_n(n - 2) = \frac{1}{2}(n^2 - 5n + 4h + 4). \]

Now let \( n \geq 2h + 1 \). We distinguish three cases.

**Case 1.** \( h \neq 2 \) or \( n \neq 5 \) and \( h \neq 3 \) or \( n \neq 7 \). Then \( \mathcal{H}_n \neq \emptyset \). Since every \( \mathcal{H}_n \)-graph is an \( \mathcal{A}_n \)-graph with \( a(n) \) edges and, by Lemma 1, \( \mu_n(n) \leq a(n) \), it follows that \( \mu_n(n) = a(n) \). It remains to be shown that \( \mathcal{A}_n \subset \mathcal{H}_n \).

Let \( G \) be an \( \mathcal{A}_n \)-graph. Then \( \langle K(G) \rangle \) is complete, otherwise an \( \mathcal{A}_n \)-graph with more edges than \( G \) would be obtained by joining two nonadjacent vertices of \( K(G) \) by an edge. Now inequality (2) holds with equality:

\[ |E(G)| = \frac{1}{2}|K(G)|(|K(G)| - 1) + h(n - |K(G)|) - \frac{1}{2}|p(G)|. \]

By Lemma 4, \( |K(G)| = k_n \). Substituting \( |K(G)| \) by \( k_n \) and \( |E(G)| \) by \( a(n) \), one deduces from (6) that \( p(G) = n - k_n = |M(G)| \) if \( n - k_n \) is even and \( p(G) = n - k_n + 1 = |M(G)| + 1 \) if \( n - k_n \) is odd. Since \( \delta(\langle M(G) \rangle) = 1 \) by definition of \( \mathcal{A} \), it
follows that \( \langle M(G) \rangle = \frac{1}{2}(n - k_n)P_2 \) if \( n \) is even and \( \langle M(G) \rangle = P_3 \cup \frac{1}{2}(n - k_n - 3)P_2 \) if \( n \) is odd. Using the definition of \( \mathcal{A} \) once more, we conclude that \( G \in \mathcal{A} \).

**Case 2.** \( h = 2 \) and \( n = 5 \). Clearly, \( C_5 \) is the only \( \mathcal{A} \)-graph with five vertices.

Hence \( \mu_{\mathcal{A}}(5) = 5 = a(5) \).

**Case 3.** \( h = 3 \) and \( n = 7 \). By Lemma 4, \( \mu_{\mathcal{A}}(7) \leq a(7) = 13 \). All graphs in \( \mathcal{H}_7 \cup \{ T_7, T_7' \} \) are \( \mathcal{A} \)-graphs with 13 edges. Conversely, suppose \( G \) is an \( \mathcal{A} \)-graph with 13 edges. By Lemma 4, \( |K(G)| = k_r = 4 \) or \( |K(G)| = k_r - 1 = 3 \). If \( |K(G)| = 4 \), then, like in Case 1, \( G \in \mathcal{A} \). If \( |K(G)| = 3 \), then (6) implies that \( \rho(G) = 4 = |M(G)| \), so that \( \langle M(G) \rangle = 2P_2 \). Since \( G \in \mathcal{A} \), it follows that \( G \cong T_7 \) or \( G \cong T_7' \).

Theorem 5 contains [1, Lemma 2].

### 3. Characterization of \( \mathcal{G} \)-graphs for \( h = 3 \)

Assume throughout this section that \( h = 3 \). We shall present some evidence for the following result.

**Theorem 6.** \( \mu_{\mathcal{G}}(n) = a(n) \) for \( n \geq 4 \) and \( \mathcal{G} = \mathcal{A} \).

Theorem 6 is equivalent to the assertion that \( \mu_{\mathcal{G}}(n) < a(n) \) for \( n \geq 4 \). It is, however, convenient to prove the following slightly stronger statement.

**Lemma 7.**

\[
\mu_{\mathcal{G}}(n) \leq \begin{cases} 
    a(n) - 1 & \text{if } n \not\equiv 0 \pmod{6}, \\
    a(n) - 2 & \text{if } n \equiv 0 \pmod{6}.
\end{cases}
\]

To get an impression of the proof of Lemma 7, which is by induction on \( n \), let \( G \) be a \( \mathcal{G} \)-graph. Then \( G \) contains a vertex \( p \) with \( N(p) \subseteq K(G) \). Let \( S \) be a 3-cut of \( G \) containing \( p \). In the proof several cases with respect to the structure of \( \langle S \rangle \) are distinguished. In each case two smaller \( \mathcal{G} \)-graphs are constructed from \( G \). Thereby an upper bound for \( |E(G)| \) is obtained via the induction hypothesis. For the proof in full detail, which is quite long, we refer to [3]. Here we only treat the case that \( \langle S \rangle \) is complete. More precisely, we shall prove the following lemma.

**Lemma 8.** Let \( G \) be a \( \mathcal{G}_n \)-graph which contains a 3-cut \( S = \{ p, q_1, q_2 \} \) such that \( N(p) \subseteq K(G) \) and \( \langle S \rangle \) is complete. If, for all \( m < n \),

\[
\mu_{\mathcal{G}}(m) \leq \begin{cases} 
    a(m) - 1 & \text{if } m \not\equiv 0 \pmod{6}, \\
    a(m) - 2 & \text{if } m \equiv 0 \pmod{6},
\end{cases}
\]

then

\[
|E(C)| \leq \begin{cases} 
    a(n) - 1 & \text{if } n \not\equiv 0 \pmod{6}, \\
    a(n) - 2 & \text{if } n \equiv 0 \pmod{6}.
\end{cases}
\]
Before proving Lemma 8 we state four additional lemmas, two of which are adopted from [4].

**Lemma 9** (Veldman [4]). If $T_1$ and $T_2$ are distinct minimum cuts of a graph, then $T_1$ interferes with $T_2$ if and only if $T_2$ interferes with $T_1$.

The following lemma is a special case of [4, Lemma 1].

**Lemma 10** (Veldman [4]). If $v$ is a vertex of degree 3 in a 3-connected graph $G$, then $N(v)$ is the only 3-cut of $G$ contained in $\{v\} \cup N(v)$.

Lemma 10 is applied in the proof of the next lemma.

**Lemma 11.** If $v$ is a vertex of degree 3 in a $\mathcal{C}$-graph, then $\langle N(v) \rangle$ is not complete.

**Proof.** Let $G$ be a $\mathcal{C}$-graph, $v$ a vertex of $G$ of degree 3 and $U$ a 3-cut of $G$ containing $v$. By Lemma 10, $U$ contains a vertex which is not in $\{v\} \cup N(v)$. Hence $N(v)$ interferes with $U$. By Lemma 9, $U$ also interferes with $N(v)$. In particular, $N(v)$ contains a pair of nonadjacent vertices. $\square$

**Lemma 12.** If some vertex of an $\mathcal{A}_{6k}$-graph $G(k \geq 2)$ has at least two neighbours in $M(G)$, then $|E(G)| \leq a(6k) - 2$.

**Proof.** Let $G$ satisfy the conditions of the lemma. From Lemma 3 and its proof it is apparent that $|K(G)| \leq 4k - 1$. Hence

$$|E(G)| \leq \binom{|K(G)|}{2} + 3|M(G)| \leq \frac{1}{2}(4k - 1)(4k - 2) + 3(2k + 1) = 8k^2 + 4a(6k) - 3k + 4 \leq a(6k) - 2. \quad \square$$

Although the upper bound in Lemma 12 is far from sharp, it is all we need in the proof of Lemma 8 (and Lemma 7).

**Proof of Lemma 8.** Assume that all conditions of Lemma 8 are satisfied. Let $\{Q_1, Q_2\}$ be a partition of $V(G) - S$ such that $\langle C_i \rangle$ is a disjoint union of one or more components of $G - S$ ($i = 1, 2$). Construct from $G$ the graphs $G_1$ and $G_2$ as depicted in Fig. 3. It is easily seen that $G_1$ and $G_2$ are 3-connected. Since $\langle S \rangle$ is complete, no 3-cut of $G$ interferes with $S$, so that, by Lemma 9, $S$ interferes with no 3-cut of $G$. Hence if $U$ is a 3-cut of $G$ with $U \cap Q_i \neq \emptyset$, then $U \subset Q_i \cup S$, implying that $U$ is a 3-cut of $G_i$ too ($i = 1, 2$). Thus all vertices of $Q_i$, being critical in $G$, are also critical in $G_i$ ($i = 1, 2$). The remaining vertices of $G_i$, having a neighbour of degree 3, are critical too ($i = 1, 2$). Hence $G_1$ and $G_2$ are $\mathcal{C}$-graphs. From Lemma 11 one easily deduces that $|Q_i| \geq 3$, so that $|V(G_i)| < |V(G)|$ ($i = 1, 2$). If $G_i \in \mathcal{A}$, then $|E(G_i)| \leq a(|V(G_i)|)$ by Theorem 5; if $G_i \in \mathcal{B}$, then
\[ |E(G_1)| \leq a(|V(G_1)|) \] by the conditions of Lemma 8. Looking at Fig. 3 we now deduce that
\[
|E(G)| \leq |E(G_1)| + |E(G_2)| - 13 \\
\leq a(|V(G_1)| - 2) + a(n - |V(G_1)| - 2) - 13 \\
= \max_{1 \leq x \leq n-h} \{a(x+7) + a(n-x) - 13\}.
\]

Let \( \phi_n(x) = a(x+7) + a(n-x) - 13 \). It is easily checked that, if \( 1 \leq i+j \leq (n-1)/2 - 3 \), \( \phi_n(i+j) \) is a decreasing function of \( i \) for each \( j \) with \( 0 \leq j \leq 5 \). Hence
\[
|E(G)| \leq \max_{1 \leq x \leq \min\{6, [(n-1)/2]-3\}} \phi_n(x).
\]

Straightforward checking yields that, for \( 1 \leq x \leq \min\{6, [(n-1)/2]-3\} \),
\[
\phi_n(x) \leq a(n) - 2 \quad \text{if} \ n = 0 \mod 6;
\]

furthermore, for \( 1 \leq x \leq \min\{6, [(n-1)/2]-3\} \),
\[
\phi_n(x) \leq a(n) - 1 \quad \text{if} \ n \neq 0 \mod 6,
\]
except in three cases. We show that \( |E(G)| \leq a(n) - 1 \) in each of these cases.

**Case 1.** \( n = 6k + 1, \ x = 1 \ (k \geq 2) \): \( \phi_{6k+1}(1) = a(6k+1) + 2 \).

In Fig. 3 there are two analogous possibilities corresponding to \( x = 1 \): either \( |V(G_1)| = 8 \) and \( |V(G_2)| = 6k \), or \( |V(G_1)| = 5k \) and \( |V(G_2)| = 8 \). We proceed with the first possibility. \( G_1 \notin \mathcal{A}_h \), since \( K(G_1) \) contains a vertex with two neighbours of degree 3. Since \( \mu_{6h}(8) \leq a(8) - 1 \), it follows that \( |E(G_1)| \leq a(8) - 1 \). From Lemma 12 and the fact that \( \mu_{6h}(6k) \leq a(6k) - 2 \) we deduce that \( |E(G_2)| \leq a(6k) - 2 \). Thus instead of \( |E(G)| \leq \phi_{6k+1}(1) \) we reach the stronger conclusion that
\[
|E(G)| \leq \phi_{6k+1}(1) - 3 = a(6k+1) - 1.
\]

**Case 2.** \( n = 6k + 1, \ x = 5 \ (k \geq 2) \): \( \phi_{6k+1}(5) = a(6k+1) - 12k + 26 \).
\[ \phi_{6k+1}(5) > a(6k+1) - 1 \text{ only if } k = 2. \] Then, however, we are back in Case 1, since \( \phi_{13}(5) = \phi_{13}(1). \)

**Case 3.** \( n = 6k + 3, x = 1 \) (\( k \geq 1 \)): \( \phi_{6k+3}(1) = a(6k+3) + 1. \)

Then in Fig. 3 either \( |V(G_1)| = 8 \text{ and } |V(G_2)| = 6k + 2 \), or \( |V(G_1)| = 6k + 2 \text{ and } |V(G_2)| = 8 \). In particular, \( |V(G_i)| = 2 \text{ mod } 6 \) (\( i = 1, 2 \)). Since \( K(G_i) \) contains a vertex with two neighbours of degree 3, it follows that \( G_i \not\in \mathcal{A} \) (\( i = 1, 2 \)). Thus, in fact,

\[ |E(G)| \leq \phi_{6k+3}(1) - 2 = a(6k + 3) - 1. \]

The proof is completed by verifying the following inequalities:

\[
\begin{align*}
\phi(6k + 1, 7) &\leq a(6k + 1) - 1 \quad (k \geq 3), \\
\phi(6k + 1, 11) &\leq a(6k + 1) - 1 \quad (k \geq 3), \\
\phi(6k + 3, 7) &\leq a(6k + 3) - 1 \quad (k \geq 3).
\end{align*}
\]

4. Discussion

In Section 3 it appeared that \( \mu_{\mathcal{A}}(n) < \mu_{\mathcal{A}}(n) \) for \( h = 3 \). For large values of \( h \), \( \mathcal{A}_n \)-graphs have a very high edge density. We expect that, for increasing values of \( h \), \( \mu_{\mathcal{A}}(n) \) will grow more rapidly than \( \mu_{\mathcal{A}}(n) \), leading us to the following conjecture.

**Conjecture 13.** For all \( h \geq 3 \), \( \mu_{\mathcal{A}}(n) = a(n) \) (\( n \geq h + 1 \)) and \( \mathcal{G} = \mathcal{A} \).

References


