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Flooding and drying in finite-element discretizations of shallow-water equations.

Part 1: One dimension

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Abstract

Free boundaries in shallow-water equations demarcate the time-dependent water line between “flooded” and “dry” topography. A novel numerical algorithm to treat flooding and drying in a formally second-order explicit space discontinuous finite-element discretization of the one-dimensional or symmetric shallow-water equations is presented. The algorithm uses fixed Eulerian flooded elements and one mixed Eulerian-Lagrangian element at each free boundary. The positivity of the mean water depth is ensured via a time step restriction based on analysis of a maximum principle for the discretized continuity equation while using an HLLC flux. The algorithm and its implementation are tested in comparison with a large and relevant suite of known exact solutions. The essence of the flooding and drying algorithm pivots around the analysis of a continuity equation with a fluid velocity and a pseudo density (in the shallow water case the depth). It therefore also applies, for example, to space discontinuous finite-element discretizations of the compressible Euler equations in which vacuum regions emerge, in analogy of the above dry regions. The approach is hypothesized to extend to finite-volume discretizations with similar mean level and slope reconstruction.

Keywords: shallow-water equations, flooding and drying, free-boundary dynamics, discontinuous finite-element method, positivity of mean water depth

AMS Subject Classification: 76M10, 65M60
1 Introduction

We consider explicit space discontinuous finite-element discretizations of the symmetric or one-dimensional shallow-water equations. This system, which is hyperbolic in its conservative limit, models the flow of a layer of water whose depth is small relative to the horizontal scales of interest. The flow domain is defined by the area where the depth $h$ is positive. In many practical applications, the boundary of the domain may consist of a “free” boundary as well as more conventional boundary conditions (e.g., impenetrable walls). The location of this free boundary is time-dependent due to the motion of the fluid, resulting in “flooding” and “drying”. Accurate prediction of this flooding and drying is important in forecasting river hydraulics, tsunamis, and near-shore surf zone dynamics in beaches and sand banks, as well as in planning dikes. This paper presents the design and verification of a discontinuous finite-element approximation to model such flows in one space dimension, with an emphasis on the treatment of flooding and drying.  

Discontinuous (Galerkin) finite-element schemes have several advantages (e.g., Cockburn et al., 1989; Cockburn and Shu, 1998):

i. The structure of the scheme allows varying orders of accuracy in the elements (so-called $p$-adaptivity).

ii. It is straightforward to use elements with local mesh refinement (so-called $h$-adaptivity).

iii. The scheme is extremely local since communication occurs entirely through fluxes at the faces between elements. This property is used to deal efficiently with the free-boundary dynamics in this paper; it also allows for efficient parallelization. The implementation of other boundary conditions such as in- and outflow conditions is efficient and accurate due to the local nature of the scheme.

Disadvantages are that the scheme is more complex, and that more degrees of freedom are involved relative to finite-volume or finite-difference schemes.

The flooding and drying algorithm consists of the following. To deal with discontinuities such as hydraulic jumps and bores, the depth and velocity times depth are the computational variables advanced in time. Nevertheless, velocity and depth are approximated by the same order polynomial basis functions per element. Consequently, the velocity times depth is approximated by using constrained, higher-order basis functions. This ensures that it becomes zero at the water line exactly when the depth becomes zero, conforming to the mathematical shallow-water model and reality. The finite-element discretization is therefore not Galerkin. The one-dimensional computational domain contains multiple disconnected “patches” of water, the flooded regions, separated by dry regions. For each patch the boundary consists either of the computational boundary or a free boundary. The algorithm uses fixed Eulerian flooded elements and one mixed Eulerian-Lagrangian element at each free boundary. Positivity of the mean water depth is ensured via a time-step restriction based on analysis.

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1A preliminary version was presented in Bokhove and Wirososetno (2003).
of a maximum principle for the discretized continuity equation while using an HLLC flux. The size of the free-boundary element is controlled by elements which merge and split using the mean and slope information of the velocity and depth. Drying in a patch occurs when a dry region emerges in a flooded region resulting in the splitting of the original patch. Merging of patches arises when a dry region between two patches becomes inundated. Drying and merging of multiple patches of fluid is introduced using mean and slope information.

The time-step restriction we have found, is similar to the Perthame’s (1999) which was derived for a flux based on kinetic theory and a first-order finite-volume method with a Euler-forward time discretization. We have adapted the kinetic flux to shallow-water equations and to our flooding and drying treatment in a higher-order discontinuous (Galerkin) finite-element method, but preliminary tests showed the HLLC flux to be more accurate.

We verified the numerical discretization of the shallow-water equations by comparing the numerical solutions with various one-dimensional exact solutions in which flooding and drying occurs: Riemann problems with drying (e.g., Smoller, 1994, and Toro, 1999), the Carrier-Greenspan (1958) solution, the parabolic bowl solution (Young, 1986), and the Peregrine-Williams (2001) solution. This suite of exact verification cases is much more extensive than usual for testing shallow-water discretizations. The design of the algorithm has benefitted from the various test criteria.

Extra numerical dissipation may be required to control unwanted oscillations at discontinuities such as hydraulic jumps and bores. The inclusion of a dissipation operator, analogous to the one in Jaffre et al. (1995) and Van der Vegt and Van der Ven (2002) for the space-time discontinuous Galerkin finite-element discretization, is preferable over a slope limiter for steady state solutions, and hence in general flow situations. Unfortunately, we were unable to derive a dissipation operator adequate for the variety of complex simulations we considered. In the end, we successfully used the slope limiting treatment of Schwanenberg (2003) for the shallow-water equations with topography, and the general shock detection and limiting approach of Krivodonova et al. (2003). Since a rigorous mathematical justification of such extra numerical dissipation appears to be lacking, the addition of extra dissipation is not discussed here as it is not required to test the flooding and drying algorithm.

The paper is organized as follows. The shallow-water equations are introduced in §2. The space discontinuous finite-element method is set up for elements with fixed and moving (free-boundary) nodes or points in §3. The flux on these faces is approximated by the HLLC flux, as explained in §4. The discretization in the special elements near free boundaries is introduced in §5. When used with a time-step restriction, the mean water depth can be guaranteed to remain non-negative. In the end, a robust and accurate method is developed that deals simultaneously with flooding and drying, and discontinuities such as hydraulic bores and jumps. In §6, we present the comparison of numerical and known exact solutions. Finally, the potential of the discretization is illustrated in a complex simulation of waves which are forced on one (offshore) boundary, steepen into bores, and subsequently flow over a dike periodically in §7. We conclude with a discussion in §8.
2 Shallow-Water Equations

The symmetric shallow-water equations (e.g., Pedlosky, 1987) are

\[
\begin{align*}
\partial_t h + \partial_x (h u) &= 0, \\
\partial_t (h u) + \partial_x (h u^2 + P) &= -g h \partial_x h_b + S_2, \\
\partial_t (h v) + \partial_x (h u v) &= S_3,
\end{align*}
\]

(2.1)

with the horizontal velocity field \((u, v) = (u(x, t), v(x, t))\); \(h(x, t)\) the depth of the layer; and \(P = P(h) = g h^2/2\) the “pressure” or potential; with \(g\) being the gravitational acceleration. The terms \(S_2, S_3\) may depend in general on \(x\) and \(t\) explicitly, and implicitly on the variables \(u, v, h\). The fluid’s bottom is at \(z = h_b(x)\). Partial derivatives are denoted by \(\partial_t = \partial/\partial t\), etc.

Let \(U = (h, h u, h v)^T\). To facilitate the numerical formulation, we rewrite (2.1) concisely as

\[
\partial_t U + \partial_x F(U) = S,
\]

(2.2)

where \(F(U) = (h u, h u^2 + P, h u v)^T\) is the flux, and \(S = (0, S_2, S_3)^T\) the additional topographic, dissipative or forcing term.

The flow domain \(\Omega_f \subset \Omega \subset \mathbb{R}\), which may be time dependent, is embedded in a fixed computational domain \(\Omega\). The boundary \(\partial\Omega_f\) (isolated points in our one-dimensional case) consists in general of fixed and free points. At the fixed boundary points, the boundary conditions specify the in- and outflow, or no through flow at walls. The free boundary is specified by \(h(x, t) = 0\). The system (2.1) or (2.2) is completed with initial conditions \(U_0 = U(x, 0)\).

3 Discontinuous Finite-Element Method

3.1 Finite elements

The flow domain \(\Omega_f \subset \Omega = [a, b]\) is partitioned by points \(x_k(t), k = 1, \ldots, N_{el} + 1\), into \(N_{el}\) open elements \(K_k = (x_k, x_{k+1})\). The result is a tessellation

\[
\mathcal{T}_h = \left\{ K_k \bigcup_{k=1}^{N_{el}} K_k = \Omega_f \text{ and } K_k \cap K_{k'} = \emptyset \text{ if } k \neq k', 1 \leq k, k' \leq N_{el} \right\}
\]

(3.1)

with \(\bar{K}_k\) the closure of \(K_k\). For convenience, we will also use the notation \(x_{k,L} := x_k\) and \(x_{k,R} := x_{k+1}\) below; we define \([K_k] = x_{k,R} - x_{k,L}\). Both the elements \(K_k\) and \(N_{el}\) may be time-dependent, and the domain \(\Omega_f\) may consist of disjoint “patches”.

We consider finite-element discretizations of (2.2) with approximations \(U_h\) and \(w_h\) to the state vector \(U\) and test functions \(w\). These are such that \(U_h\) and \(w_h\) belong to the broken space

\[
V_h = \{ w : w_{|K_k} \in P^d \rho(K_k), k = 1, \ldots, N_{el} \},
\]

(3.2)
in which \( P^{d_p}(K_h) \) denotes the space of polynomials in \( K_h \) of degree \( d_p = d_p(U) \). Hence, in each element \( K_h \), we approximate \( U \) and \( w \) by a polynomial spanned by basis functions \( \{ \psi_{n,k}(x) \} \) defined in §3.3 below. The complications arising from the free boundaries are discussed later, in §5.

### 3.2 Weak formulation

When we multiply (2.2) by an appropriate test function \( w_h = w_h(x,t) \) and integrate by parts, the following weak formulation is obtained: Find a \( U_h \in V_h \) such that \( \forall w_h \in V_h \) the following relation is satisfied

\[
\sum_{k=1}^{N} \left\{ \int_{K_k} w_h \partial_t U_h \, dx + [F(x^-_{k,R}) w_h(x^-_{k,R}) - F(x^+_{k,L}) w_h(x^+_{k,L})] \right. \\
- \left. \int_{K_k} F \partial_x w_h \, dx - \int_{K_k} S w_h \, dx \right\} = 0,
\]

(3.3)

where \( w_h(x^-_{k,R}) = \lim_{t \to t^-_{k,R}} w_h(x,t) \) and \( w_h(x^+_{k,L}) = \lim_{t \to t^+_{k,L}} w_h(x,t) \). (Note, we only denote dependencies explicitly when confusion may arise.)

Consider a given point \( x_{k+1} = x_{k,R} = x_{k+1,L} \). Since the elements \( K_k \) are isolated from one another at this stage, \( U_l := U(x^-_{k,R}) \neq U(x^+_{k+1,L}) =: U_r \) and consequently, the flux \( F(x^-_{k,R}) \neq F(x^+_{k+1,L}) \) in general. This is where the numerical flux, which is at the heart of the method, comes into play. We replace \( F(U_l) \) and \( F(U_r) \) by a numerical flux \( \tilde{F}(U_l, U_r) \) chosen to satisfy the following conditions (Shu and Osher, 1989): i) \( \tilde{F}(U_l, U_r) \) is a monotonous flux for \( F(U) \) in the sense that \( \tilde{F}(U_l, U_r) \) is Lipschitz continuous in both arguments \( U_l \) and \( U_r \); ii) \( \tilde{F} \) is consistent in the sense that \( \tilde{F}(U, U) = F(U) \); and iii) \( \tilde{F} \) is an E-flux

\[
\int_{U_l}^{U_r} (F(s) - \tilde{F}(U_l, U_r)) \, ds \geq 0.
\]

Several choices are possible for \( \tilde{F} \). In this paper we use the HLLC flux which is discussed in detail in section 4.

### 3.3 Discretized weak formulation

Within each element \( K_k \), \( U(x,t) \) is approximated by a polynomial

\[
U_h(x,t) = \sum_{m=0}^{d_p} \hat{U}_{m,k}(t) \psi_{m,k}(x,t)
\]

(3.4)

and similarly for the test function \( w \). Basis and test functions satisfy

\[
\hat{\psi}_0(\zeta) = 1, \quad \text{and} \quad \hat{\psi}_m(\zeta) = \zeta^m - c_m \quad \text{for} \quad m = 1, \ldots, d_p,
\]

(3.5)
in a reference element $\hat{K}$ with coordinate $\zeta \in (-1,1)$, were $d_p$ is the maximum degree of the polynomials used and constants $c_m$ ensure that $\hat{\psi}_m(x)$ has zero mean in $\hat{K}$ for $m \geq 1$.

Taking $d_p = 1$ or 2, we approximate $h, u, v$ and $w$ by its mean and its slope:

$$U_h(x,t) = \bar{U}_k + \bar{U}_k \psi_1,k(x,t) + \bar{U}_{2k} \psi_2,k(x,t) \quad \text{and} \quad w_h(x,t) = \bar{W}_k + \bar{W}_k \psi_1,k(x,t)$$

(3.6)

with $C_k = C(K,k)$, $h_{2k} = 0, (h,u)_{2k} = \hat{h}_k \hat{u}_k$, $(h,v)_{2k} = \hat{h}_k \hat{v}_k$. The latter two relations follow by projection and ensure that $h,h u$ and $h v$ are all zero when the depth $h$ reaches zero. Since elements are allowed to move, we define

$$\frac{d|K_k|}{dt} = dx_{k,R}/dt - dx_{k,L}/dt \quad \text{and} \quad dx_{k,R}/dt \equiv V_{k,R}.$$  

(3.7)

Since $\bar{W}_k$ and $\bar{W}_k$ are arbitrary, we obtain the following equations for the mean and fluctuating part after substituting (3.6) into (3.3), and introducing the numerical flux $\bar{F}$:

$$\frac{d}{dt} \left( |K_k| \bar{U}_k \right) = - \left[ \bar{F}_v \left( U(x^-_{k,R}), U(x^+_{k+1,L}) \right) \right. $$

$$\left. - \bar{F}_v \left( U(x^-_{k-1,R}), U(x^+_{k,L}) \right) \right] + \frac{|K_k|}{2} \int_{-1}^{1} S(U_h, \zeta, t) \, d\zeta $$

$$\frac{d\bar{U}_k}{dt} = - \frac{3}{|K_k|} \left[ \bar{F} \left( U(x^-_{k-1,R}), U(x^+_{k,L}) \right) \right. $$

$$\left. + \bar{F} \left( U(x^-_{k-1,R}), U(x^+_{k,L}) \right) \right] + \hat{U}_k \left( V_{k,R} - V_{k,L} \right) / |K_k| $$

$$+ 2 \bar{U}_{2,k} \left( V_{k,R} + V_{k,L} \right) / |K_k| + \frac{3}{|K_k|} \int_{-1}^{1} F(\zeta) \, d\zeta + \frac{3}{2} \int_{-1}^{1} S(\zeta) \, d\zeta$$

$$\frac{dx_{k,R,L}}{dt} = V_{k,R,L},$$

(3.8)

where $\bar{F}_v(U_l, U_r)$ is a numerical flux representing $F(U) - V U$, which reduces to $F(U_l, U_r)$ when the nodal velocity $V = 0$. The remaining integrals of nonlinear functions of $h,u$ and $v$ are approximated by using Gaussian quadrature. Time discretization can be, for example, with the third-order Runge-Kutta method of Shu and Osher (1989), as used in the tests in §6.

For hyperbolic systems, numerical fluxes are often based on the exact or approximate Riemann problem at the faces of the elements. It turns out that it is difficult to guarantee numerical stability near or at free boundaries where the depth is small or zero, because small errors in the numerical flux can lead to negative depths. We therefore present and analyze the HLLC numerical flux in §4 and §5.3.

### 4 HLLC Flux

As noted above, since $U$ is discontinuous across element faces, we need to approximate the flux $F(U)$ by a numerical flux $\bar{F}(U)$. Similarly, we approximate $F_v = F(U) - V U$
by a numerical flux $\hat{F}_v(U)$. In this section we describe one such approximation, the HLLC flux. In §5.3, we show that, when used in conjunction with a Euler forward time discretization, this approximation has the desirable property of keeping the mean depth in each element non-negative. In our approximation the depth $h = \bar{h} + \zeta \hat{h}$. Dry patches emerge when the depth becomes zero in a flooded region. Hence it is undesirable to enforce $\hat{h}$ to be positive. Instead, the slope information is used to indicate where dry regions emerge, that is, when $|\hat{h}| \leq \bar{h}$.

Following Godunov, an accurate numerical flux can be devised as follows: approximating the variable $U$ adjacent to a point by the constant values $U_l$ and $U_r$ immediately left and right of a face, one obtains a Riemann problem which can be solved exactly. However, since an exact Riemann solver is numerically expensive, the approximate HLLC solver developed by Toro et al. (1994) is used instead. Next, we briefly summarize the HLLC flux for later use.

First, consider the Riemann problem

$$\partial_t U + \partial_x F(U) = 0$$

with constant initial left and right states $U_l$ and $U_r$, respectively. We are seeking an expression for $F - V U$ on either side of the space-time cell edge, which velocity we denote by $V$, such that a communication between the elements is properly established. In the Riemann problem for the shallow-water equation (Toro, 1999), we can distinguish four situations without dry patches where either $P_l \neq 0, P_r \neq 0$ or $u_l + c_l > u_r + c_r$ with $c = \int^h a(w) / w \, dw = 2 \sqrt{g h}$ and gravity-wave speed $a = \sqrt{\partial P / \partial h} = \sqrt{g h}$, and three more situations where drying is possible. Drying occurs where $h = 0$.

![Figure 1: The wave configuration is sketched to define the variables and states involved in the HLLC flux approximation. The constant left and right states $U_{l,r}$ are separated by a left and right wave $S_{l,r}$ from two star regions $U_{s l,r}$. These two star regions are divided by a contact wave $S_m$.](image)

Consider the element boundary $x' = x_{k+1}(t) - x_{k+1}(t^n) = V(t^n) \tau$ for $0 < \tau = t^{n+1} - t^n < \Delta \tau$, which moves at a speed $V$. The (discontinuous) values $U_h$, immediately left and right of this face are $U_l$ and $U_r$, respectively. In the flooded case,
the HLLC approach assumes that there are four constant states from left to right \(U_l, U_l^*, U_r, U_r\), separated at \((x-x_{k+1})/(t-t^n) = S_l, S_m, S_r\). We integrate (4.1) over two control volumes ODCE and OABE to the left and right of the space-time face \(x^{'} = 0\), see Fig. 1, for the four cases (i) \(S_l < V, S_r, S_m > V\); (ii) \(S_l < V, S_r > V, S_m > V\); (iii) \(S_l, S_m, S_r < V\); and iv) \(S_l, S_m, S_r > V\). Subsequently, we calculate \(F - V U\) along the point \(x_{k+1}(t)\) as the average of the contribution on either side. After using Gauss’ theorem in space and time on (4.1), we obtain (see Van der Vegt and Van der Ven, 2002):

\[
F_{v,hlC}(U_l, U_r) = \frac{1}{2\Delta t} \left( \int_{S_-} (F - V U_h) \bar{n}_{-K} \, dl_- + \int_{S_+} (F - V U_h) \bar{n}_{+K} \, dl_+ \right)
\]

\[
= \frac{1}{2} \left\{ F_l - V (U_l + U_r) - (|S_l - V| - |S_m - V|) U_l^* + (|S_r - V| - |S_m - V|) U_r^* + |S_l - V| U_l - |S_r - V| U_r \right\},
\]

where \(F_{l,r} = F(U_l, U_r), \bar{n}_{\pm}\) are outward pointing normal vectors in space time along the face \(x_{k+1}(t)\), and \(dl_{\pm}\) are pieces along the boundary \(S_{\pm}\) of the area left and right of the face \(x_{k+1}(t)\). The intermediate states \(U_{l,r}^*\) and speed \(S_m\) follow by a) using the Rankine-Hugoniot relations and b) the intermediate one star state, \(U^*, HLL\) approach (Batten et al., 1997):

**a)** By using the Rankine-Hugoniot relations for \(\partial_t U + \partial_2 F = 0\) we find

\[
(S_l - S_m) U_{l,r}^* = (S_{l,r} - u_{l,r}) U_{l,r} + \begin{pmatrix} 0 \\ P^* - P_{l,r} \end{pmatrix}
\]

(4.3)

with

\[
P^* = P_{l,r}^* = P_{l,r} + h_{l,r} (S_{l,r} - u_{l,r}) \cdot (S_m - u_{l,r}).
\]

(4.4)

and velocities \(u_{l,r}\). By combining (4.2) and (4.3) we can deal with the cases \(S_{l,r} = S_m\) in order to avoid a division by zero. The latter handling is important when \(h = 0\) at one of the faces of an element.

**b)** Using the HLL approach with one intermediate state \(\tilde{U}^*\) to determine \(S_m\), and assuming that \(u_l^* = u_r^* = \tilde{u}^* = S_m\), we find that

\[
\tilde{u}^* = S_m = \frac{P_l - P_r + h_r u_r (S_r - u_r) - h_l u_l (S_l - u_l)}{h_r (S_r - u_r) - h_l (S_l - u_l)},
\]

(4.5)

and it may be checked that \(P^* = P_{l,r}^* = P_{l,r} + h_l (S_l - u_l)/(S_m - u_l)\) as it should be for contact discontinuities with speed \(S_m\). By retracing the derivation for \(S_m\) we find that \(S_m = S_l = S_l\) when \(h_r (S_r - u_r) - h_l (S_l - u_l) = 0\). The latter can happen when a dry patch is about to emerge.
Finally, the wave speeds are estimated based on the left and right moving rarefaction wave speeds

\[ S_l = \min(u_l - a_l, u_r - a_r), \quad S_r = \max(u_l + a_l, u_r + a_r), \]  

respectively, with \( a^2 = \partial P/\partial h \). When \( S_l > V \) the flux simplifies to \( F_l - V U_l \) and when \( S_r < V \) to \( F_r - V U_r \), i.e. the upwind cases. When we choose a Lagrangian grid velocity the choice \( V = S_m \) is appropriate since

\[ \lim_{V \to S_m} \tilde{F}_{\text{r,hlc}}(U_l, U_r) = (0, P^*, 0)^T, \]  

and \( P^* = P_r \) if \( h_l = 0 \) and \( u_r = S_m \), or \( P^* = P_l \) if \( h_r = 0 \) and \( u_l = S_m \).

## 5 Flooding and Drying

### 5.1 Fluid patches

The fluid is divided into one or more distinct patches of fluid in a bounded region, see Fig. 2. In one dimension, each patch has a left and right boundary. The boundary conditions at the external boundaries are, for example, (prescribed) in- or outflow conditions (depending on the characteristics), and no flow through moving or fixed walls.

![Figure 2: A sketch of the \( M \) patches \( P_i \) within the space \( \Omega \).](image1)

![Figure 3: A patch in the finite-element space with a free-boundary element \( K_{rh} \) on the right. Denoted is the height field \( h(x,t) \) for the case with linear basis functions. Note that in \( K_{rh} \) we have \( h = 0 \) at the free boundary.](image2)
In one dimension a dry-wet boundary is demarcated by a particle on the left or right of the patch. There are three situations to distinguish: i) a patch of fluid moves along with a dry-wet boundary on the left, on the right, or at both sides; ii) a patch of fluid breaks up into two patches in which case we may have to define two new particles; and iii) two patches of fluid merge into one patch of fluid.

5.2 Free-boundary elements

The elements of a patch consist of Eulerian fixed elements and mixed Eulerian-Lagrangian free-boundary elements. The Eulerian fixed elements of \( \Omega_f \) coincide with a subset of underlying fixed elements of the computational domain \( \Omega \).

A free, dry-wet boundary on the right of a patch is modeled with a free-boundary element which right node moves with the flow, see Fig. 3. At this right node of the free-boundary element the depth is generally zero. The left node of this free-boundary element is fixed.

When the depth is zero at this right node \( k = ri \), we constrain the linear representation \( h_k = \tilde{h}_k + \tilde{h}_k \zeta \) to be zero at \( \zeta = 1 \) such that \( \tilde{h}_{ri} = -\tilde{h}_{ri} \). From the mean part of (3.8), we find the equations of motion for \( \tilde{h}_{ri} \)

\[
|K_{ri}| \frac{d\tilde{h}_{ri}}{dt} = \tilde{F}(h u(x_{ri-1,R}), h u(x_{ri,L})) - \tilde{h}_{ri} V_R \tag{5.1}
\]

with \( V_R = dx_{ri,R}/dt \) and \( V_L = 0 \). Likewise for a free-boundary element on the left \( k = le \), and we constrain \( \tilde{h}_{le} = \tilde{h}_{le} \). From (3.8), we find the equations of motion for \( \tilde{h}_{le} \)

\[
|K_{le}| \frac{d\tilde{h}_{le}}{dt} = -\tilde{F}(h u(x_{le,R}), h u(x_{le+1,L})) + \tilde{h}_{le} V_L \tag{5.2}
\]

with \( V_L = dx_{le,L}/dt \) and \( V_R = 0 \). The discretization for the other variables remains unconstrained, which formally ensures second-order accuracy for the velocity and hence the advancement of the free boundary.

5.3 Time-step criterion for the mean depth

Although \( h \) may be positive at a free boundary as initial condition, the Riemann problem indicates that \( h = 0 \) at the free boundary immediately thereafter. When \( h > 0 \) at a free boundary, the velocity of the free boundary nodes are chosen as \( V = V_L = u_r - c_r \) and \( V = V_R = u_l + c_l \), respectively, based on the Riemann problem with a dry region on the left or right. Once \( c_{l,r} \) has become zero at the free boundary, we therefore enforce \( h = 0 \) at the free boundary, whence \( V \) is the local advective velocity. The associated flux at the front is then zero, as in the HLLC scheme with \( V = S_m \). Using an Euler forward time step scheme as an example, we next show that the mean depth remains non-negative for a suitably restricted time step.
Proposition 5.1. Given the following assumptions and definition:

A1: Use an Euler forward time stepping scheme to integrate the continuity equation \( \partial_t h + \partial_x (h \mathbf{u}) = 0 \) from time \( t^n \) to time \( t^{n+1} \).

A2: Let the depth \( h^n = \hat{h}^n_k + \zeta \hat{h}^n_k \) be non-negative at time \( t^n \). Let velocities \( \mathbf{u} = \mathbf{u} + \zeta \mathbf{u} \) and \( \mathbf{v} = \mathbf{v} + \zeta \mathbf{v} \). Define \( \tilde{m}_u = h \mathbf{u} = \bar{m}_u + \zeta \bar{m}_{u,1} + (\zeta^2 - 1/3) \bar{m}_{u,2} \) and likewise for \( \tilde{m}_v = h \mathbf{v} \). Using the weak formulation for \( \tilde{m}_u = h \mathbf{u} \) and \( \tilde{m}_v = h \mathbf{v} \), we then obtain

\[
\tilde{u} = \frac{3 \bar{m}_u \hat{h} - \bar{m}_{u,1} \hat{h}}{3 \hat{h}^2 - \hat{h}^2}, \quad \hat{u} = \frac{3 (\bar{m}_{u,1} \hat{h} - \bar{m}_u \hat{h})}{3 \hat{h}^2 - \hat{h}^2}, \quad \bar{m}_{u,2} = \hat{h} \hat{u} \quad (5.3)
\]

and likewise for \( \tilde{\mathbf{v}} \) and \( \hat{\mathbf{v}} \).

A3: Use the discontinuous finite-element discretization (3.8) for the continuity equation with the HLLC flux defined in (4.2)-(4.6).

Definition: Define the Heaviside function \( \Theta(x) \) with \( \Theta(x) = 0 \) when \( x < 0 \) and \( \Theta(x) = 1 \) for \( x \geq 0 \), and denote by \( S_{mL} \) the \( S_m \)-velocity at the left face of an element, and so forth.

Given assumptions A1–A3 and definition, the mean depth \( \tilde{h}^{n+1} \) at the next time step is positive provided the minimum of the elemental time step \( \Delta t_k \) is taken, where in each interior flooded element:

\[
\Delta t_k < \frac{1}{2} |K_k| \left( -u_{rL} \Theta(-S_{mL}) \Theta(-S_{rL}) + \frac{|S_{mL}| (S_{rL} - u_{rL})}{(S_{rL} - S_{mL})} \Theta(-S_{mL}) \Theta(S_{rL}) + \frac{|S_{mR}| (|S_{rR}| + u_{rR})}{(S_{mR} - S_{rR})} \Theta(S_{mR}) \Theta(-S_{rR}) + u_{rR} \Theta(S_{mR}) \Theta(S_{rR}) \right), \quad (5.4)
\]

and in each left and right free-boundary element:

\[
\Delta t_k < |K_k| \begin{cases} 
\max \left( V_{R} - 2 u_{rL} \Theta(-S_{mL}), 0 \right) & S_{rL} < 0 \\
\max \left( V_{R} + 2 \frac{|S_{mL}| (S_{rL} - u_{rL})}{(S_{rL} - S_{mL})} \Theta(-S_{mL}), 0 \right) & S_{rL} > 0 
\end{cases} \quad \text{and} \quad (5.5)
\]

\[
\Delta t_k < |K_k| \begin{cases} 
\max \left( -V_{L} + 2 \frac{|S_{mR}| (|S_{rR}| + u_{rR})}{(S_{mR} - S_{rR})} \Theta(S_{mR}), 0 \right) & S_{rR} < 0 \\
\max \left( -V_{L} + 2 u_{rR} \Theta(S_{mR}), 0 \right) & S_{rR} > 0 
\end{cases}, \quad (5.6)
\]

respectively. All terms on the right-hand-side in (5.4)-(5.6) are evaluated at time \( t^n \).

Proof. Consider the Euler forward time integration for the mean depth in the discontinuous finite-element discretization:

\[
\tilde{h}^{n+1}_k = \tilde{h}^n_k - \left[ \tilde{F}_h(U^n_k, U_{k+1}^n) - \tilde{F}_h(U_{k-1}^n, U^n_k) \right] / |K^*_k| + (V^n_L - V^n_R) \tilde{h}^n_k / |K^*_k|, \quad (5.7)
\]

where \( \tilde{h}^{n+1}_k \) is the next time step and \( \tilde{h}^n_k \) the previous time step in element \( k \). For an interior element, \( V_{L,R} = 0 \); for a left edge element \( V_R = 0 \) and for a right edge element
\( V_L = 0 \). Define the shorthand \( S^V_t = S_t - V \), et cetera. The subscript in \( \tilde{F}_{v,h} \) denotes that we consider the depth component \( h u \) of the flux (4.2), i.e.

\[
2 \tilde{F}_{v,h} = u_t h_t + u_r h_r - \left( |S^V_t| - |S^V_m| \right) h_t + \left( |S^V_r| - |S^V_m| \right) h_r + \frac{h_t}{S_t - S_m} + \frac{h_r}{S_r - S_m}.
\]

The flux (5.8) at a face reduces to the following four cases

\[
\tilde{F}_{v,h} = \begin{cases} 
    h_r (u_r - V) < 0 & \text{if } S^V_t < S^V_m < S^V_r < 0 \\
    h_r (V - S_m) (u_r - S_r)/(S_r - S_m) < 0 & \text{if } S^V_t < S^V_m < 0 & S^V_r > 0 \\
    h_t (u_t - V) > 0 & \text{if } 0 < S^V_t < S^V_m < S^V_r \\
    h_t (S_m - V) (u_t - S_t)/(S_m - S_t) > 0 & \text{if } S^V_t < 0 & 0 < S^V_m < S^V_r 
\end{cases}
\]

in which we used the relation (4.6) to determine the inequalities. Since the HLLC depth flux is linear in \( h \), we obtain an equation linear in \( \tilde{h}_h \) in the mean depth equation (5.7) after substituting the estimate \( \tilde{h}_{l,r} = \tilde{h}_{l,r} \pm \tilde{h}_{l,r} < 2 \tilde{h} \). By using a maximum principle (e.g. Morton and Mayers, 1994) and by collecting the contribution at both faces, time step restrictions (5.4)–(5.6) are obtained.

The minimum of these time step restrictions for each element determines a bound on the overall time step in order to ensure positivity of the mean depth. The information speed following from the magnitude of the eigenvalues of the associated Riemann invariants may provide additional time step restrictions.

**Remarks:**

(i) The time-step criterion extends to the third-order Runge-Kutta scheme by repeating the proposition for every intermediate stage with intermediate depth \( \tilde{h}^{(i)} \) assuming subsequently that \( \tilde{h}^{(i)} \pm \tilde{h}^{(i)} > 0 \) and \( \tilde{h}^{(i)} > 0 \). Unfortunately, a smaller time step for \( h^{(i)} \) may require a restart at \( t^n \).

(ii) The theorem is valid for any one-dimensional continuity equation using the HLLC flux. Hence, it is not restricted to the shallow-water case, but also valid, for example, for the compressible Euler equations by replacing \( h \) by a density \( \rho \).

(iii) The proof shows that the minimum time step per element is most easily implemented numerically in parallel with the determination of the fluxes.

(iv) The conditions in remark (i) are ensured by limiting \( m \) and \( h \) to zero whenever \( h < 0 \) at one side of an element at an intermediate stage.

### 5.4 Merging and splitting of elements

In order to maintain the mostly Eulerian nature of the numerical scheme, i) we split a free-boundary element when it becomes too large, and ii) merge a free-boundary
element with its neighbor when the former becomes too small. Therefore, the number of elements in a patch with at least one free boundary may change over time. Due to the local nature of discontinuous finite-element methods, this update in mesh topology is handled locally.

The element-splitting process (which operationally is performed when a free-boundary element becomes larger than, say, the size of the local underlying fixed element $K_k$ plus 0.6 times the size of the overflown adjacent fixed element) is quite simple: the part of the free-boundary element entirely between two of the fixed faces is made a regular element, and the remainder the new free-boundary element [see Fig. 4a]. For the constraint linear basis function, no information is lost in this process. We determine the new values simply by projection. Mean and slope values of the depth and the velocity are thus preserved in this splitting process. This operation is not performed when there is a front nearby (and a patch merger—see below—is imminent).

In the element-merging process (performed operationally when the size of an edge element falls below, say, 0.4 times that of the local regular element), we construct a new free (i.e. “triangular”) element from the old free-boundary element and its neighbor, preserving the integral of $h_h, u_h, v_h$, and the location of the original front [see Fig. 4b]. For the linear basis function at the free boundary, the slope information $\tilde{h}_k, \tilde{u}_k, \tilde{v}_k$ of the neighboring element is constrained. Only mean values are thus
5.5 Merging and splitting of patches

In addition to element merging and splitting discussed above, patches may also merge and split. When the depth of the fluid becomes zero in the interior of a patch or when a splitting criterion is met, the patch is split into two patches with free-boundary conditions at the splitting point. In the absence of source terms $\mathcal{S}$, the splitting criterion derives optimally from a Riemann splitting criterion for a flat bottom case, that is, at an edge splitting occurs when $u_l + c_l \leq u_r - c_r$ or approximately since $c_{r,l} \ll 0$, when $u_l \leq u_r$ (see, e.g., Smoller, 1994; and Toro, 1999). Alternatively, the Riemann problem may be considered for a locally uniform bottom slope. When fronts from two patches meet, we merge the patches into a single patch. For computational reasons, patches are constrained to have a minimum of two elements.

![Figure 5: Patch splitting is sketched for the depth field in case of linear basis functions.](image)

![Figure 6: Patch merging is sketched for the depth field for two generic cases where free boundaries overlap a) in one element and b) in two elements in case of linear basis functions.](image)

It is easy to see that, due to our linear basis functions, the depth of the fluid must first become zero at a face before it does in the interior of an element. Thus, given
sufficient temporal resolution as guaranteed by the time-step criterion for the mean
depth, patch splitting always occurs at a face. When this happens, the two (regular)
elements bordering the face are made free-boundary elements having the same length
as the underlying regular elements, preserving the integral of $U$ [see Fig. 5]. Note
that generically the depth vanishes only on one side of the face; in this case the
slope information of the non-zero element is limited such that at the edge the depth
reaches zero. We use a slope limiter to limit a negative depth at a face to zero at
each (intermediate) time stage.

When the free boundaries of two patches meet or overlap after a full time step,
the patches are merged: the free-boundary elements are converted into regular ele-
ments, preserving as much of the original information as possible. Due to the finite
temporal resolution in practice, we perform the patch merging procedure when two
free-boundary elements have actually overlapped. There are several scenarios for this
overlap, which can be simplified if we split the free-boundary elements before merging
the patches. After the boundary-element splitting, we are left with two possibilities:
(a) the more generic case where the free boundary elements overlap within one regular
element [see Fig. 6a], and (b) the less generic case where the free-boundary elements
overlap within two regular elements [see Fig. 6b].

A note on implementation: the set of patches (whose number should be small in
normal operation) is implemented as a linked list each of which contains information
about the boundary conditions, the location of the boundaries, of the neighboring
patches (if any), and the free-boundary elements (if any).

6 Verification

In the numerical verification, we consider the dam break problem, the drying Riemann
problem over a flat bottom, the parabolic bowl solution (Young, 1986), the Carrier-
Greenspan solution (Carrier and Greenspan, 1958) and the Peregrine-Williams solu-
tion (Peregrine and Williams, 2001). These exact solutions are summarized in
Appendix A.

In the verification cases, the numerical solution is compared with the exact solu-
tion. We use the $L^2$-error

$$L^2(u, h) = \left( \int_{\Omega_f} (u_{\text{numerical}} - u_{\text{exact}})^2 + (h_{\text{numerical}} - h_{\text{exact}})^2 \, dx/L_f \right)^{1/2}$$

$$L^2(m, h) = \left( \int_{\Omega_f} (U_{\text{numerical}} - U_{\text{exact}})^2 \, dx/L_f \right)^{1/2}$$

with $L_f$ the flooded part of the computational domain, and the $L^\infty$-error, the maxi-
mum difference between the numerical and exact solution. When possible and useful,
the error between the exact and numerical solution of frontal positions, and the error
in the break-up time of a patch of fluid is calculated as well.

Scaled equations have been used, effectively taking $g = 1$. The total number of
fixed finite elements is stated in each case below, across the computational domain.
In the actual computation, only the flooded (fixed and free boundary) elements are used in action.

6.1 Riemann solutions

A summary of the solutions to the Riemann problem for the shallow-water equations can be derived from Smoller (1994) or Toro (1999).

6.1.1 Dam break

Consider a dam break problem with as initial condition $h(x, -t_0) = H_0$ for $x - x_0 < 0$ and zero elsewhere, and $u(x, -t_0) = 0$. Since the initial condition is discontinuous, we adapted our boundary treatment by using both slope and mean as variable as long as the depth at the front is positive, while the speed of the front is the prediction $u_t + c_t$ for the Riemann problem with one one side a dry bed. Otherwise, when the depth has become zero at the front, the free-boundary treatment explained before is used.

The numerical and exact solutions are displayed in Fig. 7. From Table 1, we conclude that the solution converges at order $\sim 0.7$ in the $L^2(u, h)$-norm, at order $\sim 0.65$ in the position of the front, while it is of order 0.9 for the $L^1(m, h)$-norm and 0.4 in the $L^\infty$-norm. Both the dam break problem and the next case in which a dry patch emerges are special. These cases concern solutions, starting from a discontinuity, moving over horizontal topography in which the depth becomes tangent to the bed in a quadratic manner near the free boundary, i.e. $h \propto (x_R(t) - x)^2$ at a free boundary on the right of a patch. The depth and speed at the front are therefore estimated incorrectly with linear polynomials. Higher-order (constrained) polynomial approximations tangent to the bed are required to approximate $m$ and $h$ in the edge element. In contrast, in the other cases which are considered with topography, the free surface intersects the boundary at a finite angle, and the representation by linear polynomials is sufficient even when the actual solution is quadratic (but not tangent to the bed).

6.1.2 Drying

A dry patch occurs in Riemann problems when the constant initial data at $t = 0$ meet the drying criterion $2c_t + 2c_r - u_r + u_t < 0$. Two expansion waves then propagate away from a dry patch, which immediately appears at $t \downarrow 0$.

The numerical results in Table 2 and Fig. 8 show that the accuracy is of order 0.9 for $L^2(m, h)$, while it diverges for $L^2(u, h)$ and $L^\infty$. The initial conditions are chosen such that a small dry patch develops. The numerical solution develops this dry patch initially, but due to numerical dissipation the patch incorrectly floods again thereafter. As in the dam break problem, the velocity and depth are constant and linear in the numerical solution, while being linear and quadratically tangent to the bed in the exact solution. Again, higher-order constrained polynomials at the free boundary are hypothesized to remedy this mismatch.
\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
N & L^2(u, h) & p & L^2(m, h) & p & L^\infty & p & \epsilon_{x_b} & p \\
\hline
50 & 0.030743 & 0.019413 & 0.09060 & 1.9698 \\
100 & 0.019316 & 0.67 & 0.08300 & 0.13 & 1.3194 & 0.58 \\
200 & 0.01859 & 0.70 & 0.06351 & 0.39 & 0.8747 & 0.59 \\
400 & 0.007155 & 0.73 & 0.04631 & 0.46 & 0.5570 & 0.65 \\
800 & 0.004200 & 0.77 & 0.03438 & 0.43 & 0.3391 & 0.72 \\
\hline
\end{array}
\]

Table 1: Dam break test without extra dissipation. Error in norms \(L^2(u, h), L^2(m, h), L^\infty\); and the frontal position, \(\epsilon_{x_b} = x_b - x_b^{\text{numerical}}\), at time \(t = 4\). Parameters and initial conditions are \(x < x_0 : u(x, 0) = 0, h(x, 0) = 1; x > x_0 : u(x, 0) = 0, h(x, 0) = 0\) with \(x_0 = 10, a_0 = H_0 = 1, CFL = 0.1\).

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
N & L^2(u, h) & p & L^2(m, h) & p & L^\infty \\
\hline
50 & 0.2423 & 0.074455 & 1.630 \\
100 & 0.1558 & 0.64 & 0.026786 & 1.47 & 1.549 \\
200 & 0.1831 & -0.23 & 0.014764 & 0.86 & 1.523 \\
400 & 0.1475 & 0.31 & 0.007556 & 0.97 & 1.499 \\
800 & 0.1550 & -0.07 & 0.004023 & 0.91 & 1.496 \\
\hline
\end{array}
\]

Table 2: Drying test without extra dissipation. Error in norms \(L^2(u, h), L^2(m, h), L^\infty\); and the frontal position closest to the origin, \(\epsilon_{x_b} = x_b - x_b^{\text{numerical}}\), at time \(t = 1.5\). Parameters and initial conditions \(x < x_0 : u(x, 0) = u_l, h(x, 0) = h_l; x > x_0 : u(x, 0) = u_r, h(x, 0) = h_r\) with \(h_l = 0.5, h_r = 1, u_l = 0, u_r = 3.5, c_0 = 1, CFL = 0.5\). The drying criterion \(2c_l + 2c_r - u_r + u_l \approx -8.5786 \times 2 < 0\) is met.
Figure 7: Free surface and velocity profiles are shown as function of space and time for the dam break problem. Solid lines display the numerical solution, and dashed lines the exact solution. 400 elements are used without additional numerical dissipation.

6.2 Parabolic bowl

The results in Table 3 and Fig. 9 shows that the numerical accuracy is of order 2. The parabolic test solution is thus sufficiently smooth so that the numerical results converge to the expected second-order accuracy. The difference of the exact and numerical position of the free boundary at a fixed time is not such a good indicator, perhaps because the solution and its error are oscillatory or because the element splitting process (temporarily and intermittently) reduces accuracy (which is not visible in the graph of $x_R(t)$ versus $t$). Nevertheless, this difference converges as well, but slower. Note that the free surface in the parabolic bowl solution intersects the topography at finite angle, in contrast to the previous two solutions.
Figure 8: Free surface and velocity profiles are shown as function of space and time for the Riemann problem in which a dry patch emerges. Solid lines display the numerical solution, and dashed lines the exact solution. 400 elements are used without additional numerical dissipation.

### 6.3 Carrier-Greenspan

The numerical results are given in Table 4 and Fig. 10. The numerical accuracy in the interior is about order 2 for the $L^2$-norms and around order 1 for the $L^\infty$-norm and the error in the frontal position. The plot of the height field $h(x, t)$ shows clearly that some accuracy is lost where the gradient becomes very steep or infinite, which coincides with the region where the Jacobian of the transformation becomes small.

### 6.4 Peregrine-Williams

Peregrine and Williams (2001) derived an analytical solution for a dam break problem of a shallow-water layer on a steep slope of a dike. Eventually, the water reaches the top of the dike and falls down. The top of the dike defines a critical point in the flow. Peregrine and Williams (2001) scaled the shallow-water equations, which form
<table>
<thead>
<tr>
<th>$N$</th>
<th>$L^2(u, h)$</th>
<th>$p$</th>
<th>$L^2(m, h)$</th>
<th>$p$</th>
<th>$L^\infty$</th>
<th>$p$</th>
<th>$\epsilon_{x_b}$</th>
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Table 3: Parabolic bowl test without extra dissipation. Error in the norms $L^2(u, h)$, $L^2(m, h)$, $L^\infty$; and the error in the frontal position $x_b$, $\epsilon_{x_b} = x_b - x_b^{\text{numerical}}$, all at time $t = 3$. Parameters and initial conditions are $u(x, 0) = 0, h(x, 0) = 0.5 h_1(0) (x_b(0) - (x - 0.5)^2)$ with $B = 12, x_b(0) = \sqrt{1/6}, h_1(0) = 12, CFL = 0.01$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$L^2(u, h)$</th>
<th>$p$</th>
<th>$L^2(m, h)$</th>
<th>$p$</th>
<th>$L^\infty$</th>
<th>$p$</th>
<th>$\epsilon_{x_b}$</th>
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Table 4: Carrier-Greenspan test without extra dissipation. Error in norms $L^2(u, h)$, $L^2(m, h)$, $L^\infty$; and the error on the frontal position $x_b$, $\epsilon_{x_b} = x_b - x_b^{\text{numerical}}$, all at time $t = 4$. $CFL = 0.01$. For the other parameters and initial conditions see the Appendix.
Figure 9: Free surface and velocity profiles are shown as function of space and time for the parabolic bowl problem. Solid lines display the numerical solution, and dashed lines the exact solution and the beach topography. 40 elements are used without additional numerical dissipation.

is equivalent to shallow-water equations either on a shallow slope with a horizontal coordinate $x$ or on a steep slope in which coordinate $x$ follows the slope.

A numerical simulation with a resolution of 200 elements and no extra numerical dissipation shows the numerical (solid lines) and exact (dashed lines) in Fig. 11. The accuracy of the $L^2$-norms is about 1. The accuracy at the front is about 0.7. The prediction of the break up time converges slowly, eventually at an order 0.5. The $L^\infty$-norm is lowest at order 0.36.

7 Run-up and Overtopping

Finally, a complex simulation with wave breaking, run-up and overtopping is shown to display the potential of the presented numerical discretization of the shallow-water equations. The space-time plot of depth and velocity profiles in Fig. 12 depicts the time history of multiple waves steepening to bores, running up the beach, and spilling
water over the top of the dike. This dike top is a critical point as in the Peregrine-Williams solution. Beyond the dike top the water rushes down in broken patches of fluid. The offshore beach slope is scaled to unity, and a wave maker introduces sinusoidal waves offshore. At the left boundary \( x = 0 \), we specify the state \( U_0 \), used in the flux calculation. It is grown from rest and \( h = H_0 \) linearly in \( t \) from \( 0 < t < 0.1 \) to the solution for \( t \geq 0.1 \)

\[
    u_t(0,t) = -k A \sin(k x - \omega t)/\omega \quad h_t(0,t) = H_0 - A \sin(k x - \omega t) \tag{7.1}
\]

with \( H_0 = 1 \), \( k = \omega = 3 \pi \), \( A = 0.07 \) and initial condition \( h(x,0) = H_0 - x, u(x,0) = 0 \). The offshore boundary condition is implemented with the same HLLC flux scheme used at interior element faces. Essentially depending on the characteristics, information is thus (partly) flowing in or out the domain. The landward boundary is open, such that water is allowed to vanish.
Figure 11: Free surface and velocity profiles are shown as function of space and time for the Peregrine-Williams solution. $N = 200$ elements are used with the HLLC flux without additional numerical dissipation.

In the simulation in Fig. 12, we see initially sinusoidal waves steepen to bores when they approach the shore. After a bore has formed the slope between bores is nearly parallel to the beach topography, and the dynamics behind the last bore resembles the initial condition of the idealized overtopping solution of Peregrine and Williams. The evolution of each incoming bore then resembles intermittently the Peregrine-Williams solution. Due to the offshore driving of waves, multiple bores create multiple overtopping events. On the right side of the dike top, multiple patches with very shallow water rush down the slope rapidly to leave the computational domain.

8 Conclusions

This paper is a study of the free-boundary dynamics in one-dimensional shallow-water equations with a space discontinuous finite-element scheme. Our numerical
<table>
<thead>
<tr>
<th>( N )</th>
<th>( \frac{L^2(u,h)}{10^{-2}} )</th>
<th>( p )</th>
<th>( \frac{L^2(m,h)}{10^{-2}} )</th>
<th>( p )</th>
<th>( \frac{L^\infty}{10^{0.2}} )</th>
<th>( p )</th>
<th>( \Delta t_b )</th>
<th>( p )</th>
</tr>
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<td>2.03</td>
<td>0.11229</td>
<td>0.21</td>
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<tr>
<td>10</td>
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<td>0.99</td>
<td>0.01023</td>
<td>1.08</td>
<td>0.8717</td>
<td>0.52</td>
<td>0.09737</td>
<td>0.63</td>
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<tr>
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<td>0.12717</td>
<td>0.99</td>
<td>0.04828</td>
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<td>0.6081</td>
<td>0.52</td>
<td>0.06290</td>
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<tr>
<td>40</td>
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<td>0.91</td>
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<td>0.01160</td>
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<td>0.3877</td>
<td>0.36</td>
<td>0.02502</td>
<td>0.68</td>
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</tbody>
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Table 5: The Peregrine-Williams overtopping test without additional dissipation. Error in norms \( L^2(u,h), L^2(m,h), L^\infty \); and the frontal position of the first patch, \( \epsilon_{x_b} = x_b - x_b^{\text{numerical}} \), at time \( t = 3 \), and the time of break-up, \( \Delta t_b \). The exact solution breaks at \( t = 2 \). Parameters and initial conditions \( x < x_0 : u(x,0) = -1, h(x,0) = 1; x > x_0 : u(x,0) = 0, h(x,0) = 0 \) with \( L = 4, CFL = 0.01 \).

scheme is able to preserve the non-negative nature of the (mean) depth in combination with a free-boundary treatment in mixed Eulerian-Lagrangian elements. A variety of solutions has been considered in which the free-boundary movement is essential, including cases where free boundaries emerge or disappear.

We combined an explicit space discontinuous finite-element scheme (Cockburn and Shu, 1998) with the HLLC flux to ensure positivity of the mean water depth in each element under certain reasonable time-step restrictions. A spatial discretization was considered which is formally second-order accurate with constant and linear polynomial basis functions, representing the mean and slope of a variable in each element. The free-boundary movement and the appearance of dry patches with zero water depth in the middle of the fluid domain has been handled by using the slope information of the depth. Finally, the robustness of the method is exemplified by a complex simulation of multiple shoaling and steepening water waves running up the seawall slope of a dike, overtopping, and then breaking up in two or more patches at the downslope side.

Detailed and varied numerical verifications show that our method is second order in smooth cases and in the interior of the domain in the absence of physical discontinuities. It reduces to order 0.7 to 1.0 in the presence of discontinuities, and at the free boundary. It reduces to order 0.5 at the front when (multiple) dry patches emerge. Riemann problems with dry patches on horizontal topography lead to numerical solutions with divergent predictions of the free-boundary position, because the linear polynomial approximations cannot converge well to these solutions in which the free surface is parallel to the topography at the free boundary, and because the initial condition near (emerging) free boundaries is discontinuous.

Local refinement combined with an asymptotic analysis at the free boundary (such that the order of the basis and test function is raised) is anticipated to remedy the latter cases and raise the accuracy at the edge elements in general (see Woods et al., 2003). However, the introduction of a constrained second-order basis function in these
elements poses additional restrictions in order to ensure positivity of the mean depth. Patch splitting also reduces the accuracy. Using space-time discontinuous (Galerkin) finite-element methods can possibly lead to more accurate patch splitting because the time of splitting is then known directly. The implicit nature of the space time method and the stability of the required iterative solvers of the nonlinear algebraic problem can, however, pose drawbacks.

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References

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### A Summary of Exact Solutions

In the following, we provide all the known exact solutions used in the main text for reference.

Consider a dam break problem with as initial condition $h(x,-t_0) = H_0$ for $x-x_0 < 0$ and zero elsewhere, and $u(x,-t_0) = 0$. The evolving dam break solution in an unbounded domain is

$$g h(x,t) = \begin{cases} 
1 & x - x_0 < -a_0 t \\
\frac{1}{2} \left( 2a_0 - (x-x_0)/t \right) & -a_0 (t + t_0) < x - x_0 < 2a_0 t \\
0 & x - x_0 > 2a_0 t 
\end{cases}$$

(A.1)

$$u(x,t) = \begin{cases} 
0 & x - x_0 < -a_0 t \\
\frac{2}{3} \left( a_0 + (x-x_0)/t \right) & -a_0 t < x - x_0 < 2a_0 t \\
0 & x - x_0 > 2a_0 t 
\end{cases}$$

(A.2)

with $a_0 = \sqrt{gH_0}$.

The Riemann solution with $U = U_l$ for $x < x_0$ and $U = U_r$ for $x > x_0$ at $t = 0$ in which a dry patch appears can be stated explicitly. Drying occurs when
2 c_l + 2 c_r - u_r + u_l < 0, and the solution becomes:

\[
\begin{align*}
a(x, t) &= \begin{cases} 
  a_l & x - x_0 < (u_l - a_l) t \\
  \frac{1}{2} [u_l + c_l - (x - x_0)/t] & (u_l - a_l) t < x - x_0 < S_l t \\
  0 & S_l t < x - x_0 < S_t \\
  \frac{1}{2} [(x - x_0)/t - u_r + c_r] & x_0 + S_r t < x - x_0 < (u_r + a_r) t \\
  a_r & x - x_0 > (u_r + a_r) t 
\end{cases} \\
u(x, t) &= \begin{cases} 
  u_l & x - x_0 < (u_l - a_l) t \\
  \frac{1}{2} [u_l + c_l + 2 (x - x_0)/t] & (u_l - a_l) t < x - x_0 < S_l t \\
  0 & S_l t < x - x_0 < S_t \\
  \frac{1}{2} [2 (x - x_0)/t + u_r - c_r] & x_0 + S_r t < x - x_0 < (u_r + a_r) t \\
  u_r & x - x_0 > (u_r + a_r) t
\end{cases}
\end{align*}
\]  

(A.3)

(A.4)

with \( a = \sqrt{g h}, c = 2 a \) and \( S_l = u_l + c_l, S_r = u_r - c_r \).

Consider fluid motion in a parabolic bowl, which resides symmetrically around the origin. The following, scaled shallow water equations \( \partial_t u + u \partial_x u = -\partial_x (h + h_b) \) and \( \partial_t h + \partial_x (h u) = 0 \) can be simplified exactly by introducing the following Ansatz (Young, 1986): \( u = u_0(t) x, h = h_0(t) + h_1(t) x^2/2, \) and \( h_b = B x^2/2 \) with \( x \in [-x_b, x_b] \) and, hence, \( x_b(t) = \sqrt{2 h_0(t)/h_1(t)} \). Mass \( m = \int h(x, t) \, dx = (2/3) h_1 x_b^3 \) is conserved, that is \( \partial_M/\partial t = 0 \). Given this conserved mass and an initial condition defining \( m \), the above system can be reduced to:

\[
\frac{dx_b}{dt} = \frac{1}{2 x_b} \frac{\partial E}{\partial u_0} = u_0 x_b \quad \frac{du_0}{dt} = -\frac{1}{2 x_b} \frac{\partial E}{\partial x_b} = -u_0^2 - B + m/ x_b^3.
\]

(A.5)

Instead of finding a closed-from solution, we numerically integrate the two ordinary differential equations (A.5) directly for comparison with the solution of the finite-element discretization.

Carrier and Greenspan (1958) considered the dimensionless shallow-water equations on a uniform slope \( \partial_t h + \partial_x (h u) = 0 \) and \( \partial_t u + u \partial_x u + \partial_x h = -1 \). They introduced a hodograph transformation with \( u = \lambda/2 - t \) and \( c = \sqrt{\lambda} = \bar{c}/4 \), such that the shallow-water equations reduce to a linear equation in \( \phi \) with \( u(\bar{c}, \lambda) = -\partial_x \phi/\bar{c} \). Hence the shoreline lies at \( \bar{c} = 0 \). The solution is well defined provided the Jacobian between \( x, t \) and \( \bar{c}, \lambda \) exists, where

\[
t = \lambda/2 - u, \quad \text{and} \quad x = \partial_x \phi/4 - \bar{c}^2/16 - u^2/2.
\]

(A.6)

Assuming \( \phi \) to be harmonic in \( \lambda \), the solution is a sum of Fourier modes:

\[
\phi(\bar{c}, \lambda) = \sum_{k=0}^{\infty} A_k J_0(\omega_k \bar{c}) \sin(\omega_k \lambda + \varphi_k)
\]

(A.7)

with \( \omega_k \) the frequency and \( J_0 \) the Bessel function of the first kind and \( \varphi_k \) a phase. We consider the case where the Jacobian of the transformation is critical for a single mode such that \( A_k = 1, \omega_k = 1 \) and the front reaches infinite steepness at periodic instants.

Peregrine and Williams (2001) considered the following solution overtopping a dike. Consider initially quiescent flow with depth unity for \( x < x_0 \) and no fluid or zero
depth beyond. The critical point lies at \( x_0 + E \) such that \( 0 < E < 1 \). The following analytical solution emerges for \( x < x_0 + E \). For \( t \in [0, T_1] \), where \( T_1 = 2 - \sqrt{4 - 2E} \) the time when the front reaches the critical point at \( x_0 + E \), and for \( t \in [T_1, T_2] \), where \( T_2 = \sqrt{2E} \) the time when the flow becomes critical, one obtains:

\[
\begin{align*}
&h(x, t) = \\
&\begin{cases}
1 & x - x_0 < -t^2/2 - t \\
\frac{1}{36t^2} \left(4t - t^2 - 2(x - x_0)\right)^2 & -t^2/2 - t < x - x_0 < \min(-t^2/2 + 2t, E) \\
0 & \min(-t^2/2 + 2t, E) < x - x_0 < E \\
\end{cases} \\
&u(x, t) = \\
&\begin{cases}
\frac{2}{3} t \left(t - t^2 + (x - x_0)\right) & -t^2/2 - t < x - x_0 < \min(-t^2/2 + 2t, E) \\
\frac{1}{3} \left(2 - 2\sqrt{2(E - x + x_0)}\right) & \min(-t^2/2 + 2t, E) < x - x_0 < E \\
0 & \\
\end{cases}
\end{align*}
\]

(A.8)

For \( t \in [T_2, T_3] \) with \( T_3 = 2 \) the time when the flow starts to recede down the slope from the critical point, and for \( t > T_3 \), we find the exact solution:

\[
\begin{align*}
&h(x, t) = \\
&\begin{cases}
1 & x - x_0 < -t^2/2 - t \\
\frac{1}{36t^2} \left(4t - t^2 - 2(x - x_0)\right)^2 & -t^2/2 - t < x - x_0 < \sqrt{2E}t - t^2/2 \\
0 & \sqrt{2E}t - t^2/2 < x - x_0 < \\
\frac{1}{9} \left(2 - t + \sqrt{2(E - x + x_0)}\right)^2 & \frac{E - \frac{(t-2)^2}{2}}{3} \Theta(t - 2) < x - x_0 < E \\
\frac{2}{3t} \left(t - t^2 + (x - x_0)\right) & -t^2/2 - t < \min(-t^2/2 + 2t, E) \\
\frac{1}{3} \left(2 - 2\sqrt{2(E - x + x_0)}\right) & \min(-t^2/2 + 2t, E) < x - x_0 < E \\
0 & \\
\end{cases} \\
&u(x, t) = \\
&\begin{cases}
\frac{2}{3} t \left(t - t^2 + (x - x_0)\right) & -t^2/2 - t < x - x_0 < \min(-t^2/2 + 2t, E) \\
\frac{1}{3} \left(2 - 2\sqrt{2(E - x + x_0)}\right) & \min(-t^2/2 + 2t, E) < x - x_0 < E \\
\end{cases}
\end{align*}
\]

(A.9)

with Heaviside function \( \Theta(x) \).
Figure 12: Two free surface profiles (thick solid lines) are shown from the front and back side of a dike, as well as velocity profiles for multiple waves steepening, running-up a dike and overtopping a dike. 400 elements are used with a localized slope limiter (combining Schwanenberg, 2003, and Krivodonova et al., 2003). Note how periodic the space-time profiles become. The beach topography is indicated by the dashed lines. We observe that the patches of water are rushing down the back slope very fast and are very thin.