On an Optimality Property of Ternary Trees

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The concept of effort is defined for rooted trees. The class of rooted trees with minimal effort is determined. The asymptotic behaviour of the minimal effort is calculated. Various choices for the effort function are considered, as well as variations of the optimality criterion.

1. INTRODUCTION

In the book of Knuth (1972) an extensive discussion can be found on some problems concerning rooted trees. Given a prescribed number of end-vertices, one may ask for the binary (or ternary) tree with minimal external path length (i.e. minimal sum of path lengths from the root to end-vertices), or minimal weighted external path length (here the end-vertices have given weights). In the first case the results are the complete binary (or ternary) trees. In the second case the result is given by the Huffman algorithm. However, if one wants to construct an optimal tree, there is a problem prior to that of determining the binary tree with minimal external path length. It is the problem of choosing the structure of the tree, given some criterion derived from the employment of the tree. One might want to derive the structure of the tree instead of assuming the structure from the beginning.

Several criteria may be considered. The one considered in this paper, in several variants, is the minimality of the "effort". Intuitively, this means the following. Consider the unique path from the root to an end-vertex. Each of the vertices of this path has a number of sons. Let the internal vertex \( v \) have \( u \) sons. The effort \( e \) to find the proper son will depend on \( u \), and can reasonably be assumed to be an increasing function of \( u \). Finding the proper end-vertex will require an effort equal to the sum of the efforts at the internal vertices of the path. The effort of the tree may then be defined as the sum of the efforts of all paths to end-vertices.

However, it is clear that one might consider other criteria, e.g. the maximum (rather than the sum) of the efforts of all paths.

In section 2 some preliminary results are derived for the case where the effort of an internal vertex is simply the number of sons. In section 3 it is shown that ternary trees require minimal effort for that effort function. The asymptotic
For ease of terminology, we let our trees be (directed, rooted) out-trees. The number of sons of a vertex $v$ is then the outdegree of $v$. For the rest, our terminology is the same as in the book of Knuth (1972).

**Definition 1.** The effort of a vertex $v$ with outdegree $u$ depends on $u$ only, and is denoted by $e(u)$, where $e$ is a non-negative non-decreasing function defined on $\mathbb{N}$.

**Definition 2.** The effort $E(P)$ of a path $P = (v_1, \ldots, v_n)$ from the root $v_1$ to an end-vertex $v_n$ is equal to

$$E(P) = \sum_{i=1}^{n-1} e(u_i),$$

where $u_i$ is the outdegree of $v_i$.

**Definition 3.** The effort of a tree $T$ with $N$ end-vertices is

$$E(T) = \sum_{j=1}^{N} E(P_j),$$

where the summation extends over all $N$ paths from the root to the end-vertices.

**Definition 4.** Let $\tau_N$ be the class of all rooted trees with $N$ end-vertices. Then

$$\epsilon(N) = \min_{T \in \tau_N} E(T).$$

Until section 4 we choose the effort function $e$ to be the identity mapping. Our main goal is to determine the structure of the trees for which $\epsilon(N)$ is attained. The following lemma gives a first reduction.

**Lemma 1.** $\epsilon(N)$ is attained for trees in which all internal vertices have outdegrees 2 or 3.

**Proof.** We show that if in a tree $T$ an internal vertex $v$ has outdegree $u = 1$ or $u > 3$, the tree may be transformed into a tree $T^*$ with the same number of end-vertices, for which $E(T^*) \leq E(T)$ and in which the vertex $v$ has been either
eliminated or replaced by other vertices whose outdegrees are smaller. Successive application of this transformation gives the stated result.

The transformations are depicted in Fig. 1A for the case $u = 1$, and in Fig. 1B for the case $u > 3$.

\[ E(P^*) = E(P) - (u_1 + u_2) + u_1 + 2 = E(P) - (u_1 - 2), \]

or

\[ E(P^*) = E(P) - (u_1 + u_2) + u_2 + 2 = E(P) - (u_2 - 2). \]

Since $u_1 \geq 2$ and $u_2 \geq 2$, it follows that $E(T^*) \leq E(T)$.

In particular, we need only consider rooted trees whose roots have outdegree 2 or 3.

**Definition 5.** $\beta$-trees and $\gamma$-trees are rooted trees for which the outdegree of the root is 2 or 3, respectively.

**Definition 6.**

\[
\beta(N) = \min_\beta E(T), \\
\gamma(N) = \min_\gamma E(T),
\]

where the minimum is taken over all rooted trees of the type indicated with $N$ end-vertices.
Clearly

\[ e(N) = \min \{ \beta(N), \gamma(N) \}, \]

\[ \beta(N) = \min_{x+y=N} [e(x) + e(y)] + 2N, \]

\[ \gamma(N) = \min_{x+y+z=N} [e(x) + e(y) + e(z)] + 3N. \]

These formulae are very suitable for calculating \( e(N) \) recursively. Table I below gives the values \( \beta(N), \gamma(N) \), and \( e(N) \) for \( N = 3, \ldots, 40 \), while figure 2 gives \( \beta \)-trees and \( \gamma \)-trees with minimal effort for \( N = 3, \ldots, 10 \). It is worth noting that, for a given \( N \), a \( \gamma \)-tree (or \( \beta \)-tree) with minimal effort is not necessarily unique.

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<th>( N )</th>
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*Fig. 2.* \( \beta \)-trees and \( \gamma \)-trees with minimal effort.
TABLE I
Values of Minimal Effort for $\beta$-trees and $\gamma$-trees with 3 up to 40 End-Vertices

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<tr>
<th>$N$</th>
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The following definition will be needed.

**Definition 7.** A function $f$ defined on $\mathbb{N}^+$ is **convex at** $N \geq 3$ if

$$f(N) - f(N - 1) \geq f(N - 1) - f(N - 2).$$

Note that in the range of the table $\epsilon$ is convex and $\epsilon(N) = \gamma(N)$. Of course, $\epsilon$ is an increasing function. The $\beta$-trees and $\gamma$-trees in the table suggest many conjectures. For example, it seems that optimal trees for increasing values of $N$ are ternary trees.

3. **AN OPTIMALITY PROPERTY OF TERNARY TREES**

We recall that $\epsilon(u) = u$. Furthermore, we note that each $N \in \mathbb{N}^+$ can be written uniquely as $N = 3^k + r$ with $0 \leq r < 2 \cdot 3^k, k \geq 0$.

**Theorem 1.** For $N \geq 3$, $\epsilon(N)$ is attained for $\gamma$-trees with the following structure.
If \( N = 3^k + r, 0 \leq r < 2 \cdot 3^k, \) then

(i) all internal vertices at levels 1,..., \( k - 1 \) have outdegree 3,

(ii) the subtrees whose roots are sons of the same internal vertex have numbers of end-vertices that differ by at most 1.

Moreover, \( \epsilon \) is convex.

Proof. We use induction on \( N \). The induction hypotheses are the structure for \( M \) end-vertices as well as the convexity of \( \epsilon(M) \) with \( M \leq N \). For small values of \( M \), the validity of the theorem is shown by table I.

A \( \beta \)-tree with \( N + 1 \) end-vertices for which \( \beta(N + 1) \) is attained, has two subtrees \( T_1 \) and \( T_2 \) whose roots are the sons of the root.

Both \( T_1 \) and \( T_2 \) have the structure described in (i) and (ii) as the numbers \( N_1 \) and \( N_2 \) of end-vertices are less than \( N + 1 \). If \( N_1 > N_2 + 1 \), then the effort of a \( \beta \)-tree with subtrees having \( N_1 - 1 \) and \( N_2 + 1 \) end-vertices is at most equal to the effort of the original tree, since

\[
\epsilon(N_1) - \epsilon(N_1 - 1) \geq \epsilon(N_2 + 1) - \epsilon(N_2)
\]

by the hypothesis that \( \epsilon \) is convex at values \( M \leq N \). Thus \( \beta(N + 1) \) is attained for a \( \beta \)-tree having property (ii).

We will show by a straightforward calculation of \( \beta(N + 1) \) and \( \gamma(N + 1) \) that \( \gamma(N + 1) \leq \beta(N + 1) \) and that \( \epsilon \) is convex at \( N + 1 \). The structure of the \( \beta \)-tree suggests to write

\[
N = 3^k + r = 2 \cdot 3^{m-1} + s
\]

with \( 0 \leq r < 2 \cdot 3^k, 0 \leq s < 4 \cdot 3^{m-1} \). Depending on the value of \( N \), we have \( m = k \) or \( m = k + 1 \). In order to calculate \( \gamma(N) \), we note, by inspection of the structure, that addition of an end-vertex to a group of end-vertices contributes \( \Delta \epsilon = k + 4 \) or \( \Delta \epsilon = k + 5 \) to the effort, depending on whether an end-vertex at the \( k \)-th level is replaced by an internal vertex with outdegree 2, or an end-vertex is added to an internal vertex at the \( k \)-th level. See also Fig. 3.

\[
\Delta \epsilon = k+4
\]

\[
\Delta \epsilon = k+5
\]

Fig. 3. Addition of a new vertex.
Transformations of groups of end-vertices according to $A$ in Fig. 3 can be made if $0 \leq r \leq 3^k - 1$.

By now it is evident that

$$\gamma(N) = 3kN + 4r \quad (0 \leq r \leq 3^k).$$

For larger $r$, we read off the change in $\gamma$ from Fig. 3B. Thus,

$$\gamma(N) = 3kN + 3^k \cdot 4 + (r - 3^k) \cdot 5$$

$$= 3kN + 5r - 3^k \quad (3^k \leq r \leq 2 \cdot 3^k).$$

In a similar way one derives

$$\beta(N) = \{(3m - 1)N + 4s \quad (0 \leq s \leq 2 \cdot 3^{m-1}),$$

$$I(3m - 1)N + 5s - 2 \cdot 3^{m-1} \quad (2 \cdot 3^{m-1} \leq s \leq 4 \cdot 3^{m-1}).$$

Investigation of the value of $N$ for which one formula or the other has to be chosen, shows that the interval $I = \{3^k, \ldots, 3^{k+1} - 1\}$ is to be split into $I_1 = \{3^k, \ldots, 4 \cdot 3^{k-1} - 1\}$, $I_2 = \{4 \cdot 3^{k-1}, \ldots, 2 \cdot 3^k\}$, and $I_3 = \{2 \cdot 3^k, \ldots, 3^{k+1} - 1\}$. The equation

$$\beta(N) = \gamma(N)$$

has the solution $N = 4 \cdot 3^{k-1}$ on $I_1$, is an identity on $I_2$, and has the solution $N = 2 \cdot 3^k$ on $I_3$. For other values of $N$, one finds $\beta(N) > \gamma(N)$.

$\beta(N + 1)$ is calculated as follows. For $N = 2t$, $\beta(N + 1) = (N + 1) \cdot 2 + 2\gamma(t)$. For $N = 2t + 1$, $\beta(N + 1) = (N + 1) \cdot 2 + \gamma(t) + \gamma(t + 1)$. Straightforward calculation shows $\beta(N + 1)$ to be consistent with the formulae for $\beta(N)$ in all cases. $\gamma(N + 1)$ is calculated similarly. For $N = 3t$, $\gamma(N + 1) = (N + 1) \cdot 3 + 3\gamma(t)$. For $N = 3t + 1$, $\gamma(N + 1) = (N + 1) \cdot 3 + 2\gamma(t) + \gamma(t + 1)$. For $N = 3t + 1$, $\gamma(N + 1) = (N + 1) \cdot 3 + \gamma(t) + 2\gamma(t + 1) + \gamma(t + 1)$. Again $\gamma(N + 1)$ turns out to be consistent with the formulae for $\gamma(N)$. Thus the minimal effort is attained for $\gamma$-trees. It is easily seen that property (i) also holds in case the number of end-vertices is $N + 1$.

Finally one checks that

$$\epsilon(N + 1) - \epsilon(N) = \gamma(N + 1) - \gamma(N) \geq \gamma(N) - \gamma(N - 1)$$

$$= \epsilon(N) - \epsilon(N - 1),$$

proving that $\epsilon$ is convex at $N + 1$. This completes the proof.

It is instructive to see how property (ii) determines the structure of the groups of end-vertices. But for the permutations of isomorphic subtrees it determines the order in which new end-vertices are added to form a larger tree that has again minimal effort.
For example, for $N = 9$ the tree of Fig. 4 has minimal effort. Trees with minimal effort having 10 up to 18 end-vertices arise by transformation of end-vertices e.g. in the order indicated.

![Tree diagram](image)

**Fig. 4.** How optimal trees with 10 up to 18 end-vertices can be obtained from the optimal tree with 9 end-vertices.

### 4. Asymptotics

In this section, we study the asymptotic behaviour of $\epsilon(N)$ as $N \to \infty$, in particular the bounds between which $|\epsilon(N) - 3N \cdot \log N|$ fluctuates. We start from the formulae determined in the proof of Theorem 1,

$$
\epsilon(N) = \begin{cases} 3kN + 4r & (r \leq 3^k) \\ 3kN + 5r - 3^k & (r > 3^k) \end{cases}
$$

where $N = 3^k + r$ ($r \geq 0$, $k$ maximal).

Let $k$ be fixed. Between $3^k$ and $2 \cdot 3^k$, $\epsilon$ is a linear function of $N$, with $\epsilon(3^k) = 3k \cdot 3^k$ and $\epsilon(2 \cdot 3^k) = (6k + 4)3^k$. Hence it is obvious to consider the linear function $y(x)$, defined on $[3^k, 2 \cdot 3^k]$ which coincides with $\epsilon$ at the end-points of the interval. A simple calculation shows that the desired function is

$$
y(x) = \alpha x + \beta,
$$

with $\alpha = 3k + 4$, $\beta = -4 \cdot 3^k$.

Now we consider the difference

$$
z(x) = y(x) - 3x \log_3 x.
$$

Elementary considerations show that $z$ attains its maximum at

$$
x_0 = 3^{\alpha/\beta}e^{-1},
$$

while

$$
z(x_0) = \beta + 3^{\alpha/\beta+1}e^{-1} (\log e 3)^{-1},
$$

or, in terms of $k$:

$$
z(x_0) = \epsilon \cdot 3^k,
$$

(1) (2) (3)
where
\[ c = 3^{7/3}(e \log_e 3)^{-1} - 4 \approx 0.3465. \]
From (1) we have
\[ k = \log_3 x_0 - \frac{4}{3} + \log_3 e, \]
hence we obtain, from (3):
\[ z(x) \leq (3 \cdot \log_3 e - 4e \cdot 3^{-4/3})x \approx 0.2177x. \]
Note that \( 3^k \leq x_0 \leq 2 \cdot 3^k \), so the estimate is sharp, apart from the error introduced by considering reals instead of integers. The same calculation can be made for the interval \([2 \cdot 3^k, 3^{k+1}]\). We still have (1) and (2), but now with
\[ \alpha = 3k + 5, \]
\[ \beta = -6 \cdot 3^k. \]
The maximum of the difference turns out to be
\[ (3 \log_3 e - 6e \cdot 3^{-5/3})x \approx 0.1171x. \]
To complete the numerical estimate, we note that for each \( x \) our piecewise linear function is not below \( 3x \log_3 x \), for
(i) these two functions coincide at \( 3^k (k = 1, 2, \ldots) \),
(ii) \( 3x \log_3 x \) is convex,
(iii) at \( 2 \cdot 3^k (k = 1, 2, \ldots) \) \( 3x \log_3 x \) is less than the value attained by the linear functions.
In summary, we have
\[ 3N \log_3 N \leq \epsilon(N) \leq 3N \log_3 N + 0.2177N, \]
so \( 3N \log_3 N \) gives the first term of the asymptotic expansion, while the second term is \( O(N) \).

5. A Generalized Criterion Function

In this section we consider the general effort function \( \epsilon \) of section 1. We only assume \( \epsilon \) to be a non-decreasing, nonnegative function.
As in lemma 1, we would like to eliminate "large" outdegrees by splitting the son-sets. We denote by \( [x] \) and \( \lfloor x \rfloor \) the smallest integer greater than or equal to \( x \) and the largest integer smaller than or equal to \( x \), respectively.
Lemma 2. If there exist integers \( u \) and \( p \) with \( u > p > 1 \) and \( u > \lfloor u/p \rfloor \) such that

\[
e(u) \geq e(\lfloor u/p \rfloor) + e(p),
\]

then all outdegrees \( u \) of internal vertices can be eliminated.

Proof. Let the internal vertex \( v \) have outdegree \( u \). Split the set of sons of \( v \) into \( p \) disjoint subsets, each with cardinality \( \lfloor u/p \rfloor \) or \( \lceil u/p \rceil \). Introduce \( p \) new vertices which are the new sons of \( v \) and which are the respective fathers of the \( p \) above-mentioned subsets.

Before the split, \( v \) contributes \( e(u) \) to \( E(p) \) for any path \( p \) containing \( v \). After the split, this contribution is replaced by at most \( e(\lfloor u/p \rfloor) + e(p) \) and the proposition follows.

An illustration of the splitting for \( u = 7, p = 2 \) is given in figure 5. Given the effort function \( e \), we may use this lemma to find out for which \( u \) elimination is possible. The remaining cases indicate to which class of trees we may restrict our attention in the search for trees with minimal effort. In the subsections which follow, we consider some special choices for \( e \).

![Fig. 5. The splitting of an internal vertex with outdegree \( u = 7 \), for \( p = 2 \).](image)

5.1. The Logarithmic Case

Let

\[
e(u) = \log u,
\]

then

\[
\log u \geq \log \lfloor u/p \rfloor + \log p
\]

only if \( p \mid u \), in which case the equality sign holds. Hence, a split never yields a proper saving. On the other hand, the inverse operation of splitting may be carried out without causing a raise in the effort of the tree. The minimal effort is therefore attained by the simple trees consisting of a root having the \( N \) end-vertices as sons. The inverse operation is illustrated by Fig. 5 when the direction of the arrow is reversed.
5.2. **A Modified Linear Case**

Let

\[ e(u) = u - 1. \]

Then

\[ u - 1 \geq \lfloor u/p \rfloor - 1 + p - 1 \]

or

\[ u \geq \lfloor u/p \rfloor + p - 1 \]

is the splitting condition. It is easily seen that each outdegree \( \geq 2 \) can be eliminated. If we also omit vertices with outdegree 1 (which contribute 0 to \( E(T) \)), we see that optimal trees can be found among the binary trees. A simple consideration of path lengths shows that the balanced binary trees are optimal. It is remarkable to what extent the choice of \( e \) influences the structure of an optimal tree.

In a sense this case and the logarithmic one are extremal cases. For example the case \( e(u) = u \), leading to ternary trees, is between these two cases as is clear after normalizing the effort functions (divide by \( u(2) \), say, giving normalized effort equal to 1 in the point \( u = 2 \)).

5.3. **A Third Linear Case**

Let

\[ e(u) = u + 1. \]

Now the splitting condition reads

\[ u \geq \lfloor u/p \rfloor + p + 1. \]

For \( p = 2 \) this inequality is satisfied if \( u \geq 6 \). However, choosing a larger \( p \) will not enable us to eliminate \( u = 5 \). Checking all trees with 5 end-vertices shows that the tree in Fig. 6 is the optimal tree for \( N = 5 \). Unlike the situation in section 5.1 and 5.2, now we cannot indicate at once which trees are optimal. To find the optimal trees, one has to consider \( \beta-, \gamma-, \delta-, \) and \( \epsilon- \)trees in which the roots have outdegree 2, 3, 4 and 5, respectively, in analogy to the case \( e(u) = u \).

\[ \text{FIG. 6. The optimal tree with five end-vertices for } e(u) = u + 1. \]
Investigation of all cases for the outdegrees of the sons of the root of a $\beta$-tree shows that $\beta$-trees can be eliminated for $N \geq 6$.

5.4. The Square Root Case

Let 
\[ e(u) = u^{1/2}. \]

A simple calculation shows that all outdegrees $> 17$ can be eliminated by the splitting condition, as well as 12, 14, 16. In this case the determination of optimal trees has become a much more complicated problem, which we have not attempted to solve.

5.5. A Conjecture

Consider again the general effort function $e$. If $N$ is divisible by some integer $p$, we may assign outdegree $p$ to the root, and attach $p$ equal subtrees to it, each with $N/p$ end-vertices. For this tree

\[ e(N) \leq N \cdot e(p) + p \cdot e\left(\frac{N}{p}\right). \]

If the equality sign holds a solution of the resulting functional equation is

\[ e(N) = N \cdot \frac{\log N}{\log p} \cdot e(p). \]

It is the solution when $N = p^m$ for some $m$ provided $e(1) = 0$. It seems plausible that the equality sign will hold for a value of $p$ that is chosen optimally among the divisors of $N$. This seems to be confirmed by the result for $e(u) = u$ when 3 divides $N$.

**Conjecture.** For sufficiently large $N$, in an optimal tree, the outdegree of all vertices sufficiently close to the root is $p^*$, where $p^*$ is the integer value of $p$ for which $e(p)/\log p$ is minimum.

**TABLE II**

Values $p^*$ of Conjectured or Proved Overall Outdegrees for Various Effort Functions

<table>
<thead>
<tr>
<th>$e(p)$</th>
<th>$p^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p-1$</td>
<td>2</td>
</tr>
<tr>
<td>$p$</td>
<td>3</td>
</tr>
<tr>
<td>$p+1$</td>
<td>4</td>
</tr>
<tr>
<td>$p^{1/2}$</td>
<td>7</td>
</tr>
<tr>
<td>$\log p$</td>
<td>$p$</td>
</tr>
</tbody>
</table>
For the criterion functions considered the resulting values of \( p^* \) are given in Table II. Thus for the effort function considered in 5.3 we may expect that in the long run \( \delta \)-trees have minimal effort.

6. OTHER OPTIMALITY CRITERIA

In this section we consider the maximal effort \( M(T) \) of a tree \( T \), defined as follows.

**Definition 8.**

\[
M(T) = \max_{P_j} E(P_j),
\]

where the maximum is taken over all paths \( P_j \) from the root to the end-vertices. The minimum value of \( M(T) \) over all trees with \( N \) end-vertices will be denoted by \( \mu(N) \).

As in section 2 and 3, we consider the effort function \( e(u) = u \).

**Lemma 3.** \( \mu(N) \) is attained for trees in which all internal vertices have outdegree 2 or 3.

**Proof.** Analogous to the proof of lemma 1. 

In analogy to definition 6 of section 1, we define \( \beta \) and \( \gamma \) as follows.

**Definition 9.**

\[
\beta(N) = \min_{\beta} M(T), \quad \gamma(N) = \min_{\gamma} M(T).
\]

Clearly,

\[
\mu(N) = \min(\beta(N), \gamma(N)),
\beta(N) = \min_{x+y=N} \max\{\mu(x), \mu(y)\} + 2,
\gamma(N) = \min_{x+y+z=N} \max\{\mu(x), \mu(y), \mu(z)\} + 3.
\]

Again \( \mu(N) \) may be calculated recursively. Table III below gives the values of \( \beta(N), \gamma(N), \) and \( \mu(N) \) for \( N = 3, ..., 40 \), while Fig. 7 gives \( \beta \)-trees and \( \gamma \)-trees with minimal effort for \( N = 3, ..., 10 \).

The interesting feature of this table is the single exception to the rule that for \( \gamma \)-trees the minimum of the maximal effort is smaller than for \( \beta \)-trees, the exception being

\[
\beta(4) < \gamma(4).
\]

Due to this anomaly, the analogon of theorem 1 is slightly more complicated to state.
TABLE III

Values of Minimal Maximum Effort for $\beta$-trees and $\gamma$-trees with 3 up to 40 End-Vertices

<table>
<thead>
<tr>
<th>$N$</th>
<th>$\beta(N)$</th>
<th>$\gamma(N)$</th>
<th>$\mu(N)$</th>
<th>$N$</th>
<th>$\beta(N)$</th>
<th>$\gamma(N)$</th>
<th>$\mu(N)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>4</td>
<td>3</td>
<td>3</td>
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<td>5</td>
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<td>10</td>
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<tr>
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<td>8</td>
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<tr>
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<td>9</td>
<td>9</td>
<td>40</td>
<td>11</td>
<td>11</td>
<td>11</td>
</tr>
</tbody>
</table>

THEOREM 2. For $N \geq 5$, $\mu(N)$ is attained by $\gamma$-trees with the following structure. If $N = 3^k + r$, $0 \leq r < 2 \cdot 3^k$, then

(i) all internal vertices at levels $1, \ldots, k - 2$ have outdegree 3,

(ii) the subtrees whose roots are sons of the same internal vertex have numbers of end-vertices that differ by at most 1,

(iii) the groups of end-vertices following an internal end-vertex at level $k - 1$ have the structure of optimal trees as indicated in figure 7 for $N = 3, \ldots, 9$.

Proof. Again we use induction on $N$. The proof is essentially simpler than for theorem 1 as clearly we may restrict our attention to optimal trees that have property (ii) without needing a convexity condition. Beside noting the validity of the theorem for small values of $N$, we need only calculate $\beta(N)$ and $\gamma(N)$, assuming the structure to be as described, and then calculate $\beta(N+1)$ and $\gamma(N+1)$ to show consistency. For a $\gamma$-tree we find, putting $N = 3^k + r$,

$\gamma(N) = 3k$ \hspace{1cm} \quad (r = 0),
\gamma(N) = 3k + 1 \hspace{1cm} \quad (1 \leq r \leq 3^{k-1}),
\gamma(N) = 3k + 2 \hspace{1cm} \quad (3^{k-1} + 1 \leq r \leq 3^k),
\gamma(N) = 3k + 3 \hspace{1cm} \quad (3^k + 1 \leq r \leq 2 \cdot 3^k).
For a $\beta$-tree we find, putting $N = 2 \cdot 3^{m-1} + s$,

$$\begin{align*}
\beta(N) &= 3m - 1 \quad (s = 0), \\
\beta(N) &= 3m \quad (1 \leq s \leq 2 \cdot 3^{m-2}), \\
\beta(N) &= 3m + 1 \quad (2 \cdot 3^{m-2} + 1 \leq s \leq 2 \cdot 3^{m-1}), \\
\beta(N) &= 3m + 2 \quad (2 \cdot 3^{m-1} + 1 \leq s \leq 4 \cdot 3^{m-1}).
\end{align*}$$

The values $\mu(3) = 3$, $\mu(4) = 4$, $\mu(5) = \mu(6) = 5$, and $\mu(7) = \mu(8) = \mu(9) = 6$ have been used in the calculation, so that the formulae are considered for $N \geq 10$. Explicit comparison of $\beta(N)$ and $\gamma(N)$ gives

$$\beta(N) \geq \gamma(N).$$
Proving the consistency of $\beta(N + 1)$ and $\gamma(N + 1)$ is tedious; we only give an example, viz. we calculate $\gamma(N + 1)$ for $N + 1 = 3^k + 3r'$.

The three subtrees of the root have $N^* = 3^{k-1} + r'$ end-vertices each. Suppose e.g. that $3^{k-1} + 1 \leq 3r' \leq 3^k$, then $3^{k-2} + 1 \leq r' \leq 3^{k-1}$. Therefore $\gamma(N^*) = 3(k - 1) + 2$ for all three subtrees. The root contributes 3 to the maximal effort of a path and therefore $\gamma(N + 1) = 3k + 2$. All other cases are checked in a similar way. This completes the proof.

Next, we consider the asymptotic behaviour of $\mu(N)$ as $N \to \infty$. Let $N = 3^k + r$. Certainly, if $r = 0$, then

$$\mu(N) = 3 \log_3 N.$$ 

If $r \neq 0$, note that

$$\mu(N) = 3 \log_3 3^k + i \quad (i = 1, 2, 3)$$

for the three intervals. Comparison with $3 \cdot 3 \log_3 N$ for $r = 3^{k-1}, 3^k$ and $2 \cdot 3^k$ shows that

$$\mu(N) \geq 3 \log_3 N.$$ 

The largest differences on $[3^k, 3^{k+1}]$ occur for $r = 1, 3^{k-1} + 1$ and $3^k + 1$, namely

$$d_1 = 1 - 3 \log_3(1 + 3^{-k}),$$
$$d_2 = 2 - 3 \log_3(\frac{2}{3} + 3^{-k}),$$
$$d_3 = 3 - 3 \log_3(2 + 3^{-k}).$$

Thus

$$d_1 < 1,$$
$$d_2 < 5 - 6 \log_3 2,$$
$$d_3 < 3 - 3 \log_3 2,$$

and comparison of these bounds yields

$$3 \log_3 N \leq \mu(N) < 3 \log_3 N + 1.2145.$$ 

Remark. The problems considered in this paper can be generalized in various ways. A very obvious generalization is to attach weight $w_j$ to the $j$th end-vertex and ask for the trees for which the weighted effort

$$\sum_{j=1}^{N} E(p_j) \cdot w_j$$
is minimum. One does not obtain the same solution as when using the Huffman algorithm, because that algorithm always gives a binary tree. This is a point of further investigation.

RECEIVED: February 21, 1978; REVISED: December 27, 1978

Note added in proof. Our theorem 1, with a completely different proof, has been found earlier by D. Knuth. See p. 371, theorem L, in Volume 3 of his "The Art of Computer Programming," Addison-Wesley, Reading, Mass., 1973.

REFERENCE