EXISTENCE OF $D_\lambda$-CYCLES AND $D_\lambda$-PATHS

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A cycle $C$ of a graph $G$ is called a $D_\lambda$-cycle if every component of $G - V(C)$ has order less than $\lambda$. A $D_\lambda$-path is defined analogously. In particular, a $D_1$-cycle is a hamiltonian cycle and a $D_1$-path is a hamiltonian path. Necessary conditions and sufficient conditions are derived for graphs to have a $D_\lambda$-cycle or $D_\lambda$-path. The results are generalizations of theorems in hamiltonian graph theory. Extensions of notions such as vertex degree and adjacency of vertices to subgraphs of order greater than 1 arise in a natural way.

1. Introduction

We employ the terminology of Bondy and Murty [3] and consider only simple graphs.

In [2], Bondy stated a sufficient condition for a graph $G$ to have a cycle $C$ such that $G - V(C)$ contains no $K_k$. For $k = 1$, it coincides with Ore's condition for the existence of a hamiltonian cycle. Here we introduce another kind of generalized hamiltonian cycle. A cycle $C$ of a graph $G$ is a $D_\lambda$-cycle if all components of $G - V(C)$ have order less than $\lambda$. Alternatively, $C$ is a $D_\lambda$-cycle of $G$ if and only if every connected subgraph of order $\lambda$ of $G$ has at least one vertex with $C$ in common. Thus a $D_\lambda$-cycle dominates all connected subgraphs of order $\lambda$. Analogously, a path $P$ of $G$ is a $D_\lambda$-path if every component of $G - V(P)$ has order less than $\lambda$. Graphs containing a $D_\lambda$-cycle ($D_\lambda$-path) will be called $D_\lambda$-cyclic ($D_\lambda$-traceable). A $D_1$-cycle ($D_1$-path) is the same as a hamiltonian cycle (hamiltonian path). $D_2$-cycles were studied in [6].

In subsequent sections, existence theorems for $D_\lambda$-cycles are proved. In [6], most of them were already proved for $\lambda = 2$. We will henceforth refrain from referring to these special cases, unless this is essential. Parallel results on $D_\lambda$-paths can be obtained, using the following obvious lemma.

Lemma 1. A graph $G$ is $D_\lambda$-traceable if and only if $G \cup K_1$ is $D_\lambda$-cyclic.

The theorems derived are generalizations of known results in hamiltonian graph theory. A corresponding remark can be made about the proof techniques used. Some of the results in Section 3 are closely related to Bondy's work [2].
Extensions to subgraphs of order greater than 1 of concepts such as adjacency of vertices, independence number and vertex degree arise in correspondence with the generalization of hamiltonian cycles to $D_\lambda$-cycles.

2. A necessary condition in terms of cut sets

To start with, we generalize a necessary condition for the existence of a hamiltonian cycle.

**Theorem A** [3, Theorem 4.2]. If a graph $G$ is hamiltonian, then, for every nonempty proper subset $S$ of $V(G)$,

$$\omega(G - S) \leq |S|.$$  

Denote by $\omega_\lambda(G)$ the number of components of $G$ of order at least $\lambda$. Theorem A is then a special case ($\lambda = 1$) of

**Theorem 1.** If a graph $G$ is $D_\lambda$-cyclic, then, for every nonempty proper subset $S$ of $V(G)$,

$$\omega_\lambda(G - S) \leq |S|.$$  

The proof, being an easy extension of the proof of [6, Theorem 1], is omitted.

For future reference we denote by $\mathcal{H}_\lambda$ the class of graphs not satisfying the necessary condition of Theorem 1. Thus $G$ is in $\mathcal{H}_\lambda$ iff, for some nonempty proper subset $S$ of $V(G)$, $\omega_\lambda(G - S) > |S|$.

3. Sufficient conditions involving subgraph degrees

We now turn our attention to sufficient conditions for the existence of $D_\lambda$-cycles. One of the earliest results in hamiltonian graph theory to be generalized here is due to Dirac.

**Theorem B** [3, Theorem 4.3]. If $G$ is a graph with $\nu \geq 3$ and $\delta \geq \frac{\nu}{2}$, then $G$ is hamiltonian.

We also mention a result of Chvátal and Erdős.

**Theorem C** [4, Theorem 1]. If $G$ is a $k$-connected graph with $\nu \geq 3$ and $\alpha \leq k$, then $G$ is hamiltonian.

Bondy proved a common generalization of Theorems B and C.
**Theorem D** [2, Theorem 2]. Let $G$ be a $k$-connected graph with $v \geq 3$ such that, for every $k + 1$ mutually nonadjacent vertices $u_0, u_1, \ldots, u_k$ of $G$,

$$
\sum_{i=0}^{k} d(u_i) > \frac{1}{2}(k + 1)(v - 1).
$$

Then $G$ is Hamiltonian.

In order to extend Theorems B, C and D to results on $D_\lambda$-cycles for $\lambda > 1$ we need some additional definitions. As in [6], two subgraphs $H_1$ and $H_2$ of a graph $G$ are said to be close in $G$ if they are disjoint and there is an edge of $G$ joining a vertex of $H_1$ and one of $H_2$; if no such edge exists in $G$, then $H_1$ and $H_2$, provided they are disjoint, are remote in $G$. Thus, if $H_1$ and $H_2$ both consist of exactly one vertex, $H_1$ and $H_2$ are close (remote) iff the corresponding vertices are adjacent (nonadjacent). By $\alpha_k(G)$ (or just $\alpha_k$) we denote the maximum number of mutually remote connected subgraphs of order $\lambda$ of $G$. Thus $\alpha_1$ coincides with the independence number $\alpha$. The degree of a subgraph $H$ of $G$, denoted $d_{\ell}(H)$ or $d(H)$, is the number of vertices in $V(G) - V(H)$ adjacent to one or more vertices of $H$. In other words, considering vertices as subgraphs of order 1, $d(H)$ is the number of vertices of $G$ close to $H$. If $H$ consists of a single vertex, then $d(H)$ is just the degree of this vertex. The minimum degree of connected subgraphs of order $\lambda$ will be denoted $\delta_\lambda$, so that $\delta_1 = \delta$. If $Q$ is an oriented cycle or path in a graph and $u$ and $v$ are vertices on $Q$, then $\tilde{O}[u, v]$ and $\tilde{O}[v, u]$ denote, respectively, the segment of $Q$ from $u$ to $v$ and the reverse segment from $v$ to $u$. Furthermore, $\tilde{O}(u, v):=\tilde{O}[u, v]\setminus\{u\}$, $\tilde{O}[v, u]:=\tilde{O}[u, v]\setminus\{v\}$ and $\tilde{O}(u, v):=\tilde{O}[u, v]\setminus\{u, v\}$. Three more defining relations are obtained by reversing the arrows in the previous sentence.

We are now ready to prove a generalization of Theorem C.

**Theorem 2.** Let $k$ and $\lambda$ be positive integers such that either $k \geq 2$ or $k = 1$ and $\lambda \leq 2$. If $G$ is a $k$-connected graph, other than a tree (in case $k = 1$), with $\alpha_k \leq k$, then $G$ is $D_\lambda$-cyclic.

**Proof.** By contraposition. Let $G$ be a $k$-connected non-$D_\lambda$-cyclic graph other than a tree. We will show that $\alpha_k > k$. Put $t + 1 = \min\{i \mid G$ is $D_i$-cyclic$\}$, so that $t < \lambda$. Let $C$ be a longest $D_{t+1}$-cycle among all $D_{t+1}$-cycles $C'$ of $G$ for which $\omega_i(G - V(C'))$ is minimum. As in the proof of [6, Theorem 4] one shows that $C$ has length at least $k + 1$. Fix an orientation on $C$. By assumption, $C$ is a $D_{t+1}$-cycle, but not a $D_t$-cycle of $G$. Hence $G - V(C)$ has a component $H_0$ of order $t$. All vertices of $G$ close to $H_0$ are on $C$ and, since $G$ is $k$-connected and $|V(C)| > k$, we have that $d(H_0) > k$. Let $v_1, \ldots, v_k$ be $k$ vertices of $C$ close to $H_0$. For $i = 1, \ldots, k$, let $u_{0i}$ be a vertex of $H_0$ adjacent to $v_i$ (for $i \neq j$, $u_{0i}$ and $u_{0j}$ may coincide). Assume that $v_1, \ldots, v_k$ occur on $C$ in the order of their indices and let $u_{i1}$ be the immediate successor of $v_i$ on $C$ ($i = 1, \ldots, k$). It will prove possible to
choose, for $i = 1, \ldots, k$, a subgraph $H_i$ of $G$ satisfying the following requirements:

(i) $H_i$ is connected and has order $t$.

(ii) $H_i \cap C = \tilde{C}[u_{i1}, u_{i2}]$, where $u_{i2}$ is a vertex of $\tilde{C}[u_{i1}, v_i]$ chosen in such a way that

(iii) The length of $\tilde{C}[u_{i1}, u_{i2}]$ is minimum, i.e. if $H$ is a connected subgraph of order $t$ of $G$ with $H \cap C = \tilde{C}[u_{i1}, w_i]$, then $\tilde{C}[u_{i1}, u_{i2}]$ is a subpath of $\tilde{C}[u_{i1}, w_i]$. Note that $u_{i1}$ and $u_{i2}$ may coincide, in other words $\tilde{C}[u_{i1}, u_{i2}]$ may have length 0.

If $k = 1$, then $C$ may have length 3 and the existence of a subgraph $H_1$ with the above properties is guaranteed only if $t \leq 2$.

If $k \geq 2$, then, for $1 \leq i \leq k$,

(a) a subgraph $H_i$ with the mentioned properties exists, and

(b) $v_{i+1}$ does not belong to $\tilde{C}[u_{i1}, u_{i2}]$ (indices mod $k$).

Assuming the contrary to (a) or (b), consider the cycle

$$C' = v_i u_{i0} \bar{P}[u_{i0}, u_{i0+i}] u_{i0+i} v_{i+1} \tilde{C}[v_{i+1}, v_i],$$

where $\bar{P}$ is a $u_{i0} u_{i0+i}$-path within $H_0$ (degenerate if $u_{i0} = u_{i0+i+1}$). By assumption, $\tilde{C}(v_i, v_{i+1})$ is not contained in a component of order at least $t$ of $G - V(C')$. Since, moreover, $|H_0 - V(C')| < t$, it follows that $C'$ is a $D_{i+1}$-cycle of $G$ with $\omega_i(G - V(C')) < \omega_i(G - V(C))$, contradicting the choice of $C$.

Thus we have shown that, for $1 \leq i \leq k$, a subgraph $H_i$ satisfying the requirements (i), (ii) and (iii) indeed exists, provided $t \leq 2$ in case $k = 1$. Following an analogous reasoning one proves that $H_0$ and $H_i$ are disjoint and, a fortiori, remote.

Next we prove by contradiction that, for $1 \leq i < j \leq k$, the subgraphs $H_i$ and $H_j$ are remote. Assume that $H_i$ and $H_j$ are close or non-disjoint. Then a $u_{i2} u_{i2}'$-path $P'$ can be found such that

(1) $P' \cap C = \tilde{C}[u_{i2}, w_i] \cup \tilde{C}[w_i, u_{i2}]$, where $w_i$ and $w_j$ are vertices of $\tilde{C}[u_{i2}, u_{i1}]$ and $\tilde{C}[u_{i1}, u_{i2}]$, respectively,

(2) no vertex of $V(P') - V(C)$ is in $H_0$,

(3) the sum of the lengths of $\tilde{C}[u_{i2}, w_i]$ and $\tilde{C}[w_i, u_{i2}]$ is maximum, i.e. no $u_{i2} u_{i2}'$-path satisfying (1) and (2) has more vertices with $C$ in common than $P'$.

Now consider the cycle

$$C'' = v_i u_{0i} \bar{P}[u_{0i}, u_{0i}] u_{0i} v_i \tilde{C}[v_i, u_{i2}] \bar{P}[u_{i2}, u_{i2}] \tilde{C}[u_{i2}, v_i],$$

where $\bar{P}$ is a $u_{0i} u_{0i}'$-path in $H_0$. In Fig. 1 the cycle $C''$ is indicated by arrows.

Denote by $L_i$ and $L_j$ the components of $G - V(C'')$ containing the vertices (if any) of $\tilde{C}[u_{i1}, w_i]$ and $\tilde{C}[u_{j1}, w_j]$, respectively. If $L_i$ and $L_j$ would coincide, then a $u_{i2} u_{i2}'$-path satisfying (1) and (2) could be indicated having more vertices with $C$ in common than $P'$, a contradiction with the choice of $P'$. Thus $L_i$ and $L_j$ are distinct. Moreover, by the way $H_i$ and $H_j$ were chosen, both $L_i$ and $L_j$ have order less than $t$ (otherwise (iii) would be violated). But then $C''$ is a $D_{i+1}$-cycle with $\omega_i(G - V(C'')) < \omega_i(G - V(C))$, contradicting the choice of $C$. 
Thus we have shown that the connected subgraphs $H_0, H_1, \ldots, H_k$ of $G$ of order $t$ are mutually remote, so that $\alpha_i > k$. Since $\alpha_x$ is easily seen to be a nonincreasing function of $x$, it follows that $\alpha_k \geq \alpha_i > k$. □

For $s > \lambda$, the graph $K_k \vee (k+1)K_s$ is non-$D_\lambda$-cyclic and satisfies $\alpha_k = k+1$, showing that Theorem 2 is, in a sense, best possible.

Theorem 2 can be improved to a generalization of Theorem D. Referring to the proof of Theorem 2, it can be shown that

$$d(H_i) + d(H_j) \leq v + k - \lambda - k\lambda \quad (0 \leq i < j \leq k).$$

Bondy [2] showed these inequalities to hold in case $\lambda = 1$. The proof of the general case is completely analogous and hence omitted. Summing the above inequalities eventually yields

**Theorem 3.** Let $k$ and $\lambda$ be positive integers such that either $k \geq 2$ or $k = 1$ and $\lambda \leq 2$. If $G$ is a $k$-connected graph, other than a tree, such that, for every $k + 1$ mutually remote connected subgraphs $H_0, H_1, \ldots, H_k$ of order $\lambda$ of $G$,

$$\sum_{i=0}^{k} d(H_i) > \frac{1}{2}(k+1)(v + k - \lambda - k\lambda),$$

then $G$ is $D_\lambda$-cyclic.
From Theorem 3 one easily deduces a generalization of Theorem B: a \( k \)-connected graph with \( \delta \lambda > \frac{1}{2}(v + k - \lambda - k \lambda) \) is \( D_\lambda \)-cyclic (\( k \geq 2 \) or \( k = 1 \) and \( \lambda \leq 2 \)). However, we can do better.

**Theorem 4.** Let \( k \) and \( \lambda \) be positive integers such that either \( k \geq 2 \) or \( k = 1 \) and \( \lambda \leq 2 \). If \( G \) is a \( k \)-connected graph, other than a tree, with

\[
\delta \lambda > \begin{cases} 
(v - (k + 1)\lambda + k^2)/(k + 1) & \text{if } \lambda \geq k, \\
(v - \lambda)/(\lambda + 1) & \text{if } \lambda < k,
\end{cases}
\]

then \( G \) is \( D_\lambda \)-cyclic.

**Proof.** By contraposition. Assume that \( G \) is \( k \)-connected and non-\( D_\lambda \)-cyclic. Set \( t + 1 = \min\{i \mid G \text{ is } D_{i+1}\text{-cyclic}\} \), so that \( t \geq \lambda \). Let \( C \) be a \( D_{t+1} \)-cycle of \( G \) for which \( \omega_\lambda(G - V(C)) \) is minimum. We may assume \( C \) to have length at least \( k \). Let \( H_0 \) be a component of \( G - V(C) \) of order \( t \) and let \( v_1, \ldots, v_m \) be the vertices of \( C \) close to \( H_0 \), where \( m = d(H_0) \). Choose to each \( v_i \) a subgraph \( H_i \) of \( G \) of order \( t \) as in the proof of Theorem 2 \((i = 1, \ldots, m)\). The choice of \( C \) then implies, among other things, that the vertex sets \( V(H_0), V(H_1), \ldots, V(H_m) \) and \( \{v_1, \ldots, v_m\} \) are mutually disjoint. Thus

\[
v \geq (d(H_0) + 1)t + d(H_0),
\]

or, equivalently,

\[
d(H_0) \leq (v - t)/(t + 1)
\]

and consequently

\[
\delta \lambda \leq (v - t)/(t + 1) + t - \lambda.
\]  \hspace{1cm} (2)

Since \( G \) is \( k \)-connected, \( H_0 \) has degree at least \( k \), so (1) implies that

\[
v \geq (k + 1)t + k.
\]  \hspace{1cm} (3)

If \( \lambda \geq k \), then also \( t \geq k \). The inequality (3) is then equivalent to

\[
\frac{v - t}{t + 1} + t - \lambda \leq \frac{v - (k + 1)\lambda + k^2}{k + 1}.
\]  \hspace{1cm} (4)

Combination of (2) and (4) proves the first part of the theorem.

If \( \lambda \leq k \), then from (3) it follows that

\[
v \geq (\lambda + 1)t + \lambda.
\]  \hspace{1cm} (5)

Since \( t \geq \lambda \), the inequality (5) is satisfied if and only if

\[
\frac{v - t}{t + 1} + t - \lambda \leq \frac{v - \lambda}{\lambda + 1}.
\]  \hspace{1cm} (6)

The proof is now completed by combining (2) and (6). \( \square \)
For $\lambda \geq k$, the collection $\{K^*(v+1)K^\lambda \mid \lambda \geq \lambda\}$ consists of infinitely many $k$-connected non-$D_\lambda$-cyclic graphs with $\delta_\lambda = (v-(k+1)\lambda + k^2)/(k+1)$. If $\lambda \leq k$, then $\{K^*(v+1)K^\lambda \mid \lambda \leq \lambda\}$ is an infinite collection of $k$-connected non-$D_\lambda$-cyclic graphs with $\delta_\lambda = (v-\lambda)/((\lambda+1)$. Thus Theorem 4 is, in a sense, best possible.

In view of Theorem 4, Theorem 3 might be improved to

**Conjecture 1.** Let $k$ and $\lambda$ be positive integers satisfying either $k \geq 2$ or $k = 1$ and $\lambda \leq 2$. If $G$ is a $k$-connected graph, other than a tree, such that, for every $k+1$ mutually remote connected subgraphs $H_0, H_1, \ldots, H_k$ of order $\lambda$ of $G$,

$$\sum_{i=0}^{k} d(H_i) \geq \begin{cases} v-(k+1)\lambda + k^2 & \text{if } \lambda \geq k \\ (k+1)(v-\lambda)/((\lambda+1) & \text{if } \lambda \leq k, \end{cases}$$

then $G$ is $D_\lambda$-cyclic.

If $H$ is a subgraph of order $k$ of a graph $G$ and $v$ is a vertex of $H$, then $d(H) \geq d(v) - k + 1$. From this observation one easily deduces that the truth of Conjecture 1 (for $\lambda = k$) would imply the truth of the following, which is a weaker version of a conjecture due to Bondy.

**Conjecture A** (cf. [2, Conjecture 1]). Let $G$ be a $k$-connected graph such that the degree-sum of every $k+1$ independent vertices is at least $v+k(k-1)$, where $v \geq 3$. Then there exists a cycle $C$ of $G$ such that $G-V(C)$ contains no path of length $k-1$.

In fact, Bondy conjectured that, under the condition of Conjecture A, every longest cycle $C$ of $G$ has the property that $G-V(C)$ contains no path of length $k-1$.

So far, the truth of Conjecture 1 has been established in the following cases:

(a) $\lambda = 1$ and $k \geq 1$ (Theorem D),
(b) $\lambda = 2$ and $k = 1$ [6, Theorem 2],
(c) $\lambda = 2$ and $k = 2$ [6, Corollary 3.2].

Without giving it we mention that the proof of [6, Corollary 3.2] is easily extended to a proof of Conjecture 1 for $k = 2$ and $\lambda > 2$.

**Theorem 5.** Let $G$ be a 2-connected graph such that the degree-sum of every three mutually remote connected subgraphs of order $\lambda \geq 2$ is at least $v-3\lambda + 5$. Then $G$ is $D_\lambda$-cyclic.

By Theorem 4, a 2-connected graph $G$ has a $D_\lambda$-cycle ($\lambda \geq 2$) if $\delta_\lambda \geq 3/(v-3\lambda + 5)$. Under the assumption that $G \notin \mathcal{X}_\lambda$ the existence of a $D_\lambda$-cycle can be proved if the weaker inequality $\delta_\lambda \geq 3/(v-3\lambda + 3)$ is satisfied (the proof is a
slight extension of the proof of Theorem 4; instead of inequality (1) one demonstrates the inequality \( \nu \geq (m + 1)t + m + 2 \), where \( m = d(H_0) \), using the fact that deletion of the \( m \) vertices of \( C \) close to \( H_0 \) does not create a graph with more than \( m \) components of order at least \( t \). Thus, in particular, every 2-connected graph \( G \) satisfying \( G \not\subseteq \mathcal{K}_2 \) and \( \delta \geq \frac{1}{2} \nu - 1 \) is \( D_2 \)-cyclic, providing an extension of the following consequence of a result of Bigalke and Jung [1, Satz 1]: a graph \( G \) with \( G \not\subseteq \mathcal{K}_1 \) and \( \delta \geq \frac{1}{2} \nu \) has a \( D_2 \)-cycle. The latter result, in turn, easily implies the following, due to Nash-Williams [5, Lemma 4]: if \( G \) is a 2-connected graph and \( \delta \geq \max(\alpha, \frac{1}{3}(\nu + 2)) \), then \( G \) is hamiltonian.

References