On Skew-symmetric Preconditioning for Strongly Non-symmetric Linear Systems

To solve iteratively linear system $Au = b$ with large sparse strongly non-symmetric matrix $A$ we propose preconditioning $\hat{A}u = \hat{b}$, $\hat{A} = (I + \tau L_1)^{-1}A(I + \tau U_1)^{-1}$, $\tau > 0$ where respectively lower and upper triangular matrices $L_1$ and $U_1$ are so that $L_1 + U_1 = 1/2(A - A^*)$. Such preconditioning technique may be treated as a variant of ILU-factorization, and we call it MSSILU — Modified Skew-Symmetric ILU.

We investigate and optimize (with respect to $\tau$) convergence of preconditioned Richardson method (RM) of the following special form: $\hat{x}^{m+1} = (I - \tau \hat{A})\hat{x}^m + \tau \hat{b}$, $m \geq 0$, where $\tau$ is the same as in $\hat{A}$. For this method we give an estimate for rate of convergence in relevant Euclidean norm for the case of positive real matrix $A$.

Numerical experiments have included solving linear systems arising from 5-point FD approximation of convection–diffusion equation with dominated convection by MSSILU+RM, MSSILU+GMRES(2) and MSSILU+GMRES(10).

1. MSSILU — Modified Skew-Symmetric ILU factorization

Solving the system of the linear algebraic equations

$$Au = b$$

with large sparse strongly non-symmetric matrix $A$ we propose to apply an iterative process to the following preconditioned system

$$\hat{A}u = \hat{b}, \quad \hat{A} = (I + \tau L_1)^{-1}A(I + \tau U_1)^{-1}, \quad \hat{u} = (I + \tau U_1)u, \quad \hat{b} = (I + \tau L_1)^{-1}b,$$

where $I$ is the identity matrix, $L_1$ and $U_1$ are respectively lower and upper triangular parts of matrix $A_1 = 1/2(A - A^*)$ which is the skew-symmetric component of $A$, so that $L_1 + U_1 = A_1$, $\tau > 0$ is a scalar parameter. Such preconditioning technique may be treated as a simplified variant of ILU-factorization, and we call it MSSILU — Modified Skew-Symmetric ILU. Consider MSSILU+RM (that is the preconditioned Richardson Method) of the form

$$\hat{u}^{m+1} = G\hat{u}^m + \tau \hat{b}, \quad m \geq 0, \quad G = I - \tau \hat{A},$$

or

$$B \frac{\hat{u}^{m+1} - \hat{u}^m}{\tau} + Au^m = b, \quad B = (I + \tau L_1)(I + \tau U_1),$$

where $B \neq B^*$ is preconditioning matrix and $\tau$ is the same as in (2). For (3),(4) we provide with convergence analysis [1] and way to choose the parameter $\tau$ optimally in the case when $A$ is positive real matrix, i.e. when $A_0 = 1/2(A + A^*)$ is positive definite matrix. We establish and optimize convergence in the Euclidean norm induced by $B_0 = 1/2(B + B^*)$ that is symmetric part of the preconditioning matrix. It is very important to note that in (3),(4)

$$B_1 = \tau A_1, \quad B_1 = \frac{1}{2}(B - B^*), \quad A_1 = \frac{1}{2}(A - A^*).$$

Let us agree that vector (operator) norm without lower index is everywhere Euclidean norm. Using the approach developed in [2,1] we prove the following result:

**Theorem 1.** Let $A$ be positive real. Then for arbitrary positive real matrix $B$ satisfying (5) iterative method of the form (4) converges, so that

$$\|G\|_{B_0} = \left\| B_0^{1/2} GB_0^{-1/2} \right\| < 1,$$

as soon as

$$\left( B_0 u, u \right) > \frac{\tau}{2} (A_0 u, u) \quad (\tau > 0), \quad \forall u \in \mathbb{C}^n.$$
In assumption that spectrum of $A$ is such that $\text{sp}A_0 \subseteq [\gamma_1; \gamma_2]$, $\gamma_1 > 0$ and $\rho(A_1) = 2\gamma_3$ we may evaluate spectrum of $B$ as

$$\text{sp}B_0 \subseteq [1 - \gamma_3^2 \tau^2; 1], \quad \rho(B_1) = 2\tau\gamma_3$$

(8)

(where $B_0$ is positive definite for $\tau < \gamma_3^{-1}$) and then reformulate Theorem 1 in more constructive manner:

**Theorem 2.** Let $A$ and $B$ be positive real. Then MSSILU+RM (3),(4) converges (that means (6) holds) as soon as $\tau$ satisfy the constraint

$$0 < \tau < \hat{\tau} = \left(\sqrt{\frac{\gamma_2^2 + 16\gamma_3^2 - \gamma_2}{4\gamma_3^2}}\right).$$

Minimization of $\|G\|_{B_0}$ reveals optimal value $\hat{\tau}$ to be very close to $\hat{\tau}$, and though $\hat{\tau}$ is not available in explicit analytical form we estimate the convergence rate of MSSILU+RM (3),(4):

**Theorem 3.** Let $A$ be positive real. Then MSSILU+RM (3),(4) converges for the optimal value $\hat{\tau} \in (0; \hat{\tau})$,

$$\|e^m\|_{B_0} < \rho_0^m \|e^0\|_{B_0}, \quad \rho_0 \leq 1 - \gamma_3 \left(\sqrt{\frac{\gamma_2^2 + 16\gamma_3^2 - \gamma_2}{4\gamma_3^2}}\right).$$

(10)

where $e^m$ is the error vector on iteration $m$. The iteration number $\hat{m}$ needed to achieve the prescribed accuracy $\epsilon$ is of the form $\hat{m} < m_0(\epsilon)$, $m_0(\epsilon) = \ln\epsilon/\ln\rho_0$ (here $\ln x = \log_x x$).

2. **Practical choice of the parameter. Numerical experiments**

We show how to avoid in practice the necessity to know spectral bounds for $A$. For strongly non-symmetric linear systems (where $\gamma_2 \ll \gamma_3$) one may settle for a knowing $\gamma_3$ only (as it occurs, $\hat{\tau} \approx \gamma_3^{-1}$). Moreover one may try to substitute $\gamma_3$ by $\|L_1\|_\infty = \max_i \sum_j |l_{i,j}|$ since $\gamma_3 \leq \|L_1\|_\infty \approx \|L_1\|_\infty$. One may verify that in fact $\|L_1\|_\infty \geq \gamma_3$, so that $\|L_1\|_\infty^{-1} \leq \hat{\tau}$ and practical choice $\tau = \|L_1\|_\infty^{-1}$ may be too underestimated. Various numerical tests shows that in matrix $I + \tau L_1$ the part of the “diagonal dominant” rows (i.e. rows $i$ for which $\tau \sum_j |l_{i,j}| < 1$) observed for the fastest convergence is 0.6–0.8. This heuristics gives convenient way for defining optimal value $\hat{\tau}$ in practice.

Numerical experiments included solving linear systems derived from the 5-point FD discretization of the steady convection–diffusion equation

$$Pe^{-1}\Delta u + 0.5 \left[(v_1 u)_x + v_1 u_x + (v_2 u)_y + v_2 u_y\right] = 0$$

(11)

on the unit square with homogeneous Dirichlet boundary conditions. The Peclet number $Pe$ was taken $10^3$, $10^4$ and $10^5$. We compared performance of MSSILU+RM, MSSILU+GMRES(2) and MSSILU+GMRES(10) for the model problems. For the most “recalcitrant” problem (11) with $Pe = 10^5$ and $v_1 = \sin 2\pi x$, $v_2 = -2\pi y \cos 2\pi x$ MSSILU+GMRES(10) requires 275 iterations (restarts). MSSILU+RM requires 2389 iterations (respectively 727 s and 273 s of IBM PC 486/DX2-66 CPU time) on grid $63 \times 63$ (iterations were performed until $\|r^m\|/\|r_0\| \leq 10^{-6}$). For the same problem on the coarser grid $31 \times 31$ we observed 767 iterations (restarts) (379 s) for MSSILU+GMRES(10) and 7098 iterations (152 s) for MSSILU+RM.

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3. **References**


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