Thermodynamic Bond Graphs
and the Problem of
Thermal Inertance

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ABSTRACT: It is shown that an isolated thermal inertance does not obey the second law of thermodynamics. Consequently, such an element should not be used in physical systems theory. To eliminate the structural gap in the thermal domain of current physical systems theory, a new framework is introduced using Bond Graph concepts. These Thermodynamic Bond Graphs are the result of synthesis of methods used in thermodynamics and in mechanics.

I. Introduction

At a critical temperature ($T_c = 2.19$ K) liquid helium ($He^4$) shows a singularity (A-point) in its specific heat curve (Fig. 1), which has some analogy to a second-order phase transition. This transition causes a sort of “fourth aggregation state” called liquid He II or superfluid helium, having important thermomechanical anomalies, which led to the successful hypothesis that this “liquid” can be described as a Bose-Einstein gas (1, 2).*

On account of this microscopic theory, Tisza constructed in 1938 a macroscopic theory, called the “two-fluid model”. By this theory he predicted a new kind of wave propagation in liquid helium (2, 4). Indeed, Peshkov confirmed this prediction in 1944 by producing so-called thermal waves or second sound (2, 5). The latter terminology is somewhat misleading, because thermal waves can neither be produced nor detected by acoustic means.

A few decades later the existence of this peculiar wave phenomenon was quoted by various authors who were searching for a “missing element” (inertance) in the field of physical systems theory [(6), p. 73; (7), p. 55]. Their conclusion was that “the second type of thermal energy storage must exist and perhaps could be isolated” (6), if wave phenomena have to be described by two types of storage: capacitance and inertance.

In this paper it will be shown, however, that such an isolated thermal inertance is not compatible with the second law of thermodynamics (Section II) and that Tisza’s model does not predict “purely” thermal waves, but

*Landau argues that He II behaves more like a solid than a gas (2, 3); He represents the liquid as a quasi-continuum whose excitations are quantized.
hydromechanical waves, convecting thermal energy (Section III). The structural gap in current physical systems theory caused by the former result, leads to the formulation of a new framework for a physical systems theory, which is introduced in the form of Thermodynamic Bond Graphs (Section IV). In this framework, no elements will be "missing". Some characteristic properties of Thermodynamic Bond Graphs are discussed in the Sections V and VI.

II. A Missing Element in Current Physical Systems Theory

In current physical systems theory (6, 8-11) an inertance is characterized by

\[ \Phi(f, P) = 0 \]  \hspace{1cm} (1a)

\[ p(t) = \int_0^t e(\tau) \, d\tau + p(0) \]  \hspace{1cm} (1b)

where \( e \) and \( f \) are the conjugate power variables effort and flow, and \( p \) is the generalized momentum.

In the thermal domain, with the temperature \( T \) as effort and the entropy flux \( dS/dt = f_s \) as flow, one obtains

\[ \Phi(f_s, f_s \, T \, dt) = 0 \]  \hspace{1cm} (2)

or in explicit form

\[ \Delta T = I \frac{df_s}{dt} \]  \hspace{1cm} (3)

where \( I \) is the inertance (not necessarily constant).

Equation (3) implies that \( f_s \) may have a constant value (different from zero), if the conjugate temperature difference \( \Delta T \) is zero. However, one of the consequences of the second law of thermodynamics is (formulation of
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Clausius, e.g. [(12), p. 367]: a spontaneous heat- or entropy flow is always directed from a point with a higher temperature to a point with a lower temperature ("spontaneous" means that no other gradients (differences) which may cause cross-effects, do exist). Hence, if there is an entropy flow, there has to be a temperature difference (gradient). Evidently a thermal inertance violates this postulate of macroscopic physics (thermodynamics) and should not be used in physical systems theory.

The structural absence of a thermal inertance in the current framework of physical systems theory [Table I; "Paynterian" or Mechanical Bond Graphs (MBG) (13, 14)] demonstrates the need for another framework. Before introducing such a framework it will be shown that the macroscopic description of thermal waves does not need a thermal inertance.

III. The Two-Fluid Model of Thermal Waves

The one-dimensional wave equation:

\[
\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}
\]  

(4)

can be decomposed into

\[
\frac{\partial u_1}{\partial t} = av_p^2 \frac{\partial u_2}{\partial x}
\]  

(5a)

\[
\frac{\partial u_2}{\partial t} = a^{-1} \frac{\partial u_1}{\partial x}
\]  

(5b)

and approximated by ("lumped space")

\[
\frac{du_1}{dt} = av_p^2 \frac{\Delta u_2}{\Delta x} = C^{-1} \Delta u_2
\]  

(6a)

\[
\frac{du_2}{dt} = a^{-1} \frac{\Delta u_1}{\Delta x} = I^{-1} \Delta u_1
\]  

(6b)

where \(\Delta\) stands for a finite difference and where \(C\) and \(I\) are constants which may be conceived as capacitance and inductance of a segment ("lump") of a transmission line.

The latter two equations correspond to a network description which needs two types of storage elements, e.g. the mass-spring chain for a purely mechanical or hydraulic wave (vibrating string, elastic rod, compressible fluid, etc.) or the coil-capacitor chain for a purely electromagnetic wave (transmission line). In Bond Graphs both are represented by an I–C chain (Fig. 2).

According to the conclusion in the previous section that an isolated thermal inertance cannot exist, purely thermal waves described by a thermal I–C
### Table I. Framework of current physical systems theory (MBG)

<table>
<thead>
<tr>
<th>Effort</th>
<th>Flow</th>
<th>Generalized momentum</th>
<th>Generalized displacement</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e \ (\text{dim})$</td>
<td>$f \ (\text{dim})$</td>
<td>$p \ (\text{dim})$</td>
<td>$q \ (\text{dim})$</td>
</tr>
<tr>
<td><strong>Translation</strong></td>
<td><strong>Force</strong></td>
<td><strong>Velocity</strong></td>
<td><strong>Momentum</strong></td>
</tr>
<tr>
<td>$F \ (\text{N})$</td>
<td>$v \ (\text{m.s}^{-1})$</td>
<td>$p \ (\text{Ns})$</td>
<td>$x \ (\text{m})$</td>
</tr>
<tr>
<td><strong>Rotation</strong></td>
<td><strong>Torque</strong></td>
<td><strong>Torque velocity</strong></td>
<td><strong>Angular momentum</strong></td>
</tr>
<tr>
<td>$T \ (\text{Nm})$</td>
<td>$\Omega \ (\text{rad.s}^{-1})$</td>
<td>$b \ (\text{Nms})$</td>
<td>$\phi \ (\text{rad})$</td>
</tr>
<tr>
<td><strong>Hydraulic</strong></td>
<td><strong>Total pressure</strong></td>
<td><strong>Volume flow</strong></td>
<td><strong>Pressure momentum</strong></td>
</tr>
<tr>
<td>$p \ (\text{Nm}^{-2})$</td>
<td>$\phi_v \ (\text{m}^3\text{s}^{-1})$</td>
<td>$\Gamma \ (\text{Nm}^{-2}\text{s})$</td>
<td>$V \ (\text{m}^3)$</td>
</tr>
<tr>
<td><strong>Acoustic</strong></td>
<td><strong>Pressure</strong></td>
<td><strong>Volume velocity</strong></td>
<td><strong>Momentum</strong></td>
</tr>
<tr>
<td>$p \ (\text{Nm}^{-2})$</td>
<td>$\phi_v \ (\text{m}^3\text{s}^{-1})$</td>
<td>$\Gamma \ (\text{Nm}^{-2}\text{s})$</td>
<td>$V \ (\text{m}^3)$</td>
</tr>
<tr>
<td><strong>Electric</strong></td>
<td><strong>Voltage</strong></td>
<td><strong>Current</strong></td>
<td><strong>Flux linkage</strong></td>
</tr>
<tr>
<td>$u \ (\text{V})$</td>
<td>$i \ (\text{A})$</td>
<td>$\Phi \ (\text{Vs})$</td>
<td>$q \ (\text{C})$</td>
</tr>
<tr>
<td><strong>Chemical</strong></td>
<td><strong>Chemical potential</strong></td>
<td><strong>Molar flow</strong></td>
<td><strong>Molar mass</strong></td>
</tr>
<tr>
<td>$\mu \ (\text{J.mol}^{-1})$</td>
<td>$\dot{N} \ (\text{mol.s}^{-1})$</td>
<td>$N \ (\text{mol})$</td>
<td></td>
</tr>
<tr>
<td><strong>Thermodynamical</strong></td>
<td><strong>Temperature</strong></td>
<td><strong>Entropy flow</strong></td>
<td><strong>Entropy</strong></td>
</tr>
<tr>
<td>$T \ (\text{K})$</td>
<td>$\dot{S} \ (\text{J.s}^{-1}\text{K}^{-1})$</td>
<td>$S \ (\text{J.K}^{-1})$</td>
<td></td>
</tr>
</tbody>
</table>
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2(a)

2(b)

2(c)

Fig. 2(a). Mass–spring chain. (b). Coil–capacitor chain. (c). I–C chain.

chain are impossible, too. Of course, the question arises if this conclusion is contradictory with the experimental results of Peshkov (5), who observed a thermal wave phenomenon. This paradox is resolved by Tisza’s two-fluid model (2,4), which is consistent with the second law and by which the phenomenon has been predicted. In this theory, liquid helium with a temperature below $T_c = 2.19$ K consists of a so-called normal and a so-called superfluid, which allow an internal wave motion. One may compare this to the internal wave between two layers of non-mixing fluids, although the normal and the super fluid have no observable interface. The super fluid as a whole can be seen virtually as “one macroscopic quantum state” (2) and accordingly has no entropy. Hence, the super fluid contains no entropy and accordingly no thermal energy, while the normal fluid contains all entropy.

The thermal wave thus turns out to be not purely thermal: thermal energy (heat) is convected by the normal fluid, which performs a hydromechanical wave interaction with the super fluid.

This hydromechanical wave is not “directly” observable (i.e. mechanically or optically, e.g. by pressure- or density-differences), because liquid He II is homogeneous in ordinary space. The convected thermal energy, however,
makes the internal wave action of the normal fluid observable (measurable) via the thermal domain and gives the impression of a "purely thermal wave". Because the thermal variables are the only macroscopical variables which exhibit the wave phenomenon (the bifurcation into super and normal fluid is a microscopic phenomenon), the only way to observe or control the phenomenon macroscopically leads through the thermal domain: a thermal wave can only be produced in the thermal domain, i.e. by a varying temperature or heat flow and not by acoustic sources.

Although elimination of the hydromechanical variables in Tisza's model results in a wave equation for the absolute temperature, one should not use this result as evidence for the existence of an isolated thermal inertance, because, by the same argument, one could call a stone moving uniformly through an isothermal space, an isolated thermal inertance: an amount of entropy (stored in the stone) moves uniformly without the presence of a temperature gradient.

**IV. Thermodynamic Bond Graphs**

1. **A new framework for physical systems theory**

   The non-existence of an isolated thermal inertance separates the conventional domains of physical systems theory into two classes: domains with two types of storage and domains with one type of storage. The wish to abolish this unsatisfactory situation did result in the development of a new framework based on a generalized form of thermodynamics and formulated with the use of Bond Graph concepts, called Thermodynamic Bond Graphs (TBG) (15, 16).

   The TBG approach allows a synthesis of methods: network methods to describe the interconnection structure (electrical network theory, Newtonian mechanics) and variational methods to describe the elements (thermodynamics, Lagrangian mechanics). This allows the description of systems containing complex elements.

   The basic idea behind the TBG framework is the decomposition of those conventional domains which have two types of storage, into two new domains which have only one type of storage (Table II). Hence, in the TBG framework one does not speak of mechanical or electromagnetic domains as usual, but of kinetic, potential (elastic, gravitational, etc.), electric and magnetic domains, just as of thermal, material and chemical domains, each characterized by its own type of physical property (state variable). These isomorphic domains which are called "physical domains" accordingly, are reciprocal, and accordingly all intra-domain couplings (i.e. couplings within one domain) are reciprocal, such that all non-reciprocal couplings (e.g. gyrators) belong to the interdomain couplings (i.e. couplings between the domains). In this framework the concept of mechanical force has no unique meaning: it may be the effort of the potential domain or the flow (rate of change of momentum) of the kinetic domain.

   Consequently the velocity may be the flow (rate of change of position) of
<table>
<thead>
<tr>
<th>Physical domain</th>
<th>State variable</th>
<th>Effort</th>
<th>Flow</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Physical domain</strong></td>
<td></td>
<td>$e = \frac{\partial E_{total}}{\partial q}$ (dim)</td>
<td>$f = \frac{dq}{dt}$ (dim)</td>
</tr>
<tr>
<td>Translationally potential</td>
<td>Displacement</td>
<td>Force $F$ (N)</td>
<td>$\dot{x}$ (m.s$^{-1}$)</td>
</tr>
<tr>
<td>(elastic, gravitational)</td>
<td>$x$ (m)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Special forms:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>hydraulic</td>
<td>Volume $V$ (m$^3$)</td>
<td>Pressure $p$ (Nm$^{-2}$)</td>
<td>$\dot{V}$ (m$^3$.s$^{-1}$)</td>
</tr>
<tr>
<td>acoustic</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rotationally potential</td>
<td>Angular displacement $\phi$ (rad)</td>
<td>Moment $M$ (Nm)</td>
<td>$\dot{\phi}$ (rad.s$^{-1}$)</td>
</tr>
<tr>
<td>Translational kinetic</td>
<td>Momentum of impulse $p$ (Ns)</td>
<td>Velocity $v$ (m.s$^{-1}$)</td>
<td>$\dot{p}$ (N)</td>
</tr>
<tr>
<td>Special forms:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>hydraulic</td>
<td>Pressure momentum $\Gamma$ (Nm$^{-2}$.s)</td>
<td>Volume flow $\phi_v$ (m$^3$.s$^{-1}$)</td>
<td>$\dot{\Gamma}$ (Nm$^{-2}$)</td>
</tr>
<tr>
<td>acoustic</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rotationally kinetic</td>
<td>Angular momentum $b$ (Nms)</td>
<td>Angular velocity $\Omega$ (rad.s$^{-1}$)</td>
<td>$\dot{b}$ (Nm)</td>
</tr>
<tr>
<td>Electric</td>
<td>Charge $q$ (C)</td>
<td>Voltage $u$ (V)</td>
<td>$\dot{q}$ (A)</td>
</tr>
<tr>
<td>Magnetic</td>
<td>Magnetic flux or flux linkage $\Phi$ (V.s)</td>
<td>Current or magnetomotive force $i = Hi$ (A)</td>
<td>$\dot{\Phi}$ (V)</td>
</tr>
<tr>
<td>Material</td>
<td>Mole number $N$ (mol)</td>
<td>Total material potential $\mu^{tot}$ (J.mol$^{-1}$)</td>
<td>$\dot{N} = f_N$(mol.s$^{-1}$)</td>
</tr>
<tr>
<td>Chemical</td>
<td>Species mole numbers $N_i$ (mol)</td>
<td>Chemical potentials $\mu_i$ (J.mol$^{-1}$)</td>
<td>$\dot{N_i} = f_{N_i}$(mol.s$^{-1}$)</td>
</tr>
<tr>
<td>Thermal (&quot;thermodynamical&quot;)</td>
<td>Entropy $S$ (J.K$^{-1}$)</td>
<td>Temperature $T$ (K)</td>
<td>Entropy flow $\dot{S} = f_S$ (J.s$^{-1}$.K$^{-1}$)</td>
</tr>
</tbody>
</table>
the potential domain or the effort of the kinetic domain.* Thermodynamic Bond Graphs represent a new “thermodynamic-like” framework of state variables (properties) and physical domains, in contrast with Gyro Bond Graphs (11), which do have only one type of storage too, but are based on the mathematical requirement of a minimum set of necessary elements. In Gyro Bond Graphs the set of necessary elements is reduced to its minimum with the use of gyrators: A number of elements are replaced by their dual “seen through” a gyrator in such a way that no dual elements are left in the network (Fig. 3). In the TBG framework a (unit) gyrator appears too, but on physical grounds, as will be explained in the next subsection.

2. The Symplectic Gyrator

The synthesis between the mechanical and the thermodynamical approach is obtained in the TBG by providing all efforts and flows in principle with an asymmetrical, thermodynamical character (intensive properties and rates of change of the extensive properties) and by modelling explicitly Newton’s Second Law of Motion:

\[ F = \frac{dp}{dt} \]  

(7)

together with the identity between velocity and rate of change of displace-

\begin{align*}
\text{Mech BG:} & \quad \rightarrow 1 : m & \rightarrow R : g \\
\text{Gyro BG:} & \quad \rightarrow G Y \quad C : m & \rightarrow G Y \quad R : g \\
& \quad \rightarrow G Y \quad 1 & \rightarrow G Y \quad 1
\end{align*}

FIG. 3. The reduction of the number of elements in Gyro Bond Graphs with the use of gyrators.

*The decomposition of the mechanical (and electromagnetic) domain corresponds in analytical mechanics to the decomposition of a \(2n\)-dimensional symplectic manifold into \(n\)-dimensional Lagrangian submanifolds (17). These Lagrangian submanifolds are reciprocal and thus have a potential associated to any reciprocity relation. Quoting Abraham and Marsden (18), p. 414: “One can carry these ideas further, as Tulczyjew has done, and regard Lagrangian submanifolds as basic entities describing systems”. These basic entities correspond to the physical domains in the TBG.
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\[ v = \frac{dx}{dt} \]  

(8)

in the form of a unit gyrorator called a Symplectic GYrator (SGY). This SGY provides an interdomain coupling between the potential and the kinetic domain (Fig. 4).

The SGY gives the phase space (the space of positions and conjugate momenta, which is a subspace of the complete state space of the TBG modelling concept) its symplectic structure (19). It represents the Hamilton canonical equations:

\[
\frac{\partial H}{\partial q} = -\dot{p} \\
\frac{\partial H}{\partial p} = \dot{q}
\]

(9a)  

(9b)

and because in the TBG framework the momentum \( p \) (state variable of the kinetic domain) is just one of the state variables \( q_i \), one obtains the constitutive equations of a unit gyrorator (the minus sign appears because all flows

\[
\begin{align*}
\mathcal{G} \quad \text{(SGY)}
\end{align*}
\]

\[ e_1 \xrightarrow{t_1} \xrightarrow{e_2} t_2 \]

\[ e_1 = t_1 \]

\[ e_2 = t_2 \]

FIG. 4. The Symplectic Gyrorator.

*A symplectic matrix is

\[
J = \begin{bmatrix}
0 & I \\
-I & 0
\end{bmatrix}
\]

where \( I \) is a unit matrix. A system with \( n \) degrees of freedom is described by

\[
\frac{\partial H}{\partial q_i} = -\dot{p}_i, \quad \frac{\partial H}{\partial p_i} = \dot{q}_i \quad \forall i: 1 \leq i \leq n
\]

which corresponds to an array of symplectic gyrorators (Fig. 5), (20). The canonical transformations which result in the equations of motion of a Hamiltonian system belong to the so-called symplectic group of transformations (18, 19).
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have their positive orientation towards the gyrator):

\[ H(q, p) = H(q_1, q_2) \]  

(10)

consequently

\[ \frac{\partial H}{\partial q_1} = e_1 = -\dot{p} = -f_2 \]  

(11a)

\[ \frac{\partial H}{\partial q_2} = e_2 = \dot{q} = f_1 \]  

(11b)

so

\[ e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = Jf. \]  

(12)

Although the relations of the coupling between the electric and magnetic domain, Maxwell's second and fourth law:

\[ \nabla \times E = -\frac{\partial B}{\partial t} \]  

(13a)

\[ c^2 \nabla \times B = \frac{j}{\varepsilon_0} + \frac{\partial E}{\partial t} \]  

(13b)

are, in general, more complex than the coupling between the kinetic and the potential domain in classical mechanics [Eqs. (7), (8)] the assumptions usually made for electric networks create an analogous situation: Maxwell's second and fourth law reduce for a current loop to the characteristic equations of a symplectic gyrator:

\[ u = -\frac{d\Phi}{dt} \]  

(14a)

\[ i = Hl = \frac{dq}{dt}, \]  

(14b)

where \( u \) is the induced voltage, \( \Phi \) the magnetic flux, \( Hl \) the magnetomotive force, \( i \) the electric current and \( q \) the electric charge.

One could compare the conditions of an electric network with the condition of an inertial frame of reference: if the conditions are not satisfied, the inter-domain couplings become more complex. Hence, in the TBG concept the representation of phenomena which do not obey these conditions is a simple extension: addition of extra couplings to the symplectic coupling, whereas
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the MBG concept provides no systematic means of representation. It should be noted that this does not mean that the solution methods for the obtained models are also a simple extension of the methods used for simple Bond Graph models. The extra couplings may destroy the reciprocity of the model (cf. Section VI) which corresponds to the non-integrability of certain flows or, in other words, the non-existence of a potential function from which the equations can be derived. Furthermore, lumped representations correspond to finite difference schemes for numerical solution of partial differential equations. Requirements of numerical stability, convergence, etc. are not automatically satisfied. For instance, most stable, explicit schemes display numerical dissipation, which is not compatible with the explicit scheme corresponding to the Bond Graph representation, because of the fundamental demand of power continuity. The conclusion is that Bond Graph techniques clarify the modelling process, but generally do not automatically provide the solution technique as may be suggested by Bond Graph models of simple systems.

3. Mass-analog: capacitor or inductor?

In Section IV.1 it has been mentioned already that in the TBG the concept of mechanical force does not have a unique character: it can be the effort of the potential domain as well as the flow of the kinetic domain. This may explain why there has been a lot of confusion about the way electrical analogs of mechanical systems should be constructed: the mass being analogous to the capacitor (one-and two-point, through and across, per- and trans-variables (21-23)) or the mass being analogous to the inductor (as usual in current physical systems theory, e.g. (6)). In terms of the TBG: it just depends in which storage element the symplectic gyrator is absorbed by dualization. From the point of view of algebraic topology (one- and two-point variables) (24) it is preferable to include the symplectic gyrator in the spring and in the coil, because of the point-like character of velocity and voltage in simple mechanical and electrical networks; in this case mass and capacitance become analogs. In (analytical) mechanics, however, one easily gives a velocity the same character as a matter flow (displacement), which results in the mass-inductor analog (Fig. 6). Thus the problem is reduced to a matter of pref-
In case topological methods (e.g. electrical circuit theory) are used, an argument for the first choice is provided by Branin [(24), p. 467]: “It is not surprising to find that the mass-inductance analogy is the one which fails, since it implies that a force, which sums zero at a point, is analogous to voltage, which sums zero around a closed path. The mass-capacitance analogy, on the other hand, always applies because it is topologically consistent in making force the analog of current and velocity the analog of voltage. For pedagogical reasons, therefore, the mass-inductance analogy, owing its existence solely to the planar graph theorem, should be discarded in favor of the mass-capacitance analogy which is fundamentally the correct one.” The latter sentence of this citation is too strong, in the opinion of the present author. The planar graph theorem states that only planar graphs can be dualized. However, the fact that some systems can be dualized and some not, does not provide a criterion to choose between the mass-inductance or the mass-capacitance analogy. The choice of the mass-capacitance analogy, for instance, makes it impossible to represent a mass in a non-planar network by an inductance in the dual of the network, because it simply has no dual. If the mass-inductance analogy would have been chosen in the first place, it would have been impossible to represent a mass in a non-planar network by a capacitance in the dual of the network. Figure 7 shows a TBG interpretation of the situation. It once more makes clear why in analytical mechanics (MBG) the mass-inductance analogy is common. The structural constraint equations are usually stated in terms of the variables of the potential and electrical domain. If the structure is non-planar, the symplectic couplings between the potential and the kinetic and between the electrical and magnetic domains can
FIG. 7. How the choice of analogy depends in the MBG concept on the description of a non-planar structure.

only be eliminated by dualization of the kinetic and the magnetic domains respectively. This results in the mass-inductance analogy. Hence, no final verdict can be delivered, because the choice of analogs for mechanical systems depends on the requirements of the modelling techniques of the structure, which differ from application to application.

V. Generalized Equilibrium Thermodynamics

In the TBG framework (Table II) all state variables are in principle on equal footing as is the case in thermodynamics of simple systems [e.g. as postulated by Callen (25)]. No distinction is made between generalized coordinates and generalized momenta like in Hamiltonian mechanics. Therefore the storage of total energy $E$ (internal as well as potential and kinetic energy) can be described by a generalization of the Gibbs fundamental equation [(12), p. 125]:

$$\frac{dE}{dq_i} dq_i + \ldots + \frac{dE}{dq_i} dq_i + \ldots = e_i dq_i + \ldots + e_i dq_i + \ldots$$

(15)
Every term at the righthand side of (15) represents an amount of energy stored through a port (bond) and belonging to one of the physical domains. This storage of total energy will be represented by a multiport capacitor (Fig. 8) (20), an interdomain coupler of which the constitutive relations, i.e. the relations between the vector of state variables and the vector of corresponding efforts, satisfy the Maxwell reciprocity relations (25), such that (15) is an exact form:

$$\frac{\partial e_i}{\partial q_j} = \frac{\partial e_i}{\partial q_i}, \quad \forall i, j: i \neq j.$$  \hspace{1cm} (16)

The analysis of the multiport capacitor corresponds to what is usually called "equilibrium thermodynamics" or "thermostatics". The assumption commonly made in equilibrium thermodynamics, that the energy is a homogeneous first-order function of the (extensive) state variables (23), is not made in the TBG approach, because it excludes for instance quadratic energy forms. Consequently, the constitutive relations are not restricted to the class of homogeneous zero-order functions of the state variables as in thermodynamics, but constitutive relations can be linear, which is the common situation in mechanics and electrical circuit theory. This explains the use of the terminology: "generalized" equilibrium thermodynamics.

"Paynterian" Bond Graphs (11), equivalent with the methods of (Newtonian) mechanics and accordingly called Mechanical Bond Graphs (MBG), display a basic difference compared to Thermodynamic Bond Graphs (TBG): A Mechanical Bond Graph containing only storage elements still allows processes to take place (quantities may change with time), although the energy of the whole system is constant or conserved (e.g. a harmonic oscillator). By contrast, all time derivatives are zero in a TBG containing only storage elements (cf. an electric network containing only capacitors), which means that not only is the energy conserved, but that no processes take place (equilibrium state).* In MBG's the coupling which allows the existence of oscillatory, energy conserving processes, is contained within one of the two types of storage, namely the inertance. In the TBG all couplings in a system

*In the TRG picture steady or stationary motion is the equilibrium state of the kinetic domain (constant momentum), because the total energy includes the kinetic energy in the same way as the other energy forms. This may not correspond to what usually is conceived by equilibrium (25). In most mechanical systems the only possible equilibrium state is the state of zero momentum, because of symplectic couplings with other domains.
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Harmonic oscillator

mechanical or electrical

MBG vs TBG representation of a harmonic oscillator.

are separate elements. Like in Gyro Bond Graphs, the inertance which is used in the MBG is split into a capacitor and a symplectic gyrator (Section IV.2). This (symplectic) gyrator (interdomain coupling) allows the existence of an oscillatory process (Fig. 9).

Furthermore, the gyrator, which is usually considered as the counterpart of the transformer, because both are powerconserving or "non-energetic" (26), is considered in the TBG as the counterpart of the resistor, because the resistor as well as the gyrator allow processes to take place (diffusive and oscillatory processes, respectively). Secondly, both of these elements, which will be the subject of the next section, are described by a constitutive relation between an effort vector and a flow vector. The latter property has been called "mixing" by Brayton (27). Some authors (e.g. (26)) even call the gyration ratio "gyrational resistance".

The transformer, finally, is considered to be of the same class as the Junction Structure, relating efforts with efforts and flows with flows (0- and 1-junctions may be considered as special types of transformers). Brayton has called this property "non-mixing" (27).
VI. Generalized Non-equilibrium Thermodynamics

1. Multiport resistors and gyrators

In "thermodynamics of irreversible processes" or "non-equilibrium thermodynamics" processes are described by relations between the "generalized forces" (= efforts) and "generalized fluxes" (= flows) (28–30). These constitutive relations are called (linear) phenomenological laws, of which Ohm's law and Fick's law are simple examples (25).

In vector notation, including "cross-effects", e.g. the Peltier/Seebeck effect, these relations read:

\[ f = B \Delta e. \]  \hfill (17)

The constitutive matrix \( B \) can be decomposed into a symmetric part \( B^s \) and an antisymmetric part \( B^a \):

\[ B^s = \frac{1}{2} (B + B^T) \]  \hfill (18a)

\[ B^a = \frac{1}{2} (B - B^T) \]  \hfill (18b)

where \( B^T \) is the transposed form of \( B \).

Hence

\[ f = B^s \Delta e + B^a \Delta e \]  \hfill (19)

or

\[
\begin{align*}
  & R: \ [f' = B^s \Delta e'] \\
  & GV: \ [f'' = B^a \Delta e'']
\end{align*}
\]  \hfill (20)

0-junction array:

\[ \Delta e = \Delta e' = \Delta e''. \]  \hfill (20d)

Only the symmetric part \( B^s \) contributes to the quadratic form

\[ P_{\text{diss}} = \Delta e^T f = \Delta e^T B \Delta e = \Delta e^T B^s \Delta e \]  \hfill (21)

which represents the dissipated power \( P_{\text{diss}} \) or the entropy production \( f_{\text{sat}}^{\text{irr}} \) times the absolute temperature \( T \):

\[ P_{\text{diss}} = T f_{\text{sat}}^{\text{irr}}. \]  \hfill (22)

Accordingly, (20a) characterizes a multiport resistor \( R \) (20). This multiport resistor may be extended to an irreversible transducer RS (Fig. 10) after
Thoma’s extension of a 1-port resistor $R$ to a 2-port $RS$ (31), if the produced entropy flow and accordingly the dissipated power given by (22) have to be modelled explicitly. This is the case if the system is not isothermal, such that the energy has to be the “real total energy” and not the total free energy $F = E - TS$, which is very often used, because in many applications the assumption that the temperature is constant can be made. Note that the $RS$ is power conserving but intrinsically nonlinear due to the bond which represents the dissipated power (Fig. 10).

$B^i$ is not only symmetric, but also positive-definite according to the second law of thermodynamics:

$$P_{\text{diss}} > 0 \rightarrow \Delta e^T B^i \Delta e > 0. \quad (23)$$

The antisymmetric part $B^a$ does not contribute to the quadratic form (21): a quadratic form of an antisymmetric matrix is always zero (32). Consequently $B^a$ (20b) characterizes a power conserving element, which is conceived in the TBG as a new multiport generalization of the gyrator, to be called “multiport gyrator” $GY$ (20). This new multiport gyrator (Fig. 11a) is more general than the usual multiport generalization of the 2-port gyrator analogous to the multiport transformer (Fig. 11b), because it is characterized by any antisymmetric matrix, while the usual multiport gyrator is characterized by an antisymmetric matrix of the special form

$$B^a = \begin{bmatrix} 0 & -G^T \\ G & 0 \end{bmatrix}. \quad (24)$$

Figure 11c provides an alternate representation of an $n$-dimensional symplectic gyrator array with the use of the new multiport gyrator (cf. Fig. 5).

2. The TBG and the Onsager Casimir Reciprocal Relationships

Onsager showed in 1931 that, under the assumption of microscopic reversibility (time symmetry of the relations; detailed balance), (17) is reciprocal, i.e. the matrix $B$ is symmetric ($B^a = 0$) (33):

$$B_{ij} = B_{ji} \quad \forall i, j: i \neq j. \quad (25)$$

Ever since, these so-called “Onsager Reciprocal Relationships” (ORR) have been the object of discussion and confusion (cf. Section VI.4), but they still form the basis of non-equilibrium thermodynamics.
In 1945 Casimir, being inspired by Tellegen (the inventor of the gyrator in network theory), stated that one has two distinct types of forces, $\alpha$-forces and $\beta$-forces* (34). He showed that only relations between equal types of forces

* $\alpha$-forces are even functions of the microscopic velocities (and thus of time), while $\beta$-forces are odd functions of the microscopic velocities. By the assumption of microscopic reversibility, Onsager constrained his proof to $\alpha$-forces. Hence Casimir's results are an extension and not a correction of the ORR.
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(\alpha, -\alpha, \beta, -\beta) are reciprocal, whereas relations between different types of forces 
(\alpha, -\beta) are antireciprocal.

These extended ORR are known as the “Onsager Casimir Reciprocal Relationships” (OCRR):

\[ B_{ij} = \epsilon_{ab} B_{ji} \quad \forall i, j: i \neq j \] (26)

with

\[ \epsilon_{ab} = 1 \quad \text{if} \quad a = b \]
\[ \epsilon_{ab} = -1 \quad \text{if} \quad a \neq b \]

In terms of the TBG concept: Casimir added the usual multiport generalization of the gyrator (special form of \( B^a \)) to Onsager's multiport resistor (\( B^b \)). As has been shown in Section VI.1, this “usual multiport gyrator” does not represent every antisymmetric part of the phenomenological laws, but only matrices \( B^a \) of the type of (24). The consequences of the new multiport gyrator for the OCRR will be discussed in Section VI.3.

If two groups of domains are connected by the usual multiport gyrator, it can be eliminated by partial dualization, i.e. interchanging the roles of part of the conjugate efforts and flows in the equations (Fig. 12). In Rosenberg's terminology such a gyrator is non-essential and the system containing a non-essential gyrator is “extended reciprocal” (35). However, if the groups of domains are also connected by some other kind of coupling, the gyrator can not be eliminated and is called essential (35). Reciprocity or extended reciprocity implies that the system can be characterized by a potential function. The Brayton–Moser mixed potential (36) is an example of a potential function which characterizes an extended reciprocal (electrical) network. In this mixed potential function voltages (efforts) and currents (flows) appear in a mixed form in the equations, because some are interchanged by partial dualization.

3. The new multiport gyrator and the breakdown of the OCRR

Both Onsager and Casimir stressed that the O(C)RR break down if magnetic fields are present (Lorentz forces) or in rotating coordinate frames (Coriolis forces) (33, 34). This “breakdown” means that (17) (the matrix \( B \)) becomes asymmetric, i.e. an antisymmetric part \( B^a \) appears, which is a

![Fig. 12. Elimination of a non-essential, gyrational coupling by partial dualization.](image)
function of the magnetic flux or the angular velocity, corresponding to a modulated multiport gyrator in the TBG. Generally this matrix $B^e$ will not have the form of (24) which can be described by the usual multiport gyrator (Fig. 11b). Hence the new multiport gyrator generalization (Fig. 11a) has to be used, which is characterized by any kind of antisymmetric matrix $B^e$ and includes the usual multiport gyrator as a special case. If this multiport gyrator differs from the usual multiport gyrator, then it is "essential", i.e. it can not be eliminated by partial dualization (Sect. VI.2) and the system could be called "essentially non-reciprocal". In case the system is extended reciprocal, the "breakdown" of the OCRR can be offset by looking upon the influence of Lorentz and Coriolis forces as $\alpha-\beta$ interactions. Consequently, the "real breakdown" of the OCRR means nothing else than the appearance of a modulated, essential multiport gyrator in the TBG representation and the resulting essential non-reciprocity of the model. This means that the phenomenological laws (17) cannot be derived from a potential function. For the TBG the following corollary of the OCRR is important to note: all essential gyrators are modulated.

A well-known example of such an essential multiport gyrator is the 3-port gyrator which represents the gyrosopic forces (including Coriolis forces) corresponding to the exterior or cross products (antisymmetric operators) in Euler's equations:

$$ F = \left( \frac{dp}{dt} \right)_{\text{inertial}} = \left( \frac{dp}{dt} \right)_{\text{rotating}} + \omega \times p = m \frac{dv}{dt} + m(\omega \times v) \quad (27a) $$

$$ M = \left( \frac{db}{dt} \right)_{\text{inertial}} = \left( \frac{db}{dt} \right)_{\text{rotating}} + \omega \times b = J \frac{d\omega}{dt} + (\omega \times J\omega). \quad (27b) $$

In the MBG concept this 3-port gyrator is not recognized as a multiport gyrator, but is called "Eulerian Junction Structure", for short EJS (Fig. 13). (37).

Usually the EJS is represented in a decomposed form, i.e. in a triangle of elementary 2-port gyrators. Because of the form of the Junction Structure of this decomposition, the EJS contains at least one modulated 2-port gyrator.

![Diagram](image)

**FIG. 13.** The Eulerian Junction Structure is an example of the new (modulated) 3-port gyrator.
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Fig. 14. Essentiality: The decomposition of the EJS (3-port gyrator) contains an elementary gyrator after every possible partial dualization.

after every possible dualization, which shows the essentiality of the EJS (Fig. 14).*

On principle the Lorentz forces have the same character as the Coriolis forces: Larmor's theorem states that the equations of Lorentz and Coriolis forces are similar in case of a uniform magnetic field which is weak and constant [(39), p. 34–6].†

4. Truesdell's criticism on "Onsagerism"

The new multiport gyrator corresponds in the linear case to the notorious

*Though not by partial dualization, it still remains possible to eliminate this essential 3-port gyrator representing gyroscopic forces (EJS), namely by a coordinate transformation to an inertial coordinate frame. By looking upon an EJS as an antisymmetric gyristor (38), which is the result of a coordinate transformation from an inertial to a rotating coordinate frame, it can be eliminated by back transformation to an inertial frame of reference.

†On the basis of the similarity between the equations describing magnetic and gyroscopic forces (exterior products = antisymmetric operators), it is expected that all gyroscopic forces have a magnetic analogon, such that the analogy can be extended to strong, time variant and perhaps non-uniform magnetic fields. This analogy is still a subject of study.
"choice of forces and fluxes", the basis of the criticism Truesdell uttered on "Onsagerism" (40).

Truesdell rejects the OCRR by showing (mathematically) that the linear phenomenological laws can always be made symmetric by this choice. Suppose

$$ f = B e = B' e + B^a e \quad (28) $$

now choose

$$ f' - f - B^a e \quad (29) $$

then

$$ f' = B' e. \quad (30) $$

This "symmetrization" is represented in terms of (Vector) Bond Graphs in Fig. 15 (20). It shows that this mathematical argument is valid if one is only interested in the dissipative behaviour (multiport resistor) or—in case $R^a$ is of the form (24) such that the gyrator can be eliminated—if the flows in the rest of the system are changed as well, thus for the multiport capacitor(s) too. This means that the state variables have to be chosen in such a way that there are no antisymmetric parts (gyrators) in the phenomenological or rate equations. In general, the physical interpretation of these state variables will be difficult or even impossible, because they were obtained by a purely mathematical criterion (symmetry of the phenomenological equations). Since state variables should be the primitives of any physical systems theory, such a criterion is unacceptable: one should be able to choose the relevant physical properties as state variables. If this results in asymmetric phenomenological relations, the antisymmetric part (gyrator) is a fundamental part of the system (it allows

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**Fig. 15.** TBG interpretation of Truesdell's symmetrization of the phenomenological equations by "the choice of forces and fluxes".
oscillatory phenomena!). In simple mechanical systems which are extended reciprocal, the gyrators are non-essential and disappear if one accepts two types of state variables as is usually done. This also means, however, that the difference between effort (generalized force) and flow (generalized flux) disappears, because both can be considered as time derivatives of state variables.

From the thermodynamic viewpoint, a criterion* for such a distinction exists and it is this criterion which is obeyed in the TBG: a flow has the character of a time derivative of an (extensive) state variable, while an effort has the character of a partial derivative of the total energy with respect to a state variable. Truesdell, however, looking at non-equilibrium thermodynamics from a (continuum) mechanical viewpoint, does not make a distinction between forces and fluxes. Meixner, who applied the theory of non-equilibrium thermodynamics to electrical networks, obtained a conclusion which is equivalent with Truesdell's symmetrization [Theorem 1 of (41)]: "If the entropy production is known as a bilinear form in the thermodynamic forces and fluxes (or rates), and if one writes the factors which are even with respect to time reversal as linear functions of the factors which are uneven with respect to time reversal, the entity of phenomenological coefficients satisfies Onsager reciprocal relations."

The physical interpretation of factors which are odd or even with respect to time reversal is not given, however.†

**VII. Conclusion**

It has been shown that an isolated thermal inertance is inconsistent with the second law of thermodynamics. This result revealed that a physical systems theory which uses "the two-types-of-storage concept" forms a barrier for the unification of macroscopical theories. The paradox between the non-existence of an isolated thermal inertance and the phenomenon of thermal waves, which seemed to justify the search for this "missing" element, has been resolved with the use of Tisza's original macroscopic theory of thermal waves.

Because thermodynamics did not fit in a systems representation based on mechanics (Mechanical Bond Graphs), it was successfully tried to fit mechanics into a new systems representation (Thermodynamic Bond Graphs)

*Of course, one may object that this criterion depends on the properties or (extensive) state variables which are the primitives of the whole theory. Indeed, primitives are not the result of deduction (from general "laws" or principles), but of induction (from experience or experiment). So primitives are always more or less subjective. "Objectivity" of certain primitive concepts just depends on the magnitude of the group of people who agree about the meaning of those concepts.

†By "thermodynamic forces and fluxes" Meixner means the conjugate variables according to the "entropy scheme" (25) of which the bilinear form is the entropy production. The conjugate variables used in (Thermodynamic) Bond Graphs, effort and flow, belong to the "energy scheme" (25) of which the bilinear form is the (dissipated) power. This makes no difference for the above conclusion, however.
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based on a generalized form of thermodynamics, including oscillatory phenomena. On many problematic items in thermodynamics (e.g. irreversible processes, OCRR) as well as in mechanics (e.g. gyroscopic forces) a new light can be shed, if they are interpreted in terms of a systems theory which uses the TBG approach.

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