The explicit structure of the nonlinear Schrödinger prolongation algebra

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ABSTRACT

The structure of the nonlinear Schrödinger prolongation algebra, introduced by Estabrook and Wahlquist, is explicitly determined. It is proved that this Lie algebra is isomorphic with the direct product $H \times (A_1 \otimes \mathbb{C}[t])$, where $H$ is a three-dimensional commutative Lie algebra.

1. INTRODUCTION

In the second of their fundamental papers [1] Estabrook and Wahlquist introduced a prolongation structure for the nonlinear Schrödinger equation.

\begin{equation}
    i\psi_t + \psi_{xx} - \frac{i}{2} \varepsilon \psi \psi^2 = 0,
\end{equation}

where $\varepsilon = \pm 1$ and the bar denotes complex conjugate. This prolongation structure involves a Lie algebra generated by the eight letters $x_1, x_2, y_1, y_2, z_1, z_2, z_1, z_2$ subjected to the following commutator relations

\begin{equation}
    \begin{cases}
        [x_1, x_2] = [x_1, y_2] = [x_2, y_1] = [x_2, z_1] = [z_1, z_2] = [x_2, z_1] = [z_1, z_2] = 0, \\
        [x_1, z_1] = z_2, \quad [x_1, z_2] = z_2, \quad [z_1, z_1] = \frac{i}{2} y_1, \\
        \frac{i}{4} [x_2, z_2] + [y_1, z_1] - \varepsilon z_1 = 0, \quad [x_1, z_2] + 2[y_2, z_1] = 0, \\
        \frac{i}{4} [x_2, z_2] - [y_1, z_1] - \varepsilon z_1 = 0, \quad [x_1, z_1] - 2[y_2, z_1] = 0, \\
        [x_1, y_1] + [x_2, y_2] + 2[z_1, z_2] - 2[z_1, z_2] = 0.
    \end{cases}
\end{equation}
In the present note our aim is to establish the explicit form of this algebra in a way analogous to the analysis of the KdV-prolongation algebra described in [2].

With the help of the symbolic computation techniques developed in [3] the following table of commutators is derived from the relations (2).

Here $\mathcal{J}_2$ and $\mathcal{J}_3$ are defined by

$$
\mathcal{J}_2 = [z_1, z_2], \quad \mathcal{J}_3 = [z_1, z_3]
$$

and where we have used the notation $xy$ instead of $[x, y]$.

For our convenience we shall use this notation henceforth. Table 1 may in principle be checked by hand using the Jacobi identity.

Table 1.

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$y_1$</th>
<th>$y_2$</th>
<th>$z_1$</th>
<th>$z_2$</th>
<th>$z_3$</th>
<th>$z_4$</th>
<th>$z_5$</th>
<th>$\mathcal{J}_2$</th>
<th>$\mathcal{J}_3$</th>
</tr>
</thead>
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<tr>
<td>$x_1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$z_2$</td>
<td>$z_3$</td>
<td>$z_4$</td>
<td>$z_5$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$y_1$</td>
<td>0</td>
<td>$ez_1$</td>
<td>$ez_2$</td>
<td>$ez_3$</td>
<td>$ez_4$</td>
<td>$-ez_1$</td>
<td>$-ez_2$</td>
<td>$-ez_3$</td>
<td>$-ez_4$</td>
<td>0</td>
</tr>
<tr>
<td>$y_2$</td>
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<td>$-\frac{z_4}{2}$</td>
<td>$-\frac{1}{2} x_1 z_4$</td>
<td>$\frac{z_5}{2}$</td>
<td>$\frac{z_6}{2}$</td>
<td>$\frac{z_7}{2}$</td>
<td>$\frac{1}{2} x_1 z_5$</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
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<td>0</td>
<td>0</td>
<td>$\frac{1}{2} y_1$</td>
<td>$\mathcal{J}_2$</td>
<td>$\mathcal{J}_3$</td>
<td>$\frac{e}{2} z_2$</td>
<td>$-\frac{e}{2} z_3$</td>
<td>$\frac{e}{2} z_4$</td>
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<td>0</td>
<td>0</td>
<td>$-\mathcal{J}_2$</td>
<td>$-\mathcal{J}_3$</td>
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<td>$\frac{e}{2} z_3$</td>
<td>$-\frac{e}{2} z_4$</td>
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<td></td>
</tr>
<tr>
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<td>0</td>
<td>0</td>
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<td>$-\frac{e}{2} z_5$</td>
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<tr>
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<td>$z_1 z_4$</td>
<td>$z_1 z_5$</td>
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<td>$-e \mathcal{J}_3 z_4$</td>
<td>$-e \mathcal{J}_4 z_5$</td>
<td>$-e \mathcal{J}_5 z_6$</td>
<td>$-e \mathcal{J}_6 z_7$</td>
<td>$-e \mathcal{J}_7 z_8$</td>
<td></td>
</tr>
<tr>
<td>$z_5$</td>
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<td>0</td>
<td>0</td>
<td>$-\frac{e}{2} z_2$</td>
<td>$\frac{e}{2} z_3$</td>
<td>$-\frac{e}{2} z_4$</td>
<td>$\frac{e}{2} z_5$</td>
<td>$-\frac{e}{2} z_6$</td>
<td>$\frac{e}{2} z_7$</td>
<td></td>
</tr>
<tr>
<td>$\mathcal{J}_2$</td>
<td>$\mathcal{J}_3$</td>
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<td>0</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

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2. AN INHERENT GRADING OF THE NLS-PROLONGATION ALGEBRA

Analogous to the KdV case (see [2]) we shall show that the NLS-prolongation algebra described in the preceding section possesses an inherent grading. Let 
\[ \deg(x_1) = \delta_1, \ \deg(x_2) = \delta_2, \ \deg(y_1) = \delta_3, \ \deg(y_2) = \delta_4, \ \deg(z_1) = \delta_5, \ \deg(z_2) = \delta_6, \ \deg(z_3) = \delta_7 \text{ and } \deg(z_4) = \delta_8, \] 
the degrees being elements of a commutative, additively written, group.

In order to make the relators (2) homogeneous the following relations between the degrees must hold:

\[
\begin{align*}
\delta_1 + \delta_2 &= \delta_6 \\
\delta_3 + \delta_4 &= \delta_3 \\
\delta_2 + \delta_6 &= \delta_3 + \delta_5 = \delta_5 \\
\delta_1 + \delta_6 &= \delta_4 + \delta_5 \\
\delta_1 + \delta_3 &= \delta_2 + \delta_4 = \delta_3 + \delta_8 = \delta_7 + \delta_6,
\end{align*}
\]

yielding \(\delta_3 = 0, \ \delta_2 = -\delta_3, \ \delta_2 = -\delta_1, \ \delta_4 = 2\delta_1, \ \delta_6 = \delta_1 + \delta_2\) and \(\delta_8 = \delta_1 - \delta_3\).

Obviously we may set \(\delta_1 = (1,0)\) and \(\delta_5 = (0,1)\) and in this way obtain a \(\mathbb{Z}^2\)-grading.

\[
\begin{align*}
\deg(x_1) &= (1,0) \\
\deg(x_2) &= (-1,0) \\
\deg(y_1) &= (0,0) \\
\deg(y_2) &= (2,0) \\
\deg(z_1) &= (0,1) \\
\deg(z_2) &= (1,1) \\
\deg(z_3) &= (0,-1) \\
\deg(z_4) &= (1,-1).
\end{align*}
\]

Now let \(A\) be an arbitrary semi-simple Lie algebra and \(C\) a Cartan subalgebra of dimension \(l\). It is known by Serre's theorem (see [4]) that \(A\) can be presented by the free algebra \(L(e_1, \ldots, e_n, f_1, \ldots, f_n, h_1, \ldots, h_l)\) and the relators

\[
\begin{align*}
e_i f_j &= \delta_{ij} h_i \\
h_i e_j &= \langle \alpha_i, \alpha_j \rangle e_j \\
h_i f_j &= -\langle \alpha_j, \alpha_i \rangle f_j \\
e_i^{a_i} e_j &= 0 \\
f_i^{a_i} f_j &= 0
\end{align*}
\]

where \(\langle \alpha_j, \alpha_i \rangle\) is the \(i-j\)-th element of the Cartan matrix.
A has a natural $\mathbb{Z}'$-grading. This can be seen as follows.

Let $\deg(e_i) = e_i, \deg(f_i) = \phi_i$ and $\deg(h_i) = \eta_i$, $1 \leq i \leq l$.

The relators become homogeneous if $\eta_i = (0, \ldots, 0)$, $\phi_i = (0, \ldots, 1, \ldots, 0)$, the canonical $i$-th basis element of $\mathbb{Z}'$ and $\phi_i = -e_i$.

From this, in connection with the $\mathbb{Z}^2$-grading found for the NLS-algebra, we suspect that the latter contains a semi-simple algebra with a two-dimensional Cartan subalgebra.

Comparing the grades we find that $z_1$ plays the role of $e_2$, $x_1$ of $e_1$, $x_2$ of $f_1$, $z_1$ of $f_2$ and $y_1$ of an $h$.

However $x_1 x_2 = 0$, so $h_1$ must be zero whereas $z_1 z_1 = \frac{1}{2} y_1 = h_2 \neq 0$. This implies that the NLS-algebra contains only a semi-simple algebra with a one-dimensional Cartan subalgebra, i.e. a $\mathfrak{sl}(2)$.

3. PRESENTATION OF THE LIE ALGEBRA $A_1 \otimes \mathbb{C}[t]$

Let $E$ be the Lie algebra presented by the free algebra $L(h, e, f_1, f)$ and the relations

$$\begin{align*}
he &= 2e, \quad hf = -2f, \quad ef = h, \\
\h f_1 &= -2f_1, \quad e^3f_1 + f_1^3 e = 0, \quad ff_1 = 0.
\end{align*}$$

Then we have the following theorem

**THEOREM 1.** The Lie algebra $E$ is isomorphic with the Lie algebra $A_1 \otimes \mathbb{C}[t]$, the Lie algebra of polynomials in the indeterminant $t$ with coefficients in the simple algebra $A_1 = \mathfrak{sl}(2)$.

**PROOF.** Let $M$ be the subspace of $E$, generated by the words of the form

$$a_{i_1} (a_{i_2} \cdots (a_{i_n} a_1) \cdots)$$

with $1 \leq i_k \leq 2$ for $1 \leq k \leq n$ and $a_1 = e, a_2 = f_1$ with the exception of $a_1 = e$ itself.

We shall show that $M$ is an ideal in $E$ in 5 steps.

**STEP 1.** $eM \subset M, f_1 M \subset M$.

**PROOF.** This follows from the definition of $M$.

**STEP 2.** $hM \subset M$.

**PROOF.** $ha_1 = 2a_1$ and $ha_2 = -2a_2$ is a direct consequence of the relations (7). If $ha = \lambda a$ and $hb = \mu b$ with scalars $\lambda$ and $\mu$ then according to Jacobi identity it follows that $h(ab) = (\lambda + \mu)ab$.

The result follows by induction

**STEP 3.** $e^3 f_1 = f_1^3 e = 0$. 

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PROOF. According to (7) we have

\[ e^3f_1 + f_1^3e = 0 \]

and

\[ h(e^3f_1 + f_1^3e) = 4e^3f_1 - 4f_1^3e = 0. \]

Consequently \( e^3f_1 = f_1^3e = 0 \).

STEP 4. \( fM \subseteq M \).

PROOF. \( fa_1 = -h \) and \( fa_2 = 0 \). The result follows from step 2 by induction. Observe that \( a_1 \notin M \).

STEP 5.

PROOF. By proposition 3 of ref. [2] with \( EW \) replaced by \( E \) we know that \( M \) is an ideal of \( E \).

Secondly we shall prove that \( E = A_1 \oplus M \) with \( A_1 \) spanned by \( e, f \) and \( h \).

Obviously we have \( E = A_1 + M \). Hence it suffices to show that \( A_1 \cap M = \{0\} \).

Now the relations (7) induce a natural \( \mathbb{Z} \)-grading on \( E \) with \( \deg(h) = 0 \), \( \deg(e) = \deg(f_1) = 1 \) and \( \deg(f) = -1 \).

With respect to this grading scalar multiples of \( f_1 \) are the only elements of \( M \) of degree 1. Remember that \( e \notin M \). All other homogeneous elements of \( M \) have degree \( \geq 2 \). So clearly we have \( A_1 \cap M = \{0\} \).

According to theorem 3 of ref. [2] we know that \( M^{2n} \), the elements of \( M \) of degree \( 2n \), is spanned linearly by \( h_{2n} \) and \( M^{2n+1} \) by \( eh_{2n} \) and \( f_1h_{2n} \), \( n = 0, 1, 2, \ldots \).

Moreover we have

\[ h_{2n+2} = \frac{1}{2} f_1(eh_{2n}) = \frac{1}{2} e(f_1h_{2n}). \]

Now writing \( e_{2n+1} = -2eh_{2n} \) and \( f_{2n+1} = 2f_1h_{2n} \) we have the following formulae

\[
\begin{aligned}
    h_{2m}e_{2n+1} &= 2e_{2m+n+1} \\
    h_{2m}h_{2n} &= 0 \\
    h_{2m}f_{2n-1} &= -2f_{2m+2n-1} \\
    e_{2n+1}f_{2m-1} &= h_{2m+2n} \\
    f_{2m-1}f_{2n-1} &= e_{2m+1}e_{2n+1} = 0 \\
    \text{with } f_{-1} &= f, h_0 = h \text{ and } e_1 = e,
\end{aligned}
\]

(8) corresponding to those of ref. [2].

Finally an isomorphism of \( E \) and \( A_1 \otimes \mathbb{C}[t] \) is given by

\[
\begin{aligned}
    f_{2n-1} &\mapsto f \otimes t^n \\
    h_{2n} &\mapsto h \otimes t^n \\
    e_{2n+1} &\mapsto e \otimes t^n
\end{aligned}
\]

(9)
4. THE STRUCTURE OF THE NLS-PROLONGATION ALGEBRA

If we set for $\varepsilon = 1$: $h = 2y_1$, $e = 2z_1$, $f = 2z_1$ and $f_1 = 2z_2$ then from table 1 it follows that the relations (7) are satisfied.

For $\varepsilon = -1$ to obtain the same result we may substitute:

$$h = -2y_1, \ e = 2iz_1, \ f = 2iz_1 \text{ and } f_1 = 2iz_2.$$  

Consequently, the NLS prolongation algebra contains $A_1 \otimes \mathbb{C}[t]$ as a subalgebra. We shall show that the natural $\mathbb{Z}$-grading of $A_1 \otimes \mathbb{C}[t]$ is an homomorphic image of the $\mathbb{Z}^2$-grading of this subalgebra. In fact this homomorphism $\phi$ is defined by

$$\phi(a, b) = 2a + b.$$  

Using the $\mathbb{Z}^2$-grading of the NLS-prolongation algebra we shall search for the “radical”.

Evidently, $x_2$ belongs to it. Now the role of $x_1$ and $y_2$ is not satisfactorily determined. The $\mathbb{Z}$-grading of $x_1$ equals 2 and the $\mathbb{Z}$-grading of $y_2$ equals 4.

Now the elements of $A_1 \otimes \mathbb{C}[t]$ with same degree are multiples of $h_2$ with degree 2 and $h_4$ with degree 4.

---

![Diagram](image-url)

**Picture 1.**
So we expect that \( x_1 + \lambda z_1 z_2 \) will belong to the "radical" for appropriate \( \lambda \). Indeed, if we commute this element with the basic letters \( x_1, x_2, \ldots, z_2 \) we find that for \( \lambda = 2\varepsilon \), this element commutes with these basic letters.

For \( y_2 + \mu z_1 z_3 \) the same holds for \( \mu = \varepsilon \).

Therefore we have proved the following result

**THEOREM 2.** The NLS-prolongation algebra is isomorphic with the direct product

\[
H \times (A_1 \otimes \mathbb{C}[t]),
\]

where \( H \) is a three-dimensional commutative Lie algebra.

5. **THE CASE** \( \text{deg}(z_1) = \text{deg}(\xi_1) \)

From the physical point of view it may be requisite that \( \text{deg}(z_1) = \text{deg}(\xi_1) \) and \( \text{deg}(z_2) = \text{deg}(\xi_2) \).

In this case from the relations (4) it follows that \( \delta_5 = \delta_7 = 0 \) and we find a \( \mathbb{Z} \)-grading immediately. Setting \( \delta_3 = 1 \), we have

\[
\begin{align*}
\text{deg}(x_1) &= 1, \quad \text{deg}(x_2) = -1, \quad \text{deg}(y_1) = 0, \quad \text{deg}(y_2) = 2, \\
\text{deg}(z_1) &= 0, \quad \text{deg}(z_2) = 1, \quad \text{deg}(\xi_1) = 0, \quad \text{deg}(\xi_2) = 1.
\end{align*}
\]

With respect to this grading theorem 1 remains in force, however, picture 1 reflecting the grading has to be modified and becomes

\[
\begin{array}{c}
\begin{array}{c}
A_1 \otimes \mathbb{C}[t] \\
H
\end{array}
\end{array}
\]

\[
\begin{align*}
&f_3 \quad \ldots \ldots \quad h_3 \quad \ldots \ldots \quad e_3 \\
&f_2 \quad \ldots \ldots \quad h_2 \quad \ldots \ldots \quad e_2 \quad y_2 + \varepsilon z_1 z_3 \\
&f_1 \quad \ldots \ldots \quad h_1 \quad \ldots \ldots \quad e_1 \quad x_1 + 2\varepsilon z_1 z_2 \\
&f \quad \ldots \ldots \quad h \quad \ldots \ldots \quad e \quad x_2 \\
&\text{degree} \
\end{align*}
\]

Picture 2.
REFERENCES