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THE NORM OF AN AVERAGING OPERATOR

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ABSTRACT. We consider the operator \( A : \ell^2 \to \ell^2 \) defined by \( A(a) = b \) for \( a = (a_n) \) and \( b = (b_n) \) with \( b_n = \frac{1}{n}(a_1 + a_2 + \cdots + a_n) \). We prove that \( A \) has norm 2.

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1. INTRODUCTION

We consider the Hilbert space \( \ell^2 \) of square-summable sequences, i.e., sequences \( a = (a_1, a_2, a_3, \ldots) \), such that

\[
\|a\| = \left\{ \sum_{i=1}^{\infty} |a_i|^2 \right\}^{\frac{1}{2}} < \infty,
\]

and corresponding inner product

\[
(a, b) = \sum_{i=1}^{\infty} a_i \overline{b_i},
\]

for \( a = (a_1, a_2, a_3, \ldots) \) and \( b = (b_1, b_2, b_3, \ldots) \) both in \( \ell^2 \).

For a sequence \( a = (a_1, a_2, a_3, \ldots) \), we can define a new sequence \( A(a) \) of averages

\[
A(a) = (a_1, \frac{1}{2}(a_1 + a_2), \frac{1}{3}(a_1 + a_2 + a_3), \ldots, \frac{1}{n}(a_1 + a_2 + \cdots + a_n), \ldots).
\]

Clearly, \( A \) is a linear operator. We will prove that \( A(a) \) is again square-summable, and in fact that for \( A : \ell^2 \to \ell^2 \), we have

\[
\|A\| = \sup_{\|a\| \neq 0} \frac{\|A(a)\|}{\|a\|} = 2.
\]

2. A IS BOUNDED

We start by proving that \( A \) is bounded. Hence we take any \( a \in \ell^2 \), and consider \( \|A(a)\|^2 \).

\[
\|A(a)\|^2 = \sum_{i=1}^{\infty} \frac{1}{i} \left( \sum_{j=1}^{i} |a_j|^2 \right)^2 = \sum_{i=1}^{\infty} \frac{1}{i^2} \left( \sum_{j=1}^{i} |a_j|^2 \right)^2 = \sum_{i=1}^{\infty} \frac{1}{i^2} \left( \sum_{j=1}^{i} \frac{j^2}{i^2} |a_j|^2 \right) \leq \sum_{i=1}^{\infty} \frac{1}{i^2} \left( \sum_{j=1}^{i} \frac{1}{j} \cdot \frac{i^2}{j^2} |a_j|^2 \right).
\]

\[
(*) \quad \sum_{i=1}^{\infty} \frac{1}{i^2} \left( \sum_{j=1}^{i} \frac{1}{j} \cdot \frac{i^2}{j^2} |a_j|^2 \right) \leq \sum_{i=1}^{\infty} \frac{1}{i^2} \left( \sum_{j=1}^{i} \frac{1}{j} \cdot \sum_{j=1}^{i} \frac{j^2}{j^2} |a_j|^2 \right).
\]

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In the last step we used the Cauchy-Schwarz inequality. Now we estimate the middle term.

\[
\sum_{j=1}^{i} \frac{1}{j^{2}} \leq 1 + \int_{1}^{i} \frac{dx}{x^{2}} = 2i^{\frac{1}{2}} - 1.
\]

This result we substitute in (*) and we interchange the order of summation:

\[
\|A(a)\|^2 \leq \sum_{i=1}^{\infty} \frac{1}{i^{2}} (2i^{\frac{1}{2}} - 1) \sum_{j=1}^{i} j^{\frac{1}{2}} |a_j|^2
\]

\[
(*) = \sum_{j=1}^{\infty} \left( \sum_{i=j}^{\infty} \left( \frac{2}{i^{2}} - \frac{1}{i^{2}} \right) \right) j^{\frac{1}{2}} |a_j|^2
\]

We recall the Riemann zeta-function \(\zeta(s)\) for \(s > 1\) defined by

\[
\zeta(s) = \sum_{i=1}^{\infty} \frac{1}{i^s}.
\]

For \(j = 1\) we have that

\[
\sum_{i=1}^{\infty} \frac{2}{i^{2}} - \frac{1}{i^{2}} = 2\zeta\left(\frac{3}{2}\right) - \zeta(2) \approx 3.58 < \frac{4}{\sqrt{1}}.
\]

for \(j = 2\):

\[
\sum_{i=2}^{\infty} \frac{2}{i^{2}} - \frac{1}{i^{2}} = 2\zeta\left(\frac{3}{2}\right) - \zeta(2) - 1 \approx 2.58 < \frac{4}{\sqrt{2}}.
\]

while for \(j = 3\),

\[
\sum_{i=3}^{\infty} \frac{2}{i^{2}} - \frac{1}{i^{2}} = 2\zeta\left(\frac{3}{2}\right) - \zeta(2) - \frac{3}{4} - \frac{1}{\sqrt{2}} \approx 2.12 < \frac{4}{\sqrt{3}}.
\]

Moreover for \(j \geq 4\):

\[
\sum_{i=j}^{\infty} \frac{2}{i^{2}} - \frac{1}{i^{2}} \leq \int_{j}^{\infty} \frac{2dx}{x^{2}} - \int_{j}^{\infty} \frac{dx}{x^{2}} = \int_{j}^{\infty} \frac{2dx}{j^{2}} - \frac{1}{j} \leq \frac{4}{j^{2}},
\]

where we used in the last step that \(\frac{2}{j^{\frac{1}{2}}} - \frac{1}{j} \leq 0\), due to \(j \geq 4\). Substituting these results in (**), we obtain

\[
\|A(a)\|^2 \leq \sum_{j=1}^{\infty} \frac{4}{j^{2}} j^{\frac{1}{2}} |a_j|^2 = 4\|a\|^2.
\]

It follows that \(\|A\| \leq 2\).
3. A HAS NORM 2

It remains to show that \( \|A\| \geq 2 \). For this we consider the sequence 
\( a = (a_j) \) with \( a_j = \frac{1}{j^\alpha} \), where \( \alpha > \frac{1}{2} \). Then

\[
\|A(a)\|^2 = \sum_{i=1}^{\infty} \frac{1}{i^2} \left( \sum_{j=1}^{i} \frac{1}{j^\alpha} \right)^2.
\]

(* ***)

Now

\[
\sum_{j=1}^{i} \frac{1}{j^\alpha} > \int_{1}^{i} \frac{dx}{x^\alpha} = \frac{1}{1-\alpha} \left( i^{1-\alpha} - 1 \right)
\]

If we substitute this estimate in (* **), we obtain

\[
\|A(a)\|^2 > \sum_{i=1}^{\infty} \frac{1}{i^2} \left( \frac{1}{1-\alpha} \left( i^{1-\alpha} - 1 \right) \right)^2 = \frac{1}{(1-\alpha)^2} \sum_{i=1}^{\infty} \frac{i^{2-2\alpha} - 2i^{1-\alpha} + 1}{i^2}
\]

\[
> \frac{1}{(1-\alpha)^2} \sum_{i=1}^{\infty} \left( \frac{1}{i^{2\alpha}} - \frac{2}{i^{1+\alpha}} \right) > \frac{1}{(1-\alpha)^2} \sum_{i=1}^{\infty} (|a_i|^2 - \frac{2}{i^2})
\]

\[
= \frac{1}{(1-\alpha)^2}(\|a\|^2 - 2\zeta(\frac{3}{2})).
\]

Consequently

\[
\frac{\|A(a)\|^2}{\|a\|^2} > \frac{1}{(1-\alpha)^2} - \frac{2\zeta(\frac{3}{2})}{\|a\|^2 (1-\alpha^2)}.
\]

Now let \( \alpha > \frac{1}{2} \), so that \( \|a\| \to \infty \). Therefore

\[
\frac{\|A(a)\|^2}{\|a\|^2} \to \frac{1}{(1-\frac{1}{2})^2} = 4.
\]

Hence we obtain that \( \|A\|^2 \geq 4 \).

Remembering that we already had that \( \|A\| \leq 2 \), we obtain that \( \|A\| = 2 \), as desired.

4. REMARK

We can consider the same problem for \( A : \ell^p \to \ell^p \), so now with norm

\[
\|a\| = \left( \sum_{i=1}^{\infty} |a_i|^p \right)^{\frac{1}{p}},
\]

for \( 1 \leq p < \infty \) and

\[
\|a\| = \sup_{i} |a_i|
\]

for \( p = \infty \). Some preliminary calculations suggest that \( \|A\| = q \) in this case, where \( q \) is given by \( \frac{1}{p} + \frac{1}{q} = 1 \). Only for the cases \( p = 1 \) and \( p = \infty \) (and \( p = 2 \), the situation above), we are able to give a full proof.