Eulerian variational principles for stratified hydrostatic equations

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Abstract

The aim of this paper is to advocate the use of variational and Hamiltonian formulations of the hydrostatic equations of motion in finding new conservative numerical techniques for forecast models. For that reason, the fundamental conservative structure of the hydrostatic equations of motion is presented. Variational principles and Hamiltonian formulations of various hydrostatic equations, stratified continuously or in layers, are derived systematically from an Eulerian perspective in the horizontal and a Lagrangian or material perspective in the vertical direction. Variational formulations are derived for the hydrostatic incompressible, Boussinesq system and the hydrostatic equations in multiple isentropic and isopycnal layers. The various Hamiltonian formulations presented share similar Poisson brackets, i.e. the (contribution to the) Poisson bracket in each layer or in the continuously stratified case is the one for the shallow water equations with the depth replaced by the appropriate pseudo density, while the potential (and internal) energy in the Hamiltonian differs in each case. For hydrostatic equations in material coordinates, either stratified continuously or in layers, the coupling between layers happens solely through the Hamiltonian, an observation which may aid in searching for conservative numerical discretizations.

1 Introduction

Hydrostatic “primitive” equations of motion are widely used in atmospheric and oceanic fluid dynamics, in particular in numerical weather prediction and atmosphere-ocean climate modeling. In the atmosphere these equations are derived from the compressible Euler equations after using the hydrostatic approximation, while in the ocean the incompressibility condition and the Boussinesq approximation are often invoked in addition.

The aim of the present paper is to enable the use of variational and Hamiltonian (i.e. geometric) formulations in finding new conservative numerical techniques. For that reason, I formulate previously unknown geometric formulations of several stratified hydrostatic models used in geophysical fluid dynamics.

Variational principles and Hamiltonian formulations of various hydrostatic equations, stratified continuously or in layers, are derived from an Eulerian perspective in the horizontal and a Lagrangian or material perspective in the vertical direction. Material coordinates in the vertical are at least conceptually advantageous in that they eliminate the reference to the vertical velocity in hydrostatic systems. As a starting point, Eulerian variational formulations of the atmospheric non-hydrostatic Euler equations and the hydrostatic Euler equations in isentropic coordinates are outlined, while novel formulations are derived for the hydrostatic incompressible, Boussinesq system and the hydrostatic equations in multiple isentropic and isopycnal layers. These Eulerian variational formulations consider the equations of motion from an Eulerian viewpoint for the horizontal coordinates while a Lagrangian viewpoint is adopted in the vertical by using material coordinates. Such hydrostatic Eulerian variational formulations in an Eulerian framework (in the horizontal) can be contrasted with the more well-known Lagrangian variational formulations in a Lagrangian framework. In the former the horizontal fluid parcels or labels are functions of the fixed horizontal coordinates and time, while in the latter the positions of the fluid parcels are functions of the fluid labels and time. Boussinesq models in isopycnal coordinates are often used in ocean circulation models, such as the MICOM model (e.g. Bleck and Smith 1990). In contrast, material coordinates have been less popular in atmospheric numerical models, since the use of terrain-following or sigma-coordinates in numerical models appears to
be more efficient near the Earth’s boundary. (Boundary conditions are implemented more easily in sigma-coordinates than in material coordinates.) Nevertheless, the conceptual advantages of isentropic coordinates have been recognized in the atmospheric community (e.g. Hoskins, McIntyre and Robertson 1985).

The structure of the paper is as follows. Eulerian variational principles and Hamiltonian formulations of the continuously stratified hydrostatic systems are outlined and derived in section 2 and for the layer models in section 3. Finally, a discussion of the applicability of the Eulerian variational principles is found in section 4.

The review papers by Shepherd (1990) and Morrison (1998) provide good introductions to the realm of variational and Hamiltonian fluid dynamics. Conservation laws are related to variational and Hamiltonian formulations via Noether’s theorem, but I won’t explicitly explore their properties presently and refer to the before-mentioned review papers.

2 Continuous stratification

a. Stratified equations

The Eulerian variational principle for the Euler equations follows directly from a transformation of the Lagrangian variational principle. The variables in the Lagrangian framework are the positions $\mathbf{x} = (x, y, z) = \mathbf{x}(\mathbf{a}_3, t)$ as function of the fluid parcel labels $\mathbf{a}_3 = (a, b, c)$ and time $t$, while in the Eulerian framework their role is reversed: the fluid labels are variables $\mathbf{a}_3 = \mathbf{a}_3(x, z, t)$ as function of space and time. The following Eulerian principle for several stratified equations of motion is the difference between the kinetic energy and the potential and internal energy:

$$0 = \delta \int_{t_0}^{t_1} dt \mathcal{L}_{EHB}[\mathbf{a}_3, d\mathbf{a}_3/dt] = \delta \int_{t_0}^{t_1} dt \int_{D_H} \int_{h_b}^{h_T} dz \left\{ \rho_q \left( \frac{1}{2} u_i + R_i \right) u^i + \delta_H \rho_q \frac{1}{2} w^2 - \rho g z - \epsilon_B \rho U(s, \rho) \right\},$$

$$\rho = \rho_R(\mathbf{a}) \det \left| \frac{\partial \mathbf{a}_3}{\partial (\mathbf{x}, z)} \right|,$$

$$\frac{\partial \mathbf{a}_3}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{a}_3 + w \frac{\partial \mathbf{a}_3}{\partial z} = 0,$$

where the Coriolis parameter $f = f(x, y) = \hat{z} \cdot \nabla \times \mathbf{R}$ with $R_3 = 0$, $u_i = \delta_{ij} w^j$ is the horizontal velocity (the rule of summation over repeated indices is used and lowercase indices run from 1 to 2), $\nabla = (\partial/\partial x, \partial/\partial y)^T$, $\mathbf{v}$ and $w$ are the horizontal and vertical velocity, respectively, $\rho_R$ is a reference Lagrangian density, $g$ is the constant of gravity, $D_H$ is the horizontal extent of the domain, $U(s, \rho)$ the internal energy as function of entropy $s$ and density $\rho$, the fluid lies between the topography or Earth’s surface at $z = h_b$ and the free surface or outer atmosphere at $z = h_T$.

The following equations of motion emerge from the variational principle (2.1):

- The Euler equations emerge when we take $\delta_H = \epsilon_B = 1$ and $\rho_q = \rho$ and specify an equation of state $p = p(\rho, s)$ with pressure $p$ (e.g. Holm (1996)). To illustrate these and upcoming variational calculations, I have included the straightforward derivation of the Euler equations from a one-dimensional vertical version of principle (2.1) in the Appendix.
\[ \frac{\partial}{\partial \rho} \delta \rho = 0 \text{ and } \rho \equiv \rho_0, \text{ and specify } p = p(\rho, s), \text{ the hydrostatic primitive equations arise which are (in a modified form) still used in numerical weather and climate prediction. When we transform (2.1) from Cartesian to isentropic coordinates in the vertical, we arrive at the variational principle investigated by Bokhove (2000).} \]

\[ \text{Finally, when we take } \delta H = \epsilon_B = 0 \text{ and } \rho = \rho_0 \text{ with } \rho_0 \text{ as constant reference density, the hydrostatic incompressible Boussinesq equations appear which are often used in large-scale oceanography. In the next section, the variational principle (2.1) will be transformed from Cartesian to isopycnal coordinates in the vertical and it will be shown that the hydrostatic Boussinesq equations in isopycnal coordinates naturally follow from such a transformed variation principle.} \]

\[ \text{b. Hydrostatic incompressible Boussinesq equations} \]

To derive the variational principle for hydrostatic incompressible Boussinesq flows we take \( \delta H = \epsilon_B = 0 \) and \( \rho = \rho_0 \). Moreover, since the fluid is incompressible and Boussinesq, we have a pseudo density \( \sigma = \rho_0 \frac{\partial z}{\partial \rho} \)

\[ \text{as the Jacobian between Cartesian and isopycnal coordinates, i.e. } \rho_0 \, dx \, dy \, dz = \sigma \, dx \, dy \, dp. \]

With this definition of pseudo density, a direct transformation of principle (2.1) to isopycnal (or density) coordinates gives

\[ 0 = \delta \int_{t_0}^{t_1} dt \mathcal{L}_B[a, da/dt] \equiv \delta \int_{t_0}^{t_1} dt \int_{D_H} \int_{\rho_B(x,t)} dx \, \mathcal{L}_{\rho_T}(x,t) \, d\rho \left\{ \left( \frac{1}{2} \sigma(x,\rho, t) u_i(x,\rho, t) + R_i(x) \right) u^i(x,\rho, t) - \frac{1}{2} \rho g \frac{\partial[z(x,\rho, t)^2]}{\partial \rho} \right\}, \]

(2.5)

in which the kinetic minus the potential energy appears, and in which variations are taken with respect to fluid labels \( a = (a, b)^T(x, \rho, t) \). Fluid parcels \( a(x, \rho, t) \) are thus advected by the horizontal fluid velocity \( v \equiv (u, v)^T \) on isopycnal surfaces labeled by coordinate \( \rho \)

\[ v = -\Gamma^{-1} \frac{\partial a}{\partial t} \iff \frac{\partial a^k}{\partial t} + u^i \frac{\partial a^k}{\partial x^i} = 0 \]

(2.6)

with the tensor \( \Gamma_k^i = \frac{\partial a^k}{\partial x^i} \), and the pseudo density \( \sigma \) is defined by \( \sigma = \sigma_0(a, \rho) J(a, b) \) with the horizontal Jacobian \( J(a, b) \equiv \partial_x a \partial_y b - \partial_x b \partial_y a \). We refer to Holm (1996) and Bokhove (2000) for relations useful in the evaluation of Hamilton’s principle (2.5). In extension to the atmospheric hydrostatic equations of motion in isentropic coordinates a free surface has been included to accommodate oceanic applications. Furthermore, the system is constrained to be incompressible and the internal energy term is therefore absent (\( \epsilon_B = 0 \)), while the potential energy appears in a different form as is usual. To evaluate the variation of the potential energy, the Montgomery potential \( M = (p + pgz)/\rho_0 \) is defined. Hence with hydrostasy \( p_z = -\rho g \), we find \( z = (\rho_0/g) \partial M/\partial \rho \). Several boundary terms appear after varying (2.5), and cancel one another or are zero by using suitable boundary conditions, such as taking \( p_a = 0 \) at the free surface. Finally, variation of (2.5) with respect to \( \delta a^k \) and \( (\delta a^k)_{B,T} \) yields

\[ 0 = \int_{t_0}^{t_1} dt \int_{D_H} dx \, \int_{\rho_B} dx \, \sigma(\Gamma^{-1})^n_k \left\{ \frac{\partial u^m}{\partial t} + u^i \frac{\partial u^m}{\partial x^i} + \right\} \]

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the topography and the free surface, i.e. at motion arising from these variations are therefore horizontal advection of the isopycnals at

\[ \sigma = \frac{\partial R_n}{\partial x^n} + \frac{\partial M}{\partial x^n} \delta a^k + \]

\[
\int_{t_0}^{t_1} dt \int_{D_H} \delta \left\{ \left[ \sigma (u_n + R_n) (\Gamma^{-1})^m_k \left( \frac{\partial R_j}{\partial x^j} + u^j \frac{\partial R_j}{\partial x^j} \right) \right] \right\}_{\rho_B, \rho} \]

\[
\left\{ \left[ \sigma (u_n + R_n) (\Gamma^{-1})^m_k \left( \frac{\partial R_j}{\partial x^j} + u^j \frac{\partial R_j}{\partial x^j} \right) \right] \right\}_{\rho_B, \rho} \]

\[ \delta a^k \]

\[ \frac{\partial M}{\partial x^n} = 0, \]

where subscripts \((\cdot)_{B,T}\) denote evaluation at \(\rho = \rho_{B,T}\), respectively. The equations of motion arising from these variations are therefore horizontal advection of the isopycnals at the topography and the free surface, i.e. at \(z = h_B(x)\) and \(z = h_T(x, t)\), respectively:

\[ (\delta a^k)_{B,T} : \frac{\partial \rho_{B,T}}{\partial t} + u^k \frac{\partial \rho_{B,T}}{\partial x^k} = 0 \]  \hspace{1cm} (2.8)

and the horizontal momentum equations

\[ \delta a^k : \frac{\partial u_m}{\partial t} + u^k \frac{\partial u_m}{\partial x^k} + u^k \left( \frac{\partial R_m}{\partial x^k} - \frac{\partial R_m}{\partial x^m} + \frac{\partial M}{\partial x^m} \right) = 0, \]

where the relation between \(\sigma\) and \(M\) is \(\sigma = (\rho_B^2 / g) \partial^2 M / \partial \rho^2\). As in the isentropic case (see Bokhove 2000), the momentum equations are second-order partial differential equations in time for the fluid labels, and are Euler-Lagrange equations. The continuity equation expressed in terms of the pseudo density follows directly from the definitions (2.6) and \(\sigma = \sigma_0(a, \rho) J(a, b)\) as follows

\[ \frac{\partial \sigma}{\partial t} = \frac{\partial \sigma_0}{\partial a^k} \frac{\partial a^k}{\partial t} J(a, b) + \sigma_0 \epsilon^{ij} \epsilon_{mn} \left( \frac{\partial^2 a^m}{\partial x^i \partial t} \right) \frac{\partial a^n}{\partial x^j} = -w^j \frac{\partial \sigma}{\partial x^j} - \sigma \frac{\partial u^j}{\partial x^j}, \]

where \(\epsilon_{ij}\) and \(\epsilon^{mn}\) are permutation symbols.

The generalized interior momentum corresponding to (2.5) is (e.g. Morrison (1998))

\[ \pi_k(x, t) = \delta L_B[a, \delta a/dt] / \delta \left( \frac{\partial a^k}{\partial t} \right) = \sigma (\Gamma^{-1})^m_k \left( \delta_{mn} (\Gamma^{-1})^j_k \frac{\partial a^j}{\partial t} - R_m \right) = -\sigma (\Gamma^{-1})^m_k \left( u_m + R_m \right). \]

An Eulerian action principle follows after a Legendre transform and may be rewritten in terms of \(\pi\) and \(a\) or in terms of \(v\) and \(a\). In terms of the latter, one finds

\[ 0 = \delta \int_{t_0}^{t_1} dt \int_{D_H} \delta \left\{ \int_{\rho_B}^{\rho_T} dp \left[ -\sigma (u_m + R_m) (\Gamma^{-1})^m_k \frac{\partial a^k}{\partial t} \right] - H[u, a^k] \right\} \]

with the Hamiltonian as Legendre transform

\[ H[a, v] = H[a, v] = \int_{D_H} dx \int_{\rho_B}^{\rho_T} dp \left\{ \frac{1}{2} \sigma |v|^2 + \frac{1}{2} \rho g \frac{\partial (z^2)}{\partial \rho} \right\}. \](2.13)

Variations in (2.12) are taken with respect to \(v\) and \(a\). Similar variations as in Hamilton’s principle (2.5) yield the horizontal momentum equations (2.9) on isopycnal surfaces for variations \(\delta a\), the label advection equations (2.6) for variations \(\delta v\), and (2.8) for \((\delta a^k)_{B,T}\). The
generalized momentum corresponding to $\partial a_{B,T}^k / \partial t$ is zero which signals that the Lagrangian (2.5) is singular (see e.g. Sudarshan and Mukunda 1974) but only at the boundary.

A Hamiltonian formulation of the hydrostatic incompressible, Boussinesq equations in isopycnal coordinates, in terms of label variables $a$ and velocity $v$, may be derived directly from action principle (2.12) (Sudarshan and Mukunda (1974)) — for simplicity taking $\rho_{B,T}$ constant at the boundaries. The Hamiltonian formulation described in Holm and Long (1989), in which the advection of boundary isopycnals is not included, will subsequently appear when we apply the reduction theory developed by Marsden and Weinstein (1983) to this Eulerian Hamiltonian fluid-parcel formulation.

3 Layer models

a. Constant entropy

When we apply the hydrostatic approximation to the variational principle (2.1) for non-hydrostatic flows, by dropping the vertical part in the kinetic energy, we find

$$0 = \delta \int_{t_0}^{t_1} dt \int_{D_H} dx \int_{h_b}^{h_T} dz \rho \left\{ \left( \frac{1}{2} u_i + R_i \right) u^i - g z - U(s, \rho) \right\}. \quad (3.1)$$

We assume the atmosphere to consist of an ideal gas with $p = \rho RT$ and $R$ the gas constant, relating pressure to density and temperature. In that case $U(s, \rho) = c_v T$ with $c_v$ the specific heat at constant volume. From the first law of thermodynamics $T \, ds = c_p \, dT - \frac{1}{\rho} \, dp \equiv \theta \left( \frac{p}{p_{00}} \right)^{\kappa}$ (3.2) with $\kappa = R/c_p$, specific heat at constant pressure $c_p$, potential temperature $\theta$, and reference pressure, entropy, and temperature $p_{00}, s_{00}, T_{00}$, respectively. For an ideal gas and with hydrostasy $\partial p/\partial z = -\rho g$, variational principle (3.1) can be rewritten as

$$0 = \delta \int_{t_0}^{t_1} dt \int_{D_H} dx \int_{h_b}^{h_T} dz \left\{ \rho \left( \frac{1}{2} u_i + R_i \right) u^i - c_p \rho T + \frac{\partial(p z)}{\partial z} \right\}. \quad (3.3)$$

We will next consider an atmosphere that consists of several constant entropy layers in each of which the fluid velocity is assumed to be independent of depth initially. Hence it will remain so at subsequent times. With hydrostasy, (3.3) transforms to

$$0 = \delta \int_{t_0}^{t_1} dt \int_{D_H} dx \int_{h_b}^{h_T} dz \left\{ \frac{\partial p}{\partial z} \left( \frac{1}{2} u_i + R_i \right) u^i - g \frac{\partial(p z)}{\partial z} - c_p \theta \frac{\partial p}{\partial z} \frac{p^{\kappa}}{p_{00}^{\kappa+1}} \right\}. \quad (3.4)$$

In an $N$-layer atmosphere (see Figure 1) above topography $z = h_b$ and with pressure $p_a$ in the passive layer above the active atmospheric layers, integration of (3.4) in $z$ gives

$$0 = \delta \int_{t_0}^{t_1} dt \int_{D_H} dx \left\{ (p_N - p_b) \left( \frac{1}{2} u_i + R_i \right) u^i_N + g p_b h_b - c_p \frac{\theta_N}{p_{00}^{\kappa+1}} \left( p_N^{\kappa+1} - p_b^{\kappa+1} \right) \right\} + \cdots$$
\[
\sum_{\alpha=2}^{N-1} \left\{ \left( p_\alpha - p_{\alpha+1} \right) \left( \frac{1}{2} u_{\alpha i} + R_i \right) u_\alpha^i - c_p \frac{\theta_\alpha}{p_{00}^\alpha (\kappa + 1)} \left( p_{\alpha+1}^\kappa - p_{\alpha+1}^{\kappa+1} \right) \right\} + \\
\left( p_a - p_2 \right) \left( \frac{1}{2} u_{1i} + R_i \right) u_1^i - g p_a \left( h_1 + h_2 + \ldots + h_N + h_b \right) - \\
c_p \frac{\theta_1}{p_{00}^2 (\kappa + 1)} \left( p_{2}^{\kappa+1} - p_{2}^{\kappa+1} \right) \right\}.
\]  

(3.5)

Let us also define pseudo densities \( \sigma_N = (p_b - p_N)/g \), \( \sigma_\alpha = (p_{\alpha+1} - p_\alpha)/g \) and \( \sigma_1 = (p_2 - p_a)/g \) in layers \( \alpha = 1, 2, \ldots, N \), counting from the first active layer in the outer atmosphere down to the \( N \)-th layer at the Earth’s surface or sea bottom at \( z = h_b(x) \) and with corresponding velocities \( v_\alpha \). Each layer \( \alpha \) has thickness \( h_\alpha \). The first layer thus reaches from \( z = h_b(x) \) to \( h_b(x) + h_N(x, t) \) with corresponding bottom and interface pressure \( p_b \) and \( p_N \), respectively. The second layer starts at \( h_b + h_N \) with \( p = p_N \) and finishes at \( h_b + h_{N-1} + h_N \) with \( p = p_{N-1} \), and so forth and the last layer begins at \( h_b + h_2 + \ldots + h_N \) with \( p = p_2 \) and ends at \( h_b + h_1 + \ldots + h_N \) with fixed pressure \( p = p_a \). An element of mass \( \rho \, dx \, dz \) can be integrated along columns in the vertical across each layer to find a pseudo density for each layer. For layer \( \alpha \) we thus obtain \( [(p_{\alpha+1} - p_\alpha)/g] \, dx \).

With these pseudo densities and their interrelations the variational principle (3.5) for \( N \) layers of constant entropy or constant potential temperature can be rewritten as

\[
0 = \delta \int_{t_0}^{t_1} dt \int_{D_h} \left\{ \sigma_N \left( \frac{1}{2} u_{Ni} + R_i \right) u_N^i - g \sigma_N h_b - \\
c_p \frac{\theta_N}{g p_{00}^\alpha (\kappa + 1)} \left[ p_b^{\kappa+1} - p_N^{\kappa+1} \right] + p_a h_N \\
\sum_{\alpha=2}^{N-1} \left\{ \sigma_\alpha \left( \frac{1}{2} u_{\alpha i} + R_i \right) u_\alpha^i - c_p \frac{\theta_\alpha}{g p_{00}^\alpha (\kappa + 1)} \left[ p_{\alpha+1}^{\kappa+1} - p_\alpha^{\kappa+1} \right] - g \sigma_\alpha h_b + p_a h_\alpha \right\} + \\
\sigma_1 \left( \frac{1}{2} u_{1i} + R_i \right) u_1^i - c_p \frac{\theta_1}{g p_{00}^2 (\kappa + 1)} \left( p_2^{\kappa+1} - p_a^{\kappa+1} \right) - g \sigma_1 h_b + p_a h_1 \right\}. 
\]  

(3.6)

The pressures at the interface of each layer follow from the definition of the pseudo densities, e.g. \( p_2 = g \sigma_1 + p_a, \ldots, p_\alpha = g (\sigma_1 + \ldots + \sigma_{\alpha-1}) + p_a, \ldots, p_b = g (\sigma_1 + \ldots + \sigma_N) + p_a \). Variations in (3.6) are taken with respect to the fluid labels \( a_\alpha \) in each layer. For simplicity we take \( p_a = 0 \), although we can generalize the argument in a consistent manner, resulting in an extra term \( p_a (H_T - h_1 - \ldots - h_N - h_b) \) in the integrals of (3.6), in which \( H_T > h_T \) is the fixed height of a domain filled partially by the active fluid and partially with a passive gas of constant pressure \( p_a \).

b. Constant density

Consider the Eulerian variational principle for a hydrostatic flow in Cartesian coordinates

\[
0 = \delta \int_{t_0}^{t_1} dt \int_{D_h} \int_{h_b(x)}^{h_T(x, t)} dz \rho \left\{ \left( \frac{1}{2} u_i + R_i \right) u^i - g z \right\}, 
\]  

(3.7)

which is an incomplete variational principle for an incompressible flow in hydrostatic balance with \( i = 1, 2 \). It follows from (2.1) with \( \delta_H = \epsilon_B = 0, \rho_q = \rho \). Density can be taken constant in each layer and the velocity is defined in terms of fluid parcels, which are advected by the flow. However, the principle above is incomplete because the constraint imposing the
incompressibility condition is absent. For inviscid flows in shallow layers of constant density, the velocity in each layer remains independent of the vertical coordinate when it started off so initially. The integration over the layer depths can thus be taken without explicitly imposing the incompressibility condition. We refer again to figure 1 with the modification that the density instead of the entropy is taken to be fixed in each layer. To emphasize the analogy between the entropy and density layer models we define
\[ \sigma_{\alpha} \equiv \rho_{\alpha} h_{\alpha}. \]

An Eulerian variational principle for \( N \) shallow layer equations arises when we integrate (3.7) across layers:
\[
0 = \int_{t_0}^{t_1} dt \int_{D_H} d\mathbf{x} \left\{ \left[ \frac{1}{2} u_{Ni} + R_i \right] u_i^N - \frac{1}{2} g \left( (\sigma_N + \rho_N h_b)^2 - (\rho_N h_b)^2 \right) / \rho_N \right] + \right.
\]
\[
\sum_{\alpha=1}^{N-1} \left[ \sigma_{\alpha} \left( \frac{1}{2} u_{\alpha i} + R_i \right) u_i^\alpha - \frac{1}{2} g \left( (\sigma_{\alpha} + \rho_{\alpha} \sigma_{\alpha+1}/\rho_{\alpha+1} + \ldots + \rho_{\alpha} \sigma_N/\rho_N + \rho_{\alpha} h_b)^2 \right) / \rho_{\alpha} \right] \right\},
\]

where variations are taken with respect to the fluid labels \( a_{\alpha} \) in the \( N \) layers (\( i = 1, 2; \alpha = 1, \ldots, N \)).

c. Equations of motion

In both layer models the pseudo densities \( \sigma_{\alpha} \) and the velocities are again related to the fluid labels as
\[ \sigma_{\alpha} = \sigma_{\alpha a}(a_{\alpha}, b_{\alpha}) \quad \text{and} \quad \partial a_{\alpha}/\partial t + (v_{\alpha} \cdot \nabla) a_{\alpha} = 0. \] (3.9)
The continuity equations are again like (2.10):
\[ \partial \sigma_{\alpha}/\partial t + \nabla \cdot \left( \sigma_{\alpha} v_{\alpha} \right) = 0. \] (3.10)
The momentum equations in each layer follow from (3.6) or (3.8) and are
\[ \delta a_{\alpha} : \partial v_{\alpha}/\partial t + (v_{\alpha} \cdot \nabla) v_{\alpha} + f \hat{z} \times v_{\alpha} = -\nabla M_{\alpha}. \] (3.11)
The Montgomery potentials \( M_{\alpha} \) for the isentropic layer model are defined as
\[
M_N = g h_b + c_p \theta_N \left( \frac{p_0}{p_{00}} \right)^{\kappa},
\]
\[
M_{\alpha} = g h_b + c_p \theta_N \left[ \left( \frac{p_{\alpha}}{p_{00}} \right)^{\kappa} - \left( \frac{p_N}{p_{00}} \right)^{\kappa} \right] + \sum_{\gamma=\alpha+1}^{N-1} c_p \theta_{\gamma} \left[ \left( \frac{p_{\gamma+1}}{p_{00}} \right)^{\kappa} - \left( \frac{p_N}{p_{00}} \right)^{\kappa} \right] + c_p \theta_{\alpha} \left( \frac{p_{\alpha+1}}{p_{00}} \right)^{\kappa} \quad (\alpha = 1, \ldots, N - 1)
\] (3.12)
and the ones for the isopycnal layer model are
\[
M_{\alpha} = \begin{cases} 
  g (\sigma_1/\rho_1 + \ldots + \sigma_N/\rho_N + h_b), & \text{if } \alpha = 1 \\
  g \left( \sigma_{\alpha}/\rho_{\alpha} + \ldots + \sigma_N/\rho_N + h_b + \sum_{\gamma=1}^{\alpha-1} \sigma_{\gamma}/\rho_{\alpha} \right), & \text{if } \alpha = 2, \ldots, N.
\end{cases}
\] (3.13)
These potentials emerge from the variational principle after some manipulation, integration by parts and after using suitable boundary conditions. Suitable boundary conditions include
quiescence at infinity, periodicity, no normal flow at solid boundaries, vanishing isentropic layer thickness, or a combination thereof.

d. Hamiltonian formulation

The generalized interior momentum corresponding to (3.6) or (3.8) is (e.g. Morrison (1998))

$$\pi_{\alpha k}(x, t) = \delta L_N [a_\alpha, d a_\alpha / dt] / \delta \left( \frac{\partial a_\alpha^k}{\partial t} \right) = \sigma_{\alpha} (\Gamma_{\alpha}^{-1})^{m} \left( \delta_{mn} (\Gamma_{\alpha}^{-1})^{n} \frac{\partial a_\alpha^k}{\partial t} - R_m \right)$$

$$= -\sigma_{\alpha} (\Gamma_{\alpha}^{-1})^{m} \left( u_{\alpha m} + R_m \right)$$

(3.14)

(no summation over $\alpha$) with $(\Gamma_{\alpha})^k_i = \partial a_\alpha^k / \partial x^i$. After using the Legendre transform expressed in terms of $v$ and $a$, we find the action principle

$$0 = \delta \int_{t_0}^{t_1} dt \left\{ \int_{D_H} d\mathbf{x} \left[ \sum_{\alpha=1}^{N} \sigma_{\alpha} (u_{\alpha m} + R_m) (\Gamma_{\alpha}^{-1})^{m} \frac{\partial a_\alpha^k}{\partial t} \right] + H_N [a_\alpha, v_\alpha] \right\}.$$ (3.15)

The respective Hamiltonians as Legendre transform $H_N$ are

$$H_N[\sigma_\alpha, v_\alpha] = \int_{D_H} d\mathbf{x} \left\{ \frac{1}{2} \sigma_N |v_N|^2 + g \sigma_N h_b + \frac{\theta_N}{g p_0^\kappa (\kappa + 1)} \left( p_b^{(\kappa + 1)} - p_N^{(\kappa + 1)} \right) \right\} +$$

$$\sum_{\alpha=2}^{N-1} \left\{ \frac{1}{2} \sigma_{\alpha} |v_\alpha|^2 + g \sigma_{\alpha} h_b + \frac{\theta_{\alpha}}{g p_0^\kappa (\kappa + 1)} \left( p_{\alpha - 1}^{(\kappa + 1)} - p_{\alpha}^{(\kappa + 1)} \right) \right\} +$$

$$\frac{1}{2} \sigma_1 |v_1|^2 + g \sigma_1 h_b + \frac{\theta_1}{g p_0^\kappa (\kappa + 1)} \left( p_1^{(\kappa + 1)} \right)$$

(3.16)

for the entropy model and

$$H_N[\sigma_\alpha, v_\alpha] = \int_{D_H} d\mathbf{x} \left\{ \sum_{\alpha=1}^{N} \left( \frac{1}{2} \sigma_{\alpha} |v_\alpha|^2 + \right. \right.$$}

$$\frac{1}{2} g \left[ \left( \sigma_{\alpha} + \rho_{\alpha} \sigma_{\alpha+1} \right) + \ldots + \rho_{\alpha} \sigma_N / \rho_{\alpha} h_b \right]$$

$$= \left( \rho_{\alpha} \sigma_{\alpha+1} / \rho_{\alpha+1} + \ldots + \rho_{\alpha} \sigma_N / \rho_{\alpha} h_b \right)^2 / \rho_{\alpha} \right\}$$

(3.17)

with $h_b = \sigma_{N+1} / \rho_{N+1}$, for the density model, respectively. Similar variations as in Hamilton’s principles (3.6) or (3.8) yield the $N$ horizontal momentum equations (3.11) in each isentropic or isopycnal layer for $a_\alpha$, and the advection equations (3.9)b for $\delta v_\alpha$.

A Hamiltonian formulation of the hydrostatic equations in the $N$ layers has the form

$$\frac{dF}{dt} = [F, H_N].$$ (3.18)

In terms of the three sets of variables $\{a_\alpha, \pi_\alpha\}$, $\{a_\alpha, v_\alpha\}$ and $\{\sigma_\alpha, v_\alpha\}$, we can find the following generalized Poisson brackets

$$[F, G] = \sum_{\alpha=1}^{N} \int_{D_H} d\mathbf{x} \left\{ \frac{\delta F}{\delta a_\alpha} \frac{\delta G}{\delta \pi_\alpha} - \frac{\delta F}{\delta \pi_\alpha} \frac{\delta G}{\delta a_\alpha} \right\}.$$ (3.19)
We notice that the coupling between the layers is facilitated only through the Hamiltonian, and that the contribution from one layer to the (generalized) Poisson bracket is disconnected from the others. The first bracket in terms of the variables or fields \{\pi_\alpha, a_\alpha\} (x, t) is an infinite-dimensional canonical bracket, which is skew-symmetric \[[F, G] = -[G, F]\] and satisfies Jacobi’s identity \[[K, [F, G]] + [F, [G, K]] + [G, [K, F]] = 0\] (e.g. Morrison 1998). This bracket yields Hamilton’s equations and follows directly from the corresponding action principle (cf. the finite-dimensional case of classical mechanics). The second and third brackets, which are expressed in terms of the variables \{v_\alpha, a_\alpha\} and \{v_\alpha, \sigma_\alpha\}, are transformations of this canonical bracket and are therefore also skew-symmetric and satisfy Jacobi’s identity. The third bracket only includes three variables \(v_\alpha\) and \(\sigma_\alpha\) per layer, indicating that there is a symmetry, associated with Noether’s theorem, which allows for this reduction (see e.g. Morrison 1998). The shallow-water equations have a similar bracket (e.g. Shepherd 1990) as each layer has here.

4 Summary and Discussion

Starting with the Eulerian variational principle for the Euler equations, I have systematically derived the variational and Hamiltonian formulations of the hydrostatic incompressible, Boussinesq equations in isopycnal coordinates, and the hydrostatic multi-layer isentropic or isopycnal systems (sections 2 and 3).

The various geometric formulations derived in the present paper all share similar Poisson brackets, i.e. the (contribution to the) Poisson bracket in each layer or in the continuously stratified case is the one for the shallow water equations with the depth \(h\) replaced by the appropriate pseudo density \(\sigma\), while the potential (and internal) energy in the Hamiltonian differs in each case. For both the layer and continuous models, the coupling between each (infinitesimal) layer happens solely through the Hamiltonian (e.g. Salmon 1982), which is interesting from a numerical viewpoint: any numerical progress in preserving part of the Poisson bracket structure for the shallow water equations, say, can perhaps extend to these similar hydrostatic systems. Moreover, when we discretize only the vertical material coordinate in the Poisson bracket for the continuously stratified hydrostatic models in a straightforward manner, we can derive the Poisson bracket for the layer models.

All the hydrostatic models considered have material or layer coordinates in the vertical. Careful attention must therefore be paid to the movement of the isentropes and isopycnals or the (internal) interfaces over the (sloping) boundaries. The movement of these material
surfaces or interfaces at boundaries tends to be difficult to model numerically. For numerical benchmark purposes, it is thus important to validate numerical simulations against exact solutions in which the “water line” movement is an essential feature. Both the one-layer shallow-water and the one-layer isentropic model admit such exact solutions. It may be interesting to investigate whether all these special solutions can be inferred from their variational or Hamiltonian formulations, and henceforth whether it would lead us to novel solutions.

This paper provides the conservative structure, which can form the starting point for the development of conservative numerical integration techniques in atmospheric and oceanic forecasting models. The Eulerian variational principles may form an alternative route for attempting to find conservative discretizations. Consider for example the Eulerian variational principle for the one-layer shallow water equations, i.e. (3.8) with $N = 1$ and $\sigma = h$:

$$
0 = \int_{t_0}^{t_1} dt \int_{D_H} dx \left\{ h \left( \frac{1}{2} u_i + R_i \right) u^i - \frac{1}{2} g \left( (h + h_b)^2 - h_b^2 \right) \right\} \\
= \int_{t_0}^{t_1} dt \int_{D_H} dx \left\{ \left( \frac{1}{2} u_i + R_i \right) u^i - g (h + h_b) \right\} \delta h + h u_i \delta u^i.
$$

In weighted residual methods such as the finite-element method the general approach is to find a weak formulation, which arises as an integral of the relevant equations of motion multiplied by test functions when there is no variational principle known. Nobody appears to have tried to base a discretization on Eulerian variational principles for (inviscid) fluid equations. However, with a weighted residual or finite difference approach the spatial part of (4.1) is readily discretized since no spatial derivatives appear. Due to the presence of fluid labels in the formulation one can expect Lagrangian chaotic motion of the fluid labels, in the absence of Eulerian chaotic motion, which may hamper numerical stability and accuracy. The crucial and difficult problem lies therefore in discretizing the Jacobian (3.9a) of the fluid label variables and the label advection equations (3.9b), which define the height and the velocity, such that there is no reference to the fluid labels required in the continuity and momentum equations. It is unclear whether this is possible or whether one has to settle for an approximation, for example, one in which the velocity and height are defined as an average over the particle motions.

Finally, Eulerian variational principles can form the starting point for deriving approximate balanced models which describe large-scale low Froude- or Rossby-number dynamics (Holm 1996; Bokhove 2001). For global dynamics in the atmosphere and oceans, the results of this paper need to be extended to the spherical case and to quasi-hydrostatic approximations (White 1999; Staniforth 2000). However, there appear to be no serious obstacles in using the presented techniques in such a global context (see also Roulstone and Brice 1995).

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A Eulerian variational principle in one dimension

Consider the one-vertical-dimensional form of the Eulerian variational principle (2.1) for Euler flows

\[ 0 = \delta \int_{t_0}^{t_1} dt \mathcal{L}_E[a, da/dt] = \delta \int_{t_0}^{t_1} dt \int dz \rho \left\{ \frac{1}{2} w^2 - g z - U(s, \rho) \right\}, \quad (A.1) \]

where the entropy \( s \), the density \( \rho \) and vertical velocity \( w \) are economic shorthands for expressions in terms of the vertical fluid parcel variable \( a(z, t) \), as follows:

\[ s = s(a), \quad \rho = \frac{\partial a}{\partial z}, \quad w = - \left( \frac{\partial a}{\partial z} \right)^{-1} \frac{\partial a}{\partial t}. \quad (A.2) \]

Hence, we can substitute (A.2) into (A.1) and consider

\[ 0 = \delta \int_{t_0}^{t_1} dt \int dz \left\{ \frac{1}{2} \left( \frac{\partial a}{\partial z} \right)^{-1} \left( \frac{\partial a}{\partial t} \right)^2 - \frac{\partial a}{\partial z} g z - \frac{\partial a}{\partial z} U \left( s(a), \frac{\partial a}{\partial z} \right) \right\}; \quad (A.3) \]

variations are thus taken with respect to fluid parcel variable \( a(z, t) \) only. Note that label \( a(z, t) \) thus plays the role of the generalized coordinates \( q_i \) in classical mechanics (e.g. Morrison 1998). Alternatively, we can first take the variations of (A.1) with respect to \( w \), \( \rho \) and \( s \) and subsequently use the expressions in (A.2), which yields

\[ 0 = \delta \int_{t_0}^{t_1} dt \int dz \left\{ - \delta \mathcal{L}_E \delta w \frac{\partial a}{\partial t} + \frac{\partial a}{\partial z} \right\} \]

\[ = \int_{t_0}^{t_1} dt \left\{ - \delta \mathcal{L}_E \delta \left[ \frac{1}{2} \left( \frac{\partial a}{\partial z} \right)^{-1} \left( \frac{\partial a}{\partial t} \right)^2 + \frac{\partial a}{\partial z} g z - \frac{\partial a}{\partial z} U \left( s(a), \frac{\partial a}{\partial z} \right) \right] \right\}. \quad (A.4) \]

The former approach (i.e. using (A.3)) emphasizes the role of the key label variable and the analogy with the finite-dimensional realm, while the latter approach (i.e. using (A.4)) underscores that the variations of \( w \), \( \rho \) and \( s \) are constrained to independent variations of the label \( a \) only. We also need to use the first law of thermodynamics and its variations:

\[ \delta U = T \delta s + \frac{p}{\rho^2} \delta \rho = T \frac{\partial s}{\partial a} \delta a + p \left( \frac{\partial a}{\partial z} \right)^2 \frac{\partial a}{\partial z} \quad \text{and} \quad \frac{1}{\rho} \delta p = c_p \delta T - T \delta s. \quad (A.5) \]

When we vary (A.3), we then find

\[ 0 = - \delta \int_{t_0}^{t_1} dt \int dz \left\{ w \frac{\partial \delta a}{\partial t} - \rho T \frac{\partial a}{\partial a} \delta a + \left( \frac{1}{2} w^2 + g z + U + p/\rho \right) \frac{\partial \delta a}{\partial z} \right\} \]

\[ = \int_{t_0}^{t_1} dt \int dz \left\{ \frac{\partial w}{\partial t} - T \frac{\partial a}{\partial a} \delta a + \frac{\partial a}{\partial z} \left( \frac{1}{2} w^2 + g z + U + p/\rho \right) \right\} \delta a + \ldots, \quad (A.6) \]

where I have left issues regarding boundary contributions due to integration by parts to the reader (e.g. take \( \delta a = 0 \) at a solid boundary). Hence we find the vertical momentum equation as an Euler-Lagrange equation:

\[ \frac{\partial}{\partial t} \left( - \left[ \frac{\partial a}{\partial z} \right]^{-1} \frac{\partial a}{\partial t} \right) + \frac{\partial}{\partial z} \left( \frac{1}{2} w^2 + g z \right) + \frac{1}{\rho} \frac{\partial p}{\partial z} = 0, \quad (A.7) \]
which is second order in time. If we rewrite (A.7) with the use of (A.2), we find the more familiar form of the momentum equation:

\[
\frac{\partial w}{\partial t} + \frac{\partial}{\partial z} \left( \frac{1}{2} w^2 + g z \right) + \frac{1}{\rho} \frac{\partial p}{\partial z} = 0.
\] (A.8)

An equation of state and boundary and initial conditions on \(a(z,t)\) are required in addition to (A.7) and a separate continuity equation is then not required. Nevertheless, it arises from parcel advection as follows. With

\[
\rho = \frac{\partial a}{\partial z} \quad \text{and} \quad \frac{\partial a}{\partial t} = -w \frac{\partial a}{\partial z} = -\rho w,
\] (A.9)

we find the usual continuity equation

\[
0 = \frac{\partial}{\partial t} \left( \rho - \frac{\partial a}{\partial z} \right) = \frac{\partial \rho}{\partial t} - \frac{\partial}{\partial z} \left( \frac{\partial a}{\partial t} \right) = \frac{\partial \rho}{\partial t} + \frac{\partial (\rho w)}{\partial z}.
\] (A.10)

Since entropy \(s(z,t) = s(a(x,t))\), we can derive the entropy advection equation from (A.9) and with an equation of state the Euler equations are complete.

Taking variations of the various variational principles in this paper is essentially similar to the above variation, except that the algebra becomes more involved when more than one spatial dimension is encountered. It may be helpful to verify part of the presented results in the main text by first restricting attention to only one horizontal dimension.

REFERENCES


Figure Caption

Figure 1.
Sketch of the $N$ material layers above the Earth’s surface or ocean bottom, which resides at $z = h_b$ and where the pressure is $p_b$. “Material” constant $\nu_i$ in layer $i$ is either entropy $s_i$ in isentropic or density $\rho_i$ in isopycnal layer models. The pressure at the top of layer $i$ is $p_i$ and this layer has thickness $h_i$. The passive atmospheric pressure above the ocean or in the outer atmosphere is $p_1 = p_a$. 
active fluid

\[ \nu \]

\[ z = 0 \]

passive atmosphere

\[ p_a \]

\[ h_J \]

\[ v_J \]

\[ p_2 \]

\[ h_2 \]

\[ v_2 \]

\[ p_3 \]

active fluid

\[ p_{N-1} \]

\[ h_{N-1} \]

\[ v_{N-1} \]

\[ p_N \]

\[ h_N \]

\[ v_N \]

Earth

\[ p_b \]

\[ h_b \]

z = 0