A RELIABLE AND EFFICIENT IMPLICIT A POSTERIORI
ERROR ESTIMATION TECHNIQUE FOR THE TIME
HARMONIC MAXWELL EQUATIONS

FERENC IZSÁK AND JAAP J.W. VAN DER VEGT

Abstract. We analyze an implicit a posteriori error indicator for the time
harmonic Maxwell equations and prove that it is both reliable and locally effi-
cient. For the derivation, we generalize some recent results concerning explicit
a posteriori error estimates. In particular, we relax the divergence free con-
straint for the source term. We also justify the complexity of the obtained
estimator.

1. Introduction

A posteriori error estimates are of particular importance in the numerical so-
lution of the Maxwell equations. Physical domains with non-trivial geometries,
discontinuous material coefficients and non-smooth source terms result in con-
sideerable computational problems, which require an adaptive solution technique. The
cornerstone of such an algorithm is a proper a posteriori error estimate which marks
the regions for refinement or delivers reliable stopping criteria.

Implicit error estimation techniques proved themselves to be particularly useful in
the a posteriori error analysis. Implicit a posteriori error estimates as the solution
of a local problem are really sensitive to the differential operator of the underlying
PDE and strongly depend on the shape of the corresponding subdomain.

The objective of this article is to prove that the implicit a posteriori technique
developed in [5, 7] provides both an upper and lower bound for the true error in
the finite element solution if two additional terms are included in the local equation
for the error. Hence the algorithm is both efficient and reliable. The main step in
the analysis is to link the implicit error estimator to explicit estimators for which
recently new important results are obtained.

In particular, the paper [3], in which the reliability of an error indicator has been
proved, and some numerical results have been provided. Its analysis is, however, re-
cicted to the case of the curl-elliptic Maxwell equations and divergence-free source
terms. The results in [3] have been further improved in [11], where also the elliptic-
ity condition could be removed. By using a recently developed quasi interpolation
 technique the author proved the efficiency of the error indicator. Another basic
ingredient of the proof was a decomposition lemma in [9], which is different from

Date: December 2, 2007.

1991 Mathematics Subject Classification. 65N30.

Key words and phrases. Maxwell equations, a posteriori error estimation.

This research was supported by NSF, the Dutch government through the national program
BSIK: knowledge and research capacity, in the ICT project BRICKS, Theme MSV1.

Supported by OTKA, grant No. K68253.
the classical Helmholtz decomposition. At the same time, a restriction corresponding to the source term remained: only the case of a divergence-free source term was investigated. The main improvement which makes the analysis possible is the quasi interpolation technique (see also [10]), which is an outstanding tool for the approximation of (possibly non-smooth) functions with a well defined curl. At the same time, the decomposition lemma in [9] strongly requires the divergence free property.

In this article, we will first remove the restriction of divergence free source terms. We will use these results then to prove also reliability and efficiency estimates of an implicit error estimation technique. This will essentially complete the analysis which we discussed in [5, 7].

The article is organized as follows. After some mathematical preliminaries we formalize an explicit error indicator which is the basis of our construction. We justify its complexity: we point out that the additional terms compared to the simpler error indicator in [7] are really necessary, without them we can not have a reliable error estimate. Then using a bubble function technique we prove that the localized error indicator is a lower bound of the exact error. In the subsequent section, after extending Lemma 2.2 in [9], we modify the proof in [11] such that reliability of the new error indicator is ensured also in the case of a source term with a non-zero divergence. Using all of these, an implicit error estimation technique will be derived in Section 4, which is both reliable and locally efficient.

2. Preliminaries

We investigate the time harmonic Maxwell equations for the electric field \( \mathbf{E} \)

\[
\nabla \times \nabla \times \mathbf{E} - k^2 \mathbf{E} = \mathbf{J} \quad \text{in} \quad \Omega, \\
\nu \times \mathbf{E} = 0 \quad \text{on} \quad \partial \Omega,
\]

where \( \Omega \subset \mathbb{R}^3 \) is a polyhedral Lipschitz domain with \( \nu \) the outward normal and \( k \) the wave number of the electromagnetic waves. We assume that \( \text{div} \, \mathbf{J} \in L^2(\Omega) \) holds for a given \( \mathbf{J} \in [L^2(\Omega)]^3 \). In electromagnetics, \( \text{div} \, \mathbf{J} \) gives the charge density (see [8], Section 1.2), therefore in real applications, where the electric charge is distributed on a three-dimensional manifold, this contribution will in general not be zero.

For the weak form of the time harmonic Maxwell equations we use the Hilbert space

\[
H(\nabla \times, \Omega) = \{ \mathbf{u} \in [L^2(\Omega)]^3 : \nabla \times \mathbf{u} \in [L^2(\Omega)]^3 \},
\]
equipped with the curl norm

\[
\| \mathbf{u} \|_{\text{curl}, \Omega} = (\| \mathbf{u} \|_{L^2(\Omega)}^2 + \| \nabla \times \mathbf{u} \|_{L^2(\Omega)}^2)^{1/2}
\]
and corresponding to the (perfectly conducting) boundary condition in (1) we also need the Hilbert space

\[
H_0(\nabla \times, \Omega) = \{ \mathbf{u} \in H(\nabla \times, \Omega) : \nu \times \mathbf{E} = 0 \quad \text{on} \quad \partial \Omega \}.
\]

We will also use the Hilbert space

\[
H(\text{div}, \Omega) = \{ \mathbf{u} \in [L^2(\Omega)]^3 : \text{div} \, \mathbf{u} \in L^2(\Omega) \},
\]

which is equipped with the div norm

\[
\| \mathbf{u} \|_{\text{div}, \Omega} = (\| \mathbf{u} \|_{L^2(\Omega)}^2 + \| \text{div} \, \mathbf{u} \|_{L^2(\Omega)}^2)^{1/2}
\]
Remark: Taking the divergence of both sides in (1) we obtain that $E \in H(\text{div}, \Omega) \cap H(\text{curl}, \Omega)$.

For the standard Sobolev norm of the space $H^{s}(K)$ we use the notation $\| \cdot \|_{s,K}$ and $(\cdot, \cdot)_{K}$ for the $L_{2}(K)$ and $[L_{2}(K)]^{3}$ scalar products in the domain $K$. In the case $s = 0$ or $K = \Omega$ the corresponding subscripts are dropped, which is also applied for the curl and div norms above.

Using the Green theorem, one can rewrite (1) into a weak form: Find $E \in H_{0}(\text{curl}, \Omega)$ such that for all $v \in H_{0}(\text{curl}, \Omega)$

\[
B(E, v) := (\text{curl}(E), \text{curl}(v)) - k^{2}(E, v) = (J, v)
\]

In this article we assume that the finite element approximation $E_{h}$ has been obtained using Nédélec type conforming elements. For details on these spaces, we refer to [8]. Note that on a tessellation with elements $K$ we need for existence of the finite element interpolation of $J$ and the residual that $J \in H^{2+ \delta}_{\text{curl}}(K)$ and curl$J \in [L_{2+ \delta}(K)]^{3}$ for some $\delta > 0$ (see Lemma 5.38 in [8]). In this way, for a well defined finite element method the assumption $\text{div}J \in [L_{2}(\Omega)]^{3}$ does not result in a strict smoothness requirement, see [2], Proposition 3.7.

A Nédélec type a finite element space is denoted with $H_{0,\text{curl}}(\Omega) \subset H_{0}(\text{curl}, \Omega)$ and we rewrite (2) in the following form:

Find $E_{h} \in H_{0,h}(\text{curl}, \Omega)$ such that for all $v_{h} \in H_{0,h}(\text{curl}, \Omega)$

\[
(curl E_{h}, curl v_{h}) - k^{2}(E_{h}, v_{h}) = (J, v_{h}).
\]

We will use the assumption $H_{0,h}(\text{curl}, \Omega) \supset N_{0,h}$, where $N_{0,h}$ denotes the lowest order Nédélec type finite element space.

2.1. Bilinear form for the error. We investigate an explicit a posteriori estimate for the error

\[
e_{h} = E - E_{h}.
\]

The key point in the analysis of the Maxwell equations is to apply a Helmholtz-decomposition both for the error and the exact solution. In concrete terms, we use the decomposition:

\[
v = \nabla \Phi + z,
\]

where $\Phi \in H_{1}^{1}(\Omega)$ and $z \in [\nabla H_{1}^{1}(\Omega)]^{3}$. Since curl$\circ$grad $= 0$ this orthogonality can be understood both with respect to the $H(\text{curl}, \Omega)$ and the $L_{2}(\Omega)$ scalar product. Using this decomposition and the Green formula for the curl operator (see [8], Theorem 3.31) applied to the subdomains $K \in T_{h}$, the bilinear form for the error can be rewritten as:

\[
B(e_{h}, v) = (J, v) - ((\text{curl} E_{h}, \text{curl} v) - k^{2}(E_{h}, v))
\]

\[
= (J, \nabla \Phi + z) - ((\text{curl} E_{h}, \text{curl} z) - k^{2}(E_{h}, \nabla \Phi + z))
\]

\[
= (J, \nabla \Phi + z) - \sum_{K \in T_{h}} ((\text{curl} E_{h} - k^{2}E_{h}, z)_{K} - k^{2}(E_{h}, \nabla \Phi)_{K})
\]

\[
+ \sum_{K \in T_{h}} \sum_{l_{j} \subset \partial K} \gamma_{l_{j}}(\text{curl} E_{h}, \pi_{l_{j}} z)_{l_{j}},
\]

where $l_{j}$ denotes an arbitrary face of $\partial K$. The operators $\pi_{l}$ and $\gamma_{l}$ denote the extension of the trace operators, which are defined for smooth functions $u \in [C^{\infty}(\bar{K})]^{3}$ as $\pi_{l}u = (\nu_{l} \times u|_{\partial K}) \times \nu_{l}$ and $\gamma_{l}u = \nu_{l} \times u|_{\partial K}$, to functions in $H(\text{curl}, K)$. Accordingly, in the finite element formulation we use the notations $u|_{\partial K}$ and $\nu_{l} \times u|_{\partial K}$.
respective. For their analysis, we refer to [4]. After elementwise integration by parts of the gradient operator we obtain the identity

$$B(e_h, v) = \sum_{K \in T_h} (J - (\text{curl } \mathbf{E}_h - k^2 \mathbf{E}_h), \mathbf{z})_K - \left(\text{div } (J + k^2 \mathbf{E}_h), \Phi\right)_K$$

(6)

$$+ \sum_{K \in T_h, l_j \subset \partial K} (\nu_j \times \text{curl } \mathbf{E}_h, \pi_r \mathbf{z})_{l_j} + (\nu_j \cdot (J + k^2 \mathbf{E}_h), \Phi)_{l_j}.$$ 

To rewrite the above formula, we introduce the interelement jumps for \(\mathbf{x} \in \partial K_i \cap \partial K_j:\)

$$[g](\mathbf{x}) = \lim_{x_n \to x} g(x_n) - \lim_{x_n \to x} g(x_n) .$$

For \(\mathbf{x} \in \Omega\) we take the outward limit zero. Using this notation the summation over the interior faces can be assembled and we obtain that

$$\sum_{K \in T_h, l_j \subset \partial K} (\nu_j \times \text{curl } \mathbf{E}_h, \pi_r \mathbf{z})_{l_j} + (\nu_j \cdot (J + k^2 \mathbf{E}_h), \Phi)_{l_j}$$

$$= \sum_{l \in \Gamma_h} (\nu \times [\text{curl } \mathbf{E}_h]_l, \pi_r \mathbf{z})_l + (\nu \cdot [J + k^2 \mathbf{E}_h]_l, \Phi)_l,$$

where \(\Gamma_h\) denotes the set of element faces corresponding to the finite element tessellation \(T_h\) and \(\nu_j\) is a unit vector normal to \(l_j\) corresponding to the sign of the jump. In order to simplify the forthcoming analysis, we introduce the following notations for the residuals in (6)

$$r_1|K = J - \text{curl } \mathbf{E}_h + k^2 \mathbf{E}_h|_K,$$

$$r_2|K = \text{div } (J + k^2 \mathbf{E}_h)|_K,$$

$$R_1|K = \sum_{l_j \subset \partial K} \mathbf{R}_{1,l_j} = \sum_{l_j \subset \partial K} \nu_j \times [\text{curl } \mathbf{E}_h]|_{l_j},$$

$$R_2|K = \sum_{l_j \subset \partial K} \mathbf{R}_{2,l_j} = \sum_{l_j \subset \partial K} \nu_j \cdot [J + k^2 \mathbf{E}_h]|_{l_j}.$$ 

If it is not confusing, the subscripts \(K\) will be dropped. With these notations we can rewrite (6) to obtain

$$B(e_h, v) = \sum_{K \in T_h} (r_1, \mathbf{z})_K - (r_2, \Phi)_K + \sum_{l \in \Gamma_h} (\mathbf{R}_{1,l}, \mathbf{z})_l + (\mathbf{R}_{2,l}, \Phi)_l.$$

3. Error estimation

The quality of an a posteriori error estimator \(\eta\) is determined by several factors. In the optimal case, it provides both a lower and an upper bound for the error \(e_h\), in our case, with respect to the curl norm:

$$C_{\text{eff}} \eta \leq ||e_h|| \leq C_{\text{rel}} \eta.$$

If there exist such mesh-independent constants \(C_{\text{eff}}\) and \(C_{\text{rel}}\) then the estimate \(\eta\) is called efficient and reliable, respectively.

According to the elliptic theory (see [1], formula (2.19)) we define a local a posteriori error indicator

$$\eta^2_K = h_K^2 (||r_1||^2_{L^2(K)}) + ||r_2||^2_{L^2(K)} + h_K (||\mathbf{R}_1||^2_{L^2(\partial K)}) + ||\mathbf{R}_2||^2_{L^2(\partial K)}.$$

$$\eta^2_K = h_K^2 (||r_1||^2_{L^2(K)}) + ||r_2||^2_{L^2(K)} + h_K (||\mathbf{R}_1||^2_{L^2(\partial K)}) + ||\mathbf{R}_2||^2_{L^2(\partial K)}.$$
and a global indicator as

\[ \eta_{T_h}^2 = \sum_{K \subset T_h} \eta_K^2, \]

where \( h_K \) is the mesh size of \( K \). Before its detailed analysis, we show that the additional terms in (10) compared to the estimate in [7] are necessary.

### 3.1. A comparison of error indicators.

The error indicator (9) is more complicated than the similar error indicator

\[ \hat{\eta}_K^2 = h_K^2 \| r_1 \|_{L^2(K)}^2 + h_K \sum_{l_j \subset K} \| R_1 \|_{L^2(l_j)}^2, \]

which was derived in [7] and used for the derivation of an implicit error estimate which has been successfully used to control an \( h \)-adaptive method [5]. Moreover, in (8) we have to use the assumption that \( \text{div}J \in L^2(\Omega) \). Could (9) be augmented with a simpler indicator? In particular, are all of the residual terms in (10) necessary and are they of a different magnitude? Also, one could ask if it is not possible to modify the powers of the mesh parameters in \( \hat{\eta}_K \) such that it also provides an upper bound, which is simpler than (9)?

According to classical elliptic theory (see [1], Chapter 2.2), the coefficients \( h_K \) and \( h_K^2 \) in \( \eta_K \) arise from (quasi) interpolation theorems. Therefore, due to the lack of smoothness of functions in \( H(\text{curl}, \Omega) \), it seems to be appropriate to change the powers of the mesh parameter \( h_K \).

To investigate the above questions in precise terms, we introduce a scaled error indicator

\[ \zeta_{K,\alpha,\beta}^2 = h_K^{2\alpha} \| r_1 \|_{L^2(K)}^2 + h_K^\beta \sum_{l_j \subset \partial K} \| R_1 \|_{L^2(l_j)}^2, \]

and define \( \zeta_{T_h,\alpha,\beta} \) as a global error indicator according to (10).

In the subsequent analysis, we assume that the finite element discretization satisfies the following:

- [H1] \( \Omega \) is a polyhedral domain.
- [H2] The family of tetrahedral finite element meshes \( \{T_h\} \) is consecutively refined such that

\[ \lim_{h \to 0} \left( \max_{K \subset T_h} \text{diam} K \right) = 0, \]

where the parameters \( h > 0 \) form a decreasing zero sequence as the mesh is refined.
- [H3] \( H_h(\text{curl}, \Omega) \) consists of a family of \( H^1(\Omega) \) elements.

Note that the above assumptions are quite general, not even the regular or quasi uniform property of the mesh is required. The main result of this section is the following.

**Theorem 1.** If we approximate \( E \) in the Maxwell equations (1) with \( H^1(\Omega) \) elements such that [H1]-[H3] are satisfied, then the scaled error indicator \( \zeta_{T_h,\alpha,\beta} \) does not provide an upper bound for the error since for any \( \alpha, \beta > 0 \) one can always find a \( J \in H(\text{div}, \Omega) \) in (1) such that

\[ \lim_{h \to 0} \frac{\zeta_{T_h,\alpha,\beta}}{\| e_h \|_{\text{curl}}} = 0. \]
The proof can be found in Appendix A.

3.2. The efficiency of the error indicator. A standard bubble function technique will show that \( \eta_h \) provides a lower bound for the error. For each residual, the overbar denotes its finite element approximation. For each \( K \in \mathcal{T}_h \) we also use the following notations:

- \( \Psi_K \) - the element bubble function corresponding to element \( K \).
- \( \Phi_l \) - the face bubble function corresponding to face \( l \).
- \( \tilde{K} = \text{int}\{\cup \bar{K}_0 : \bar{K}_0 \subset \mathcal{T}_h, \bar{K}_0 \cap \bar{K} \neq \emptyset\} \), which is also called the patch of element \( K \).
- \( K_j \) - a neighboring element of \( K \) with the common face \( l_j \).
- \( \bar{R}_1, \bar{R}_2, \bar{R}_1 \) and \( \bar{R}_2 \) denote the finite element approximation of the corresponding residuals. On an interelement face \( l_j \), we use the traces of the finite element functions defined in \( K \) and \( K_j \), respectively. This gives a natural extension of \( \bar{R}_1 \) and \( \bar{R}_2 \) to the adjacent elements \( K \) and \( K_j \), which will be denoted also with \( \bar{R}_1 \) and \( \bar{R}_2 \), respectively.

In the consecutive estimates we will use the following inequalities.

**Lemma 1.** Let \( K \in \mathcal{T}_h \) and \( h_K = \text{diam} \ K \). Then there exist positive constants \( C, \bar{C} \), which depend only on the shape regularity of element \( K \), such that the following inequalities are valid:

\[
\begin{align*}
\| \bar{r}_2 \|_{L^2(K)}^2 & \leq C \| \bar{r}_2, \Psi_K \bar{r}_2 \|_K, \\
\| \Psi_K \bar{r}_2 \|_{L^2(K)} & \leq C \| \bar{r}_2 \|_{L^2(K)}, \\
\| \bar{R}_2 \|_{L^2(l)}^2 & \leq C \| \bar{R}_2, \Phi_l \bar{R}_2 \|_l, \\
\| \Phi_l \bar{R}_2 \|_{L^2(K)} & \leq \bar{C} h_K^{-\frac{1}{2}} \| \bar{r}_2 \|_{L^2(K)}, \\
\| \bar{R}_2 \|_{L^2(l)} & \leq \bar{C} \| \bar{R}_2 \|_{L^2(l)}, \\
\| \nabla (\bar{r}_2) \|_{L^2(K)}^2 & \leq \bar{C} h_K^{-1} \| \bar{r}_2 \|_{L^2(K)}, \\
\| \nabla (\bar{R}_2) \|_{L^2(l)}^2 & \leq \bar{C} h_K^{-\frac{2}{3}} \| \bar{R}_2 \|_{L^2(l)}.
\end{align*}
\]

**Proof** The proof can be carried out using scaling arguments and the fact that the finite element spaces are finite dimensional. For an overview on the bubble function technique and the corresponding estimates we refer to [12] and [1]. □

In the next lemma we point out how the bilinear form (8) can be simplified for some special functions \( v \).

**Lemma 2.** For any \( w \in H^1(K) \) with \( \text{supp} \ w \subset K \), (8) simplifies into

\[
B(e_h, \nabla w) = -(r_2, w)_K.
\]

Similarly, for any \( w \in H^1(\tilde{K}) \) with \( \text{supp} \ w \subset K \cup K_j \) (8) simplifies into

\[
B(e_h, \nabla w) = -(r_2, w)_{K \cup K_j} + (R_2, w)_{l_j},
\]

where \( l_j = \tilde{K} \cap K_j \neq \emptyset \) for all \( K, K_j \in \mathcal{T}_h \).

**Proof** We use (8) with \( v = \nabla w \). Since \( w \in H^1_0(K) \) we have \( z = 0 \) in the decomposition (4) and therefore

\[
B(e_h, \nabla w) = -(r_2, w)_K.
\]
as stated in the lemma. Similarly, if \( \text{supp} \, w \subset K \cup K_j \) then \( w|_{l_i} = 0 \) for any \( i \neq j \) and we obtain (22). \( \square \)

We can now prove the reliability of the error indicator \( \eta_h \). We use the standard bubble function technique, see [1]. A similar proof has been carried out in [3] for curl-elliptic Maxwell equations with a divergence free source term \( J \).

The bilinear form restricted to the element \( K \) is denoted with \( B_K \), and we frequently use the continuity estimate

\[
\|B_K(u, v)\| \leq \sqrt{2}(1 + k^2)\|u\|_{\text{curl}, K} \|v\|_{\text{curl}, K} \quad \forall \ u, v \in H(\text{curl}, K).
\]

In the sequel, the overbar denotes the finite element approximation of the appropriate error indicators. In the estimates, \( C \) denotes different constants, which are all independent of the element size \( h \) and the wave number \( k \).

**Theorem 2.** The error indicator \( \eta_K \) provides a local lower bound of the real error up to some remainders

\[
\eta^2_K \leq C \left( (1 + k^2)^2\|e_h\|_{\text{curl}, K}^2 + h^2(\|\hat{r}_1 - r_1\|_{L^2(K)}^2 + \|\hat{r}_2 - r_2\|_{L^2(K)}^2) + h(\|\hat{R}_1 - R_1\|_{L^2(\partial K)}^2 + \|\hat{R}_2 - R_2\|_{L^2(K)}^2) \right),
\]

where \( h \) denotes the mesh size and \( C \) is a generic constant which does not depend on \( h \) and \( k \).

**Proof** The terms in (9) will be estimated separately. The estimate (58) in [7] gives for the first component

\[
\|r_1\|_{L^2(K)}^2 \leq C(\|r_1\|_{L^2(K)} + (1 + k^2)h^{-1}\|e_h\|_{\text{curl}, K}).
\]

Similarly, for the third component estimate (64) in [7] provides

\[
\|R_1\|_{L^2(K)}^2 \leq C(h^{-1}(1 + k^2)^2\|e_h\|_{\text{curl}, K}^2 + h\|\hat{R}_1 - R_1\|_{L^2(K)}^2 + \|\hat{R}_1 - R_1\|_{L^2(\partial K)}^2).
\]

For the estimation of the second term in (9), we use the following inequality:

\[
\|\hat{r}_2\|_{L^2(K)} \leq C(\hat{r}_2, \psi_K \hat{r}_2) = C((r_2 - r_2, \psi_K \hat{r}_2) + (r_2, \psi_K \hat{r}_2))
\]

\[
\leq C(\|\psi_K \hat{r}_2\|_{L^2(K)}\|r_2\|_{L^2(K)} - B(e_h, \nabla(\psi_K \hat{r}_2)))
\]

\[
\leq C(\|\psi_K \hat{r}_2\|_{L^2(K)}\|r_2\|_{L^2(K)} + (1 + k^2)\|e_h\|_{\text{curl}, K}\|\nabla(\psi_K \hat{r}_2)\|_{L^2(K)}^2)
\]

\[
\leq C(\|\hat{r}_2 - r_2\|_{L^2(K)} + (1 + k^2)h^{-1}\|e_h\|_{\text{curl}, K}\|\hat{r}_2\|_{L^2(K)}),
\]

where in the first line (14) and the triangle inequality, in the second line (21), in the fourth line the continuity estimate (23), and in the fifth line (15) and (19) have been used. Dividing by \( \|\hat{r}_2\|_{L^2(K)} \), and using the triangle inequality \( \|r_2\|_{L^2(K)} \leq \|\hat{r}_2\|_{L^2(K)} + \|r_2 - \hat{r}_2\|_{L^2(K)} \) we obtain that

\[
\|r_2\|_{L^2(K)} \leq C(\|\hat{r}_2 - r_2\|_{L^2(K)} + (1 + k^2)h^{-1}\|e_h\|_{\text{curl}, K}).
\]
For the estimation of the fourth term we use the following inequality:
\begin{equation}
\| \tilde{R}_2 \|_{L_2(l)}^2 \leq C(\Phi_1 \tilde{R}_2, \tilde{R}_2)_l = C(\Phi_1 \tilde{R}_2, \tilde{R}_2 - R_2)_l + C(\Phi_1 \tilde{R}_2, R_2)_l \\
= C((\Phi_1 \tilde{R}_2, \tilde{R}_2 - R_2)_l + B_K(\mathbf{e}_h, \nabla (\Phi_1 \tilde{R}_2)) + (r_2, \Phi_1 \tilde{R}_2)_K) \\
\leq C(||\Phi_1 \tilde{R}_2||_{L_2(l)} \| \tilde{R}_2 - R_2 \|_{L_2(l)}) \\
+ (1 + k^2)\| \mathbf{e}_h \|_{\text{curl}, K} \| \nabla (\Phi_1 \tilde{R}_2) \|_{L_2(K)}^3 + ||\Phi_1 \tilde{R}_2||_{L_2(K)} \| r_2 \|_{L_2(K)} \\
\leq C(||\tilde{R}_2||_{L_2(l)} \| \tilde{R}_2 - R_2 \|_{L_2(l)}) \\
+ h^{-\frac{1}{2}}(1 + k^2)\| \mathbf{e}_h \|_{\text{curl}, K} \| \tilde{R}_2 \|_{L_2(l)} + h^\frac{1}{2} \| \tilde{R}_2 \|_{L_2(l)} \| r_2 \|_{L_2(K)},
\end{equation}
where in the first line (16), in the second line (22), in the fourth line (23), and in the sixth line (20) and (17) have been used. Dividing both sides by \( ||\tilde{R}_2||_{L_2(l)} \) and using the triangle inequality \( ||\tilde{R}_2||_{L_2(l)} \leq ||\tilde{R}_2 - R_2||_{L_2(l)} + ||\tilde{R}_2||_{L_2(l)} \) gives that
\begin{equation}
||\tilde{R}_2||_{L_2(l)} \leq C(||\tilde{R}_2 - R_2||_{L_2(l)}) \\
+ h^{-\frac{1}{2}}(1 + k^2)\| \mathbf{e}_h \|_{\text{curl}, K} + h^\frac{1}{2} \| r_2 \|_{L_2(K)},
\end{equation}
which can be further estimated using (28) and we obtain
\begin{equation}
||\tilde{R}_2||_{L_2(l)} \leq C(||\tilde{R}_2 - R_2||_{L_2(l)}) \\
+ h^{-\frac{1}{2}}(1 + k^2)\| \mathbf{e}_h \|_{\text{curl}, K} + h^\frac{1}{2} \| r_2 \|_{L_2(K)}.
\end{equation}
Taking the square of (25), (26), (28) and (30) and summing the last two contributions over the faces of the element we obtain
\begin{equation}
\frac{1}{C} \eta_K^2 = h^2 \| r_1 \|_{L_2(K)}^3 + h^2 \| r_2 \|_{L_2(K)}^3 + h \| \mathbf{R}_1 \|_{L_2(\partial K)}^2 + h \| R_2 \|_{L_2(\partial K)}^2 \\
= (1 + k^2)^2 \| \mathbf{e}_h \|_{\text{curl}, K}^2 + k^2 (||r_1 - \tilde{r}_1||_{L_2(K)}^3 + ||r_2 - \tilde{r}_2||_{L_2(K)}^3) \\
+ \sum_{l_j \subset \partial K} h (||\mathbf{R}_1 - \tilde{R}_1||_{L_2(l_j)}^2 + ||R_2 - \tilde{R}_2||_{L_2(l_j)}^2)
\end{equation}
as stated in the theorem. \( \square \)

**Remark:** According to the definition of the residuals in (7) the residual terms in (24) can be rewritten as
\[ r_1 - r_1|_K = \mathbf{J} - \mathbf{J}|_K \quad \text{and} \quad \tilde{r}_2 - r_2|_K = \text{div}\mathbf{J} - \text{div}\mathbf{J}|_K. \]

### 3.3. The reliability of the error indicator

Following the method in [11] we prove the reliability of the global error indicator but now for a current density \( \mathbf{J} \) which is not assumed to be divergence free. For this purpose we first generalize the decomposition result in Lemma 2.2 in [9]. The main extension concerns source terms with non-zero divergence terms. We adopt the notations and the second half of the proof in [9], but restrict for brevity the analysis to simply connected Lipschitz domains. We also drop the subscript \( \Omega \) in the norms.

**Lemma 3.** For any simply connected Lipschitz domain \( \Omega \) a vector field \( \mathbf{v} \in H_0(\text{curl}, \Omega) \cap H(\text{div}, \Omega) \) can be decomposed as
\[ \mathbf{v} = \mathbf{z} + \nabla \Phi \]
such that \( \Phi \in H_0^1(\Omega) \) and \( \mathbf{z} \in [\nabla H_0^1(\Omega)]^1 \) and the following estimates hold:
\begin{equation}
||\mathbf{z}|| + ||\Phi||_1 \leq C ||\mathbf{v}|| \quad \text{and} \quad ||\mathbf{z}||_1 \leq C ||\text{curl} \mathbf{v}||.
\end{equation}
Proof For the estimation we define

\[ \mathbf{v}_0 := \mathbf{v} - \nabla \phi, \]

where \( \phi \) is the solution of the boundary value problem

\[ \Delta \phi = \nabla \cdot \mathbf{v} \quad \text{in } \Omega \]
\[ \phi = 0 \quad \text{on } \partial \Omega. \]

The simple equality

\[ (\nabla \phi, \nabla \phi) = -(\Delta \phi, \phi) = -(\nabla \cdot \mathbf{v}, \phi) = (\mathbf{v}, \nabla \phi) \]

implies that

\[ \| \nabla \phi \| \leq \| \mathbf{v} \| \]

and therefore,

\[ \| \mathbf{v}_0 \| = \| \mathbf{v} - \nabla \phi \| \leq \| \mathbf{v} \| + \| \nabla \phi \| \leq 2 \| \mathbf{v} \|. \tag{33} \]

We also have that

\[ \nabla \cdot \mathbf{v}_0 = \nabla \cdot (\mathbf{v} - \nabla \phi) = \nabla \cdot \mathbf{v} - \Delta \phi = 0, \tag{34} \]

therefore by Corollary 3.19 in [2] we obtain that

\[ \| \mathbf{v}_0 \|_{\text{curl}} \leq C \| \text{curl } \mathbf{v}_0 \|. \tag{35} \]

From this point we adopt the proof of Lemma 2.2 in [9]. Since \( \mathbf{v}_0 \in H_0(\text{curl}, \Omega) \), its extension \( \tilde{\mathbf{v}} \) by zero to an open ball \( B(0, r) \supset \Omega \) will be in \( H_0(\text{curl}, B(0, r)) \).

Using Lemma 2.1 in [9], there exists \( \tilde{\mathbf{w}} \in [H^1(B(0, r))]^3 \) such that

\[ \text{curl } \tilde{\mathbf{w}} = \text{curl } \tilde{\mathbf{v}} \quad \text{and} \quad \text{div } \tilde{\mathbf{w}} = 0, \tag{36} \]

\[ \| \tilde{\mathbf{w}} \|_{B(0, r)} \leq \| \mathbf{v}_0 \| \quad \text{and} \quad \| \tilde{\mathbf{w}} \|_{1, B(0, r)} \leq \sqrt{2} \| \mathbf{v}_0 \|_{\text{curl}, \Omega} \leq C \| \text{curl } \mathbf{v}_0 \| \tag{37} \]

are valid, where in the last estimate we used (35). The first equality in (36) gives that \( \tilde{\mathbf{w}} - \tilde{\mathbf{v}} = \nabla \tilde{\Psi} \) for some \( \tilde{\Psi} \in H^1_0(B(0, r)) \) such that together with the Poincaré inequality we obtain

\[ \| \tilde{\Psi} \|_{1, B(0, r)} \leq C \| \tilde{\mathbf{w}} - \tilde{\mathbf{v}} \|_{B(0, r)}. \tag{38} \]

Since \( \nabla \tilde{\Psi} = \tilde{\mathbf{w}} \) on \( \Omega^c = B(0, r) \setminus \Omega \) and \( \tilde{\mathbf{w}} \in [H^1(B(0, r))]^3 \) we obtain that \( \tilde{\Psi} |_{\Omega^c} \in H^2(\Omega^c) \), which has an extension \( \Psi \) on \( B(0, r) \) such that with respect to (38) we have

\[ \| \Psi \|_{1, B(0, r)} \leq C \| \tilde{\Psi} \|_{1, B(0, r)} \leq C \| \tilde{\mathbf{w}} - \tilde{\mathbf{v}} \|_{B(0, r)}. \tag{39} \]

Using the equality \( \nabla \tilde{\Psi} = \tilde{\mathbf{w}} \) on \( \Omega^c \) again we also have

\[ \| \Psi \|_{2, B(0, r)} \leq C \| \tilde{\Psi} \|_{2, \Omega^c} \leq \| \tilde{\Psi} \|_{1, B(0, r)} + \| \tilde{\mathbf{w}} \|_{1, \Omega^c}. \tag{40} \]

We define then

\[ \mathbf{z} := (\tilde{\mathbf{w}} - \nabla \tilde{\Psi}) |_{\Omega} \quad \text{and} \quad \Phi := (\tilde{\Psi} - \Psi) |_{\Omega}. \]

Then using the definition of \( \mathbf{z} \), (39), (37) and (33) we obtain

\[ \| \mathbf{z} \| + \| \Phi \|_1 \leq C (\| \tilde{\mathbf{w}} \| + \| \nabla \tilde{\Psi} \| + \| \nabla \tilde{\Psi} \| + \| \nabla \Psi \|) \leq C (\| \tilde{\mathbf{w}} \| + 3 \| \nabla \tilde{\Psi} \|) \leq \| \tilde{\mathbf{w}} \| + \| \tilde{\mathbf{w}} - \tilde{\mathbf{v}} \|_{B(0, r)} \leq 3C \| \mathbf{v}_0 \| \leq 6C \| \mathbf{v} \|, \]

where we have used that \( \tilde{\mathbf{v}} \) is the extension of \( \mathbf{v}_0 \) by zero.
Similarly, the definition of $\phi_i$ (40), (39) (37) and 35 give that

$$
\|z\|_1 \leq \|\tilde{w}\|_1 + \|\nabla \Psi\|_1 \leq C \|\text{curl } v_0\| + \|\tilde{w}\|_{2,B(0,r)}
$$

$$
\leq C(\|\text{curl } v_0\| + \|\tilde{w}\|_{1,B(0,r)} + \|\tilde{w}\|_{1,\Omega})
$$

$$
\leq C(\|\text{curl } v_0\| + \|\tilde{w} - \tilde{v}\|_{B(0,r)} + \|\tilde{w}\|_{1,\Omega})
$$

$$
\leq C(\|\text{curl } v_0\| + \|\tilde{w}\|_{B(0,r)} + \|v_0\| + \|\text{curl } v_0\|) \leq C(\|v_0\| + \|\text{curl } v_0\|)
$$

$$
\leq C(\|v_0\|^2 + \|\text{curl } v_0\|^2)^{\frac{1}{2}} \leq C\|\text{curl } v_0\| = C\|\text{curl } v\|
$$

which proves (32). □

Using Lemma 3 we can prove an approximation formula, which implies the efficiency of $\eta_K$.

**Lemma 4.** There exists a quasi interpolation operator $\Pi_h : H_0(\text{curl}, \Omega) \cap H(\text{div}, \Omega) \rightarrow N_{0,h}$, such that $\Phi_h \in H_0^1(\Omega)$ and $z_h \in [\nabla H_0^1(\Omega)]^\perp$ and the following decomposition holds

$$
\nu - \Pi_h \nu = z_h + \nabla \Phi_h,
$$

where

$$
\frac{1}{h_K} \|\Phi_h\|_{L_2(K)} + \|\nabla \Phi_h\|_{[L_2(K)]^3} \leq C\|\nu\|_K
$$

and

$$
\frac{1}{h_K} \|z_h\|_{[L_2(K)]^3} + \|\nabla z_h\|_{[L_2(K)]^3} \leq C\|\text{curl } \nu\|_{[L_2(K)]^3}.
$$

Remarks:

(1) Theorem 1 in [11] seems to be more general, but indeed, it is valid only for divergence free functions $v$ since Lemma 2.2 in [9] has been used in its proof.

(2) The superscript $h$ yields the $h$ dependence of the components, but they are in general not in $H_{0,h}(\text{curl}, \Omega)$.

**Proof** For the proof we refer to [11]. Summarized, the decomposition techniques in Lemma 7 and Lemma 10 in [11] should be applied, which are valid for any function in the $H(\text{curl}, \Omega)$ space. Along with these, Lemma 3, which is valid in $H_0(\text{curl}, \Omega) \cap H(\text{div}, \Omega)$ should be used with the proper scalings, and we obtain (41) and (42). □

We use also an inequality for the trace $v|_{l}$ of a function $v \in H^1(\Omega)$ stated in the following

**Lemma 5.** For any non-degenerate family of meshes the following trace inequality is valid:

$$
\|v\|_{L_2(l)} \leq C \frac{1}{h_K} \|v\|_{L_2(K)}^2 + h_K \|\nabla v\|_{L_2(K)}^2
$$

where $C$ is independent on the subdomain $K$.

For a simple proof we refer to Appendix B.

Obviously (43) can be rewritten as

$$
\|v\|_{L_2(l)} \leq C h_K^{\frac{1}{2}} \left( \frac{1}{h_K^2} \|v\|_{L_2(K)}^2 + \|\nabla v\|_{L_2(K)}^2 \right)^{\frac{1}{2}}
$$

which will be used subsequently. To keep the notation simple we have used $v$ also for its trace.
Theorem 3. For any non-degenerate family of meshes $\mathcal{T}_h$, the error indicator $\eta_{\mathcal{T}_h}$ is reliable:

\begin{equation}
\|e_h\|_{\text{curl}} \leq C_{\text{rel}}\eta_{\mathcal{T}_h}.
\end{equation}

Proof. We only have to slightly modify the proof in [11] such that we can incorporate the source term $I$ with nonvanishing trace. For the completeness, we give the whole proof. Using the Galerkin orthogonality relation and the inf-sup property of the bilinear form $B$ (see [6], (5.9)), we obtain for some $v \in H_0(\text{curl}, \Omega)$ the inequality

\begin{equation}
\|e_h\|_{\text{curl}}\|v\|_{\text{curl}} \leq B(e_h, v) = B(e_h, v - \Pi_h v) = \sum_{K \in \mathcal{T}_h} B_K(e_h, v - \Pi_h v),
\end{equation}

where $\Pi_h v \in \mathcal{N}_{h,h}$ is an arbitrary element. The spirit of the proof is that $\Pi_h v$ is not necessarily an interpolation of $v$. Using (8), Lemma 4, the Cauchy-Schwarz inequality, (44), (41) and (42) we can rewrite (46) as

\[
\|e_h\|_{\text{curl}}\|v\|_{\text{curl}} \leq \sum_{K \in \mathcal{T}_h} B_K(e_h, v - \Pi_h v)
\]

\[
= \sum_{K \in \mathcal{T}_h} (r_1, z_h)_K - (r_2, \Phi_h)_K + \sum_{l \in \Gamma_h} (R_1, z_h)_l + (R_2, \Phi_h)_l
\]

\[
\leq \sum_{K \in \mathcal{T}_h} \|r_1\|_{L^2(K)}\|z_h\|_{L^2(K)} + \|r_2\|_{L^2(K)}\|\Phi_h\|_{L^2(K)}
\]

\[
+ \sum_{l \in \Gamma_h} \|R_1\|_{L^2(l)}\|z_h\|_{L^2(l)} + \|R_2\|_{L^2(l)}\|\Phi_h\|_{L^2(l)}
\]

\[
\leq \sum_{K \in \mathcal{T}_h} h_K\|r_1\|_{L^2(K)}^2 \left( \frac{1}{h_K} \|z_h\|_{L^2(K)}^2 + \frac{1}{h_K} \|r_2\|_{L^2(K)}^2 \Phi_h\|_{L^2(K)}^2 \right)
\]

\[
+ \sum_{l \in \Gamma_h} h_{l/2}^2\|R_1\|_{L^2(l)}^2 \left( \frac{1}{h_{l/2}} \|z_h\|_{L^2(l)}^2 + \|\nabla z_h\|_{L^2(l)}^2 \right)^{1/2}
\]

\[
+ h_{l/2}^2\|R_2\|_{L^2(l)} \left( \frac{1}{h_{l/2}} \|\Phi_h\|_{L^2(l)}^2 + \|\nabla \Phi_h\|_{L^2(l)}^2 \right)^{1/2}
\]

\[
\leq \sum_{K \in \mathcal{T}_h} \left( \frac{1}{h_K^2} \|z_h\|_{L^2(K)}^2 + \|\nabla z_h\|_{L^2(K)}^2 \right)^{1/2}
\]

\[
\cdot \left( \sum_{l \in \Gamma_h} h_{l/2}^2\|R_1\|_{L^2(l)}^2 + \sum_{K \in \mathcal{T}_h} h_K\|r_1\|_{L^2(K)}^2 \right)
\]

\[
+ \sum_{l \in \Gamma_h} h_{l/2}^2\|\Phi_h\|_{L^2(K)}^2 + \|\nabla \Phi_h\|_{L^2(K)}^2 \right)^{1/2}
\]

\[
\leq C \sum_{K \in \mathcal{T}_h} \|\text{curl} v\|_{L^2(K)}^2 \left( \sum_{l \in \Gamma_h} h_{l/2}^2\|R_1\|_{L^2(l)}^2 + \sum_{K \in \mathcal{T}_h} h_K\|r_1\|_{L^2(K)}^2 \right)
\]

\[
+ \sum_{K \in \mathcal{T}_h} \|v\|_{L^2(K)}^2 \left( \sum_{l \in \Gamma_h} h_{l/2}^2\|R_2\|_{L^2(l)}^2 + \sum_{K \in \mathcal{T}_h} h_K\|r_2\|_{L^2(K)}^2 \right)
\]

\[
\leq C \sum_{K \in \mathcal{T}_h} \left( \|\text{curl} v\|_{L^2(K)}^2 + \|v\|_{L^2(K)}^2 \right)^{1/2}
\]
\[
\left( \sum_{l \in T_h} h_K \left( \| R_1^l \|^2_{L^2(\Omega)} + \| R_2^l \|^2_{L^2(\Omega)} \right) + \sum_{K \in T_h} h_K^2 \left( \| R_1 \|^2_{H^1(K)} + \| R_2 \|^2_{H^1(K)} \right) \right)^{1/2}.
\]

Dividing both sides by \( \| \nu \|_{\text{curl}} \) gives the statement of the theorem. \( \square \)

4. Implicit a posteriori error estimation

In this section, we will provide an implicit error estimator which is equivalent with the residual based error estimator \( \eta_K \). The implicit error estimate will be defined as the solution of a local boundary value problem for the exact error, where the unknown boundary conditions are obtained by an approximation using the computational data. This will be a Neumann type problem for the time harmonic Maxwell equations, which has been analyzed in [7]. At first sight, this approach may seem to be heuristic, but it turns out that the implicit error estimate \( \hat{\eta}_K \) also solves the localization of the variational problem (8). This interpretation makes possible the comparison of \( \hat{\eta}_h \) with the explicit residual based error indicator \( \eta_K \) such that using the results of the preceding sections we obtain the desired efficiency and reliability property of \( \hat{\eta}_h \).

Using the Helmholtz-decomposition (4) and the Green formula we can rewrite the bilinear form for the error on an element \( K \)

\[
B_K(\mathbf{e}_h, \mathbf{v}) = (\text{curl } \mathbf{E}, \text{curl } \mathbf{z})_K - k^2(\mathbf{E}, \mathbf{z} + \nabla \Phi)_K - B_K(\mathbf{E}_h, \mathbf{v})
\]

\[
= (\text{curl } \mathbf{curl } \mathbf{E}, \mathbf{z})_K - (\nu \times \text{curl } \mathbf{E}, \nu \times \mathbf{z})_{\partial K} - k^2(\mathbf{E}, \mathbf{z})_K
\]

\[
+ k^2(\text{div } \mathbf{E}, \Phi)_K - k^2(\nu \cdot \mathbf{E}, \Phi)_{\partial K} - B_K(\mathbf{E}_h, \mathbf{v})
\]

\[
= (\mathbf{J}, \mathbf{z})_K - k^2(\text{div } \mathbf{J}, \Phi)_K - (\nu \times \text{curl } \mathbf{E}, \mathbf{z})_{\partial K}
\]

\[
- k^2(\nu \cdot \mathbf{E}, \Phi)_{\partial K} - B_K(\mathbf{E}_h, \mathbf{v}),
\]

which should be solved numerically. However, on the right hand side the traces \( \nu \times \text{curl } \mathbf{E} \) and \( \nu \cdot \mathbf{E} \) are unknown such that for a well-defined error equation we have to use some estimates for these terms:

\[
\nu \times \text{curl } \mathbf{E}_{|_{t_j}} \approx \{ \nu \times \text{curl } \mathbf{E} \}_{t_j} := \frac{1}{2}(\nu_j \times \text{curl } \mathbf{E}_{h,K} + \nu_j \times \text{curl } \mathbf{E}_{h,K_j})
\]

\[
\nu \cdot \mathbf{E}_{|_{t_j}} \approx \{ \nu \cdot \mathbf{E} \}_{t_j} := \frac{1}{2}(\nu_j \cdot \mathbf{E}_{h,K} + \nu_j \cdot \mathbf{E}_{h,K_j}).
\]

Using the above averages, we define the implicit a posteriori error estimations as the solution \( \hat{\mathbf{e}}_h \) of the following variational equation:

Find \( \hat{\mathbf{e}}_h \in V_{h,K} \) such that for all \( \mathbf{z}_h + \nabla \Phi_h = \mathbf{v}_h \in V_{h,K} \) the following equality holds

\[
B_K(\hat{\mathbf{e}}_h, \mathbf{v}_h) = (\mathbf{J}, \mathbf{z}_h)_K - (\text{div } \mathbf{J}, \Phi_h)_K - B_K(\mathbf{E}_h, \mathbf{v}_h)
\]

\[
- \sum_{t_j \subset \partial K} \left( \{ \nu \times \text{curl } \mathbf{E} \}_{t_j}, \mathbf{z}_h \}_{t_j} + k^2(\{ \nu \cdot \mathbf{E} \}_{t_j}, \Phi_h)_{t_j} \right),
\]

where \( V_{h,K} \) is a suitably chosen finite element space on \( K \). Applying the Green formula also to \( B_K(\mathbf{E}_h, \mathbf{v}_h) \) in (50) this relation can be rewritten as (cf. also with...
This has a crucial influence on the quality of the error estimate. It is advised (see [1]) that it has to be different from the original finite element space. On the other hand, the discrete inf-sup condition must be satisfied for the space \( V_{h,K} \) which states:

\[
\mathbf{\Phi}_{K,h} = (\mathbf{J}, \mathbf{z}_h)_K - (\text{div} \mathbf{J}, \Phi_h)_K - (\text{curl} \mathbf{E}_h - \mathbf{k}^2 \mathbf{E}_h, \mathbf{z}_h)_K
\]

\[
- k^2 (\text{div} \mathbf{E}_h, \nabla \Phi_h)_K
\]

\[
= - \sum_{l_j \subset K} \left( \left( \{ \nu \times \text{curl} \mathbf{E} \} |_{l_j}, \mathbf{z}_h \right)_{l_j} + k^2 \left( \{ \nu \cdot \mathbf{E} \} |_{l_j}, \Phi_h \right)_{l_j}
\]

\[
+ (\nu \times \text{curl} \mathbf{E}_h, \mathbf{z}_h)_{l_j} + k^2 (\nu \cdot \mathbf{E}_h, \Phi_h)_{\partial K}
\]

\[
= (r_1, \mathbf{z}_h)_K - (r_2, \Phi_h)_K + \frac{1}{2} (R_1, \mathbf{z}_h)_{l_j} + \frac{1}{2} (R_2, \Phi_h)_{\partial K}.
\]

Special choices of \( \mathbf{v}_h \) in (51) deliver formulas which will be useful in the subsequent analysis:

**Corollary 1.** For any \( w \in H^1(K) \) with \( \text{supp} \ w \subset K \) (8) simplifies into

\[
B_K(\mathbf{\Phi}_h, \nabla w) = -(r_2, w)_K.
\]

Similarly, for any \( w \in H^1(K) \) with \( \text{supp} \ w \subset K \cap K_j \) (here \( l_j = K \cap K_j \neq \emptyset \) for all \( K, K_j \in T_h \)) (8) simplifies into

\[
B_K(\mathbf{\Phi}_h, \nabla w) = -(r_2, w)_{K \cap K_j} + (R_2, w)_{l_j}.
\]

**Proof** The proof is an easy modification of Lemma 2. □

First, we establish that the implicit error estimate is a lower bound of \( \eta_K \). For the proof we have to use the following estimates, where different norms of finite element functions are compared. For this we consider a finite element space \( V_h \) on a reference element \( \hat{K} \) and use a non-degenerate family of meshes \( T_h \) such that each \( K \) in any mesh can be obtained with an affine mapping \( B_K : \hat{K} \rightarrow K \). The corresponding finite element space on \( K \) is denoted by \( V_{K,h} \) and we use the notation

\[
\Phi_{K,h} = \{ \phi_h \in L^2(K) : \nabla \phi_h \in V_{K,h} \}.
\]

**Lemma 6.** For any subdomain \( K \) with the mesh parameter \( h \) and any \( \mathbf{v}_h \in V_{K,h}, \phi_h \in \Phi_{K,h} \) we have

\[
\| \mathbf{v}_h \|_{L^2(K)^3} \leq C h \| \text{curl} \mathbf{v}_h \|_{L^2(K)^3}
\]

\[
\| \phi_h \|_{L^2(K)} \leq C h \| \nabla \phi_h \|_{L^2(K)}
\]

\[
\| \mathbf{v}_h \|_{L^2(\partial K)^3} \leq C h^{\frac{3}{2}} \| \text{curl} \mathbf{v}_h \|_{L^2(K)^3}
\]

\[
\| \phi_h \|_{L^2(\partial K)} \leq C h^{\frac{1}{2}} \| \nabla \phi_h \|_{L^2(K)}.
\]

where the constant \( C \) does not depend on the mesh size \( h \).

**Proof** One has to use the non-degenerate properties of the family of meshes and standard scaling arguments. □

Before comparing the implicit error estimator \( \mathbf{\Phi}_h \) with the explicit estimator \( \eta_K \) we have to fix the finite element space \( V_{h,K} \), which is used for the solution of (50). This has a crucial influence on the quality of the error estimate. It is advised (see [1]) that it has to be different from the original finite element space. On the other hand, the discrete inf-sup condition must be satisfied for the space \( V_{h,K} \) which states:
There exists a positive constant $C$, which is independent of $h$ such that for all $K$ and $w_h \in V_{h,K}$ we have

$$\|w_h\|_{\text{curl},K} \leq C \sup_{v_h \in V_{h,K}} \frac{B_K(w_h,v_h)}{\|v_h\|_{\text{curl},K}}. \tag{58}$$

This is a powerful tool in the analysis of the finite element discretization and is not automatically satisfied in every scale of finite dimensional spaces $V_{h,K}$. Even the proof for standard Nédélec spaces is quite involved (see [6], (5.10)). Both in case of rectangular and tetrahedral tessellations we developed spaces $V_{h,K}$ which satisfy an inf-sup condition, see [7] and [5], and serve as a concrete example in the subsequent analysis.

**Lemma 7.** Assume that the finite element spaces $V_{h,K}, K \in \mathcal{T}_h$ satisfy the discrete inf-sup condition (58). Then the implicit error estimate $\tilde{e}_h$ gives a lower bound for the error indicator $\eta_K$:

$$\|\tilde{e}_h\|_{\text{curl},K} \leq C\eta_K.$$

**Proof** In the proof we use the decomposition $V_{K,h} \ni v_h = z_h + \nabla \phi_h$, see Lemma 4, and the fact that a discrete inf-sup condition (58) is satisfied in $V_{h,K}$. According to (51) and the estimates (54)-(57) we have

$$\|\tilde{e}_h\|_{\text{curl},K} \leq C \sup_{v_h \in V_{h,K}} \frac{B_K(\tilde{e}_h,v_h)}{\|v_h\|_{\text{curl},K}}$$

$$= C \sup_{v_h \in V_{h,K}} \frac{1}{\|v_h\|_{\text{curl},K}} \left(\|\mathbf{r}_1, z_h\|_K - \|\mathbf{r}_2, \phi_h\|_K + \|\mathbf{R}_1, z_h\|_{\partial K} + \|\mathbf{R}_2, \phi_h\|_{\partial K}\right). \tag{59}$$

$$\leq C \sup_{v_h \in V_{h,K}} \frac{1}{\|v_h\|_{\text{curl},K}} \left(\|\mathbf{r}_1, z_h\|_K \|L_2(K)\|^3 \|\nabla z_h\|_{L_2(K)}^3 + \|\mathbf{r}_2, \phi_h\|_K \|L_2(K)\| \|\phi_h\|_{L_2(K)} \right)

+ \|\mathbf{R}_1, z_h\|_{L_2(\partial K)}^3 \|\nabla z_h\|_{L_2(\partial K)}^3 + \|\mathbf{R}_2, \phi_h\|_{L_2(\partial K)} \|\phi_h\|_{L_2(\partial K)} \right)

\leq C \sup_{v_h \in V_{h,K}} \frac{1}{\|v_h\|_{\text{curl},K}} \left(\|\mathbf{r}_1, z_h\|_K \|L_2(K)\|^3 \|\nabla z_h\|_{L_2(K)}^3 + \|\mathbf{r}_2, \phi_h\|_K \|L_2(K)\| \|\phi_h\|_{L_2(K)} \right)

+ \|\mathbf{R}_1, \phi_h\|_{L_2(\partial K)} \|\nabla \phi_h\|_{L_2(\partial K)} \|L_2(\partial K)\|^3)

\leq C \sup_{v_h \in V_{h,K}} \frac{1}{\|v_h\|_{\text{curl},K}} \left(\|\mathbf{r}_1, z_h\|_K \|L_2(K)\|^3 \|\nabla z_h\|_{L_2(K)}^3 + \|\mathbf{R}_1, \phi_h\|_{L_2(\partial K)} \|\nabla \phi_h\|_{L_2(\partial K)} \|L_2(\partial K)\|^3 \right)$$

which proves the lemma. □

Following the proof of Theorem 2 and using Corollary 1 one can prove that $\tilde{e}_h$ gives an upper estimate of the error indicator $\eta_K$.

**Lemma 8.** The implicit error estimate $\tilde{e}_h$ gives an upper bound for the error indicator $\eta_K$:

$$\eta_K \leq C\|\tilde{e}_h\|_{\text{curl},K}.$$

**Proof** We estimate separately the terms in $\eta_K^2$. Again we estimate one of the boundary terms and one of the internal residuals, the remaining terms can be
estimated in the same way. The second term in (51) will be estimated as follows:

\[
\|\tilde{r}_2\|_{L^2(K)} \leq C(\psi_2, \psi_K \tilde{r}_2) = C((\tilde{r}_2 - r_2, \psi_K \tilde{r}_2)_K + (r_2, \psi_K \tilde{r}_2)_K)
\]
\[
\leq C(\|\psi_K \tilde{r}_2\|_{L^2(K)}\|\tilde{r}_2 - r_2\|_{L^2(K)} - B_K(\partial_h, \nabla \psi_K \tilde{r}_2))
\]
\[
\leq C(\|\psi_K \tilde{r}_2\|_{L^2(K)}\|\tilde{r}_2 - r_2\|_{L^2(K)}
\]
\[
+ (1 + k^2)\|\partial_h\|_{\text{curl}K} \|\nabla \psi_K \tilde{r}_2\|_{[L^2(K)]^3})
\]
\[
\leq C(\|\tilde{r}_2 - r_2\|_{L^2(K)} + (1 + k^2)h^{-1}\|\partial_h\|_{\text{curl}K})\|\tilde{r}_2\|_{L^2(K)},
\]

where in the first line (14) and the triangle inequality, in the second line (52), in the third line the continuity estimate (23), and in the fifth line (19) has been used. Dividing by \(\|\tilde{r}_2\|_{L^2(K)}\), and using the triangle inequality gives that

\[
\|r_2\|_{L^2(K)} \leq C(\|\tilde{r}_2 - r_2\|_{L^2(K)} + (1 + k^2)h^{-1}\|\partial_h\|_{\text{curl}K}).
\]

The fourth term in (51) can be estimated as follows:

\[
\|\tilde{R}_2\|_{L^2(I)}^2 \leq C(\psi_1 \tilde{R}_2, \tilde{R}_2)_I = C((\psi_1 \tilde{R}_2, \tilde{R}_2 - r_2)_I + C(\psi_1 \tilde{R}_2, R_2)_I)
\]
\[
= C((\psi_1 \tilde{R}_2, \tilde{R}_2 - r_2)_I + B_K(\partial_h, \nabla \psi_1 \tilde{R}_2) + (r_2, \psi_1 \tilde{R}_2)_K)
\]
\[
\leq C(\|\psi_1 \tilde{R}_2\|_{L^2(I)}\|\tilde{R}_2 - r_2\|_{L^2(I)}
\]
\[
+ (1 + k^2)\|\partial_h\|_{\text{curl}K} \|\nabla \psi_1 \tilde{R}_2\|_{[L^2(K)]^3} + \|\psi_1 \tilde{R}_2\|_{L^2(K)}\|r_2\|_{L^2(K)}
\]
\[
\leq C(\|\tilde{R}_2\|_{L^2(K)}\|\tilde{R}_2 - r_2\|_{L^2(I)}
\]
\[
+ h^{-\frac{1}{2}}(1 + k^2)\|\partial_h\|_{\text{curl}K} \|\tilde{R}_2\|_{L^2(I)} + h^{\frac{1}{2}}\|\tilde{R}_2\|_{L^2(I)}\|r_2\|_{L^2(K)},
\]

where in the first line (16), and the triangle inequality, in the second line (53), in the fourth line (23), and in the sixth line (20) and (17) have been used. Dividing both sides by \(\|\tilde{R}_2\|_{L^2(I)}\) and using the triangle inequality gives that

\[
\|R_2\|_{L^2(I)} \leq C(\|\tilde{R}_2 - r_2\|_{L^2(I)}
\]
\[
+ h^{-\frac{1}{2}}(1 + k^2)\|\partial_h\|_{\text{curl}K} + h^{\frac{1}{2}}\|r_2\|_{L^2(K)}),
\]

which can be further estimated using (61) and we obtain

\[
\|R_2\|_{L^2(I)} \leq C(\|\tilde{R}_2 - r_2\|_{L^2(I)}
\]
\[
+ h^{-\frac{1}{2}}(1 + k^2)\|\partial_h\|_{\text{curl}K} + h^{\frac{1}{2}}\|r_2\|_{L^2(K)}).
\]

With a straightforward modification one can prove the inequalities

\[
\|\mathbf{r}_1\|_{L^2(K)}^2 \leq C(\|\mathbf{r}_1 - \mathbf{r}_1\|_{L^2(K)}^2 + (1 + k^2)h^{-1}\|\partial_h\|_{\text{curl}K}),
\]

and

\[
\|\mathbf{R}_1\|_{L^2(I)}^2 \leq C(h^{-1}(1 + k^2)^2\|\partial_h\|_{\text{curl}K}^2
\]
\[
+ h\|\mathbf{r}_1 - \mathbf{r}_1\|_{L^2(K)}^2 + \|\mathbf{R}_1 - \mathbf{R}_1\|_{L^2(I)}^2).\]

Taking the square of (64), (65), (61) and (63) and summation of the last two contributions of the faces of the elements we obtain

\[
\frac{1}{C} \tilde{h}_K^2 = C(\|\mathbf{r}_1\|_{L^2(K)}^2 + h^2\|r_2\|_{L^2(I)}^2 + h\|\mathbf{R}_1\|_{L^2(O\partial K)}^2 + h\|R_2\|_{L^2(O\partial K)}^2)
\]
\[
\leq (1 + k^2)^2\|\partial_h\|_{\text{curl}K}^2 + h^2(\|\mathbf{r}_1 - \mathbf{r}_1\|_{L^2(K)}^2 + \|r_2 - r_2\|_{L^2(K)}^2)
\]
\[
+ h(\|\mathbf{R}_1 - \mathbf{R}_1\|_{L^2(I)}^2 + \|R_2 - R_2\|_{L^2(I)}^2)\]
as stated in the theorem. □

Using the results of Section 3 and 4 we can now state the reliability and efficiency of the implicit error indicator $\hat{e}_h$ in the following sense.

**Theorem 4.** The implicit error indicator $\hat{e}_h$ is reliable, i.e. there is a constant $C_{\text{rel}}$ independent of the mesh size such that

\begin{equation}
\|e_h\|_{\text{curl}} \leq C_{\text{rel}} \hat{e}_h \|e_h\|_{\text{curl}} + R
\end{equation}

and it is also efficient, i.e. for some constant $C_{\text{eff}}$ the following estimate holds

\begin{equation}
\|\hat{e}_h\|_{\text{curl}} \leq C_{\text{eff}} \|e_h\|_{\text{curl}} + R,
\end{equation}

where $R$ denotes residual terms which are higher order in $h$ compared to $\|e_h\|_{\text{curl}}$ if $J$ can be approximated well within the finite element space.

**Proof** The statement is a direct consequence of Theorem 2, Theorem 3, Lemma’s 7 and 8. □

**Remark:** Using the average of the traces in (48) and (49) implies that the bilinear form for the error (8) can be localized (51) and used in an adaptation algorithm. For more details see [5].

**APPENDIX A**

For the proof of Theorem 1 we need the following.

**Lemma 9.** If the right hand side of the Maxwell equations (1) is a gradient, i.e. $J = \nabla p$ for some $p \in H^1_0(\Omega)$ then the exact error can be written as follows:

\begin{equation}
\|e_h\|_{\text{curl}}^2 = \|\nabla \times E_h\|_{L^2(\Omega)}^2 + \frac{1}{k^2} \|r_1\|_{L^2(\Omega)}^3
\end{equation}

and the global error indicator $\zeta_{T,\alpha,\beta}$ corresponding to (11) has the following form:

\begin{equation}
\zeta_{T,\alpha,\beta} = \left( \sum_{K \in T_h} h_{K}^{2\alpha} \|r_K\|_{L^2(\Omega)}^2 \sum_{l_j \subset \partial K} h_K^\beta \|\nu_j \times [\nabla \times E_h]_{l_j}\|_{L^2(l_j)}^2 \right)^{\frac{1}{2}}.
\end{equation}

**Proof** If $J = \nabla p$ for some $p \in H^1_0(\Omega)$ in (1), then its unique solution is $E = -\frac{1}{k^2} J$ and the exact error is

\begin{equation}
e_h = E - E_h = -\frac{1}{k^2} J - E_h = -\frac{1}{k^2} (J + k^2 E_h) \quad \text{on } K.
\end{equation}

The residual can be written as follows:

\begin{equation}
r_1 = J - \nabla \times \nabla \times E_h + k^2 E_h = J + k^2 E_h \quad \text{on } K.
\end{equation}

Using (71) and (72) a straightforward computation gives (69). The formula in (70) can be obtained simply by summation of the terms in (11). □

**Corollary 2.** If the numerical solution $E_h$ in (3) is a gradient on $\Omega$, then (69) and (70) reduce into

\begin{equation}
\|e_h\|_{\text{curl},K}^2 = \frac{1}{k^2} \|r_1\|_{L^2(K)}^2
\end{equation}

and

\begin{equation}
\eta_{h,\alpha,\beta}^2 = \sum_{K \in T_h} h_{K}^{2\alpha} \|r_1\|_{L^2(K)}^2.
\end{equation}
In the sequel, we will construct a function $J = \nabla p$ with $p \in H^1_0(\Omega)$ and $J \in H(\text{div}, \Omega)$ such that its projection is a discrete gradient for all finite element spaces corresponding to a family of meshes. We will also ensure that the numerical solution is a gradient. For the construction of such a source term $J$, we need the following decomposition of the Nédélec spaces and the consecutive lemmas.

Let us consider the discrete Helmholtz (orthogonal) decomposition (see [8], Section 7.2.1) of the Nédélec type edge elements of order 1:

\begin{equation}
\mathcal{N}_{1,h} = H_{1,h} \oplus H_{2,h},
\end{equation}

with

\begin{equation}
H_{1,h} = \{ \nabla p_h : p_h \in H^1_0(\Omega), p_h|_K \in P_{1,K} \},
\end{equation}

where $P_{1,K}$ denotes the set of linear polynomials on $K$. In other words, $H_{1,h}$ consists of discrete gradients and $H_{2,h}$ is its orthogonal complement

\[ H_{2,h} = \{ u \in \mathcal{N}_{p,h} : u \perp H_{1,h} \}, \]

which is also called the discrete divergence free component.

**Remarks:**

1. The orthogonality corresponding to the direct sum $\oplus$ (see (75)) is understood with respect to the scalar product of the Hilbert space $H(\text{curl}, \Omega)$. Note that for $u \in \ker(\text{curl})$ the orthogonality relation $u \perp v$ is equivalent with orthogonality in the $L^2$ sense.

2. The local decomposition

\[ u|_K(x, y, z) = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_{2z} - b_{3y} \\ b_{3x} - b_{1z} \\ b_{1y} - b_{2x} \end{pmatrix}, \]

which is an easy representation of the first order Nédélec spaces on a reference tetrahedron $K$, does not coincide with the decomposition in (75) for two reasons:

- The second term is not orthogonal to the first one.
- This decomposition does not reflect how the constant terms should be assembled when a finite element is defined globally.

Also for the higher order Nédélec elements, the direct sum in their construction (see Chapter 5.5 in [8]) does not coincide with the Helmholtz decomposition in (75).

3. Note that the function $J$ to be constructed is not contained in any of the finite element spaces $\mathcal{N}_{p,h}$.

4. Here the tessellations are parameterized with the positive numbers $h$, which decrease when the tessellation is refined.

We state some basic properties of the decomposition in (75).

**Lemma 10.** The components in (75) have the following properties:

(i) $H_{1,h_1} \subsetneq H_{1,h_2}$ for any $h_1 < h_2$.

(ii) $\dim H_{1,h_n} \to \infty$ as $h_n \to 0$.

**Proof**
Let us choose $\hat{q}$.

Then the same inclusion and orthogonality holds for $q$ chosen according to (ii) in Lemma 10. Then the same inclusion and orthogonality this is equal to the number of the internal nodes hence $\dim H_{1,h_n} \to \infty$ as $h_n \to 0$. □

Lemma 11. There is a function $J \in H(\text{curl}, \Omega) \cap H(\text{div}, \Omega)$ such that for all $h$ we have $J \perp H_{2,h}$ and $J \notin H_{1,h}$.

Proof: Let us consider $0 \neq \hat{q}_1 \in H_{1,h_1}$. Then $\hat{q}_1 \perp H_{2,h_1}$ and the same holds for $q_1 = \frac{1}{2} \parallel \hat{q}_1 \parallel_{\text{curl}} + \parallel \hat{q}_1 \parallel_{\text{div}}$

For an appropriate $h_2$ we define $q_2$ as follows:

Let us choose $q_2 \in H_{1,h_2}$ such that $q_2 \perp H_{1,h_1}$ and $q_2 \perp H_{2,h_1}$. This choice is possible if $\dim H_{1,h_2} > \dim H_{1,h_1} + \dim H_{2,h_1}$ and this holds according to (ii) in Lemma 10.

Then the same inclusion and orthogonality holds for $q_2 = \frac{1}{2} \parallel q_2 \parallel_{\text{curl}} + \parallel q_2 \parallel_{\text{div}}$.

Accordingly, we define $q_n$ as follows:

Let us choose $q_n \in H_{1,h_n}$ such that $q_n \perp H_{1,h_{n-1}}$ and $q_n \perp \bigcup_{j=1}^{n-1} H_{2,h_j}$. This choice is always possible if $\dim H_{1,h_n} > \dim H_{1,h_{n-1}} + \sum_{j=1}^{n-1} \dim H_{2,h_j}$ and such $h_n$ can be chosen according to (ii) in Lemma 10. Then the same inclusion and orthogonality holds for $q_n = \frac{1}{2} \parallel q_n \parallel_{\text{curl}} + \parallel q_n \parallel_{\text{div}}$.

We define $J$ with the series:

$$J = \sum_{i=1}^{\infty} q_i$$

and verify that it satisfies all properties listed in the lemma. Note that $J$ makes sense both in $H(\text{curl}, \Omega)$ and $H(\text{div}, \Omega)$ since $\parallel q_i \parallel_{\text{curl}} \leq \frac{1}{2}$ and $\parallel q_i \parallel_{\text{div}} \leq \frac{1}{2}$ hold by the above construction.

1. Since the terms $q_i$ are orthogonal by the construction, we can decompose

$$J = \sum_{k=1}^{j} q_k + \sum_{k=j+1}^{\infty} q_k,$$

where $\sum_{k=1}^{j} q_k \in H_{1,h_j}$ according to (i) in Lemma 10 and $0 \neq (\sum_{k=j+1}^{\infty} q_k) \perp H_{1,h_j}$ according to the construction. Consequently, the first term in (77) is in $H_{1,h_j}$, while the second one is orthogonal to $H_{1,h_j}$ and therefore, $J \notin H_{1,h_j}$ for any $j$.

2. Using again (i) in Lemma 10 we have that $q_j \in H_{1,h_k}$ for any $k \geq j$, therefore, $q_j \perp H_{2,h_k}$ for $k \geq j$.

On the other hand, by the above construction $q_j \perp H_{2,h_k}$ for any $k < j$. Consequently, $q_j \perp H_{2,h_k}$ for any $k$. Since this holds for an arbitrary $j$, also $J \perp H_{2,h_k}$ for any $k$ as stated.

3. Since the differential operators curl and div are closed, $J \in H(\text{curl}, \Omega) \cap H(\text{div}, \Omega)$. □

Proof of Theorem 1. Since $q_n$ is a gradient, i.e. $q_n \in \nabla H^1_0(\Omega)$ for all $n$, the closedness of $\nabla H^1_0(\Omega)$ gives that $J \in \nabla H^1_0(\Omega)$. 

The discretized variational form (3) corresponding to the tessellation $T_h$ is:

Find $E_{h_j} \in N_{1,h_j}$ such that for all $v_{h_j} \in N_{1,h_j}$

$$\left( \nabla \times E_{h_j}, \nabla \times v_{h_j} \right)_\Omega - k^2 \left( E_{h_j}, v_{h_j} \right)_\Omega = \left( J, v_{h_j} \right)_\Omega. \tag{78}$$

Observe that $J \perp H_{2,h_j}$ as stated in the last point in the proof of Lemma 11. Moreover, by the construction of $q$, we have that $q_i \perp H_{1,h_j}$ for any $i > j$. Therefore, for all $v_{h_j} \in N_{1,h_j}$

$$\left( J, v_{h_j} \right)_\Omega = (q_1 + q_2 + \cdots + q_j, v_{h_j}),$$

which gives that

$$E_{h_j} = -\frac{1}{k^2}(q_1 + q_2 + \cdots + q_j)$$

is the (unique) solution of (78), which gives that $\nabla \times E_{h_j} = 0$. The assumption [H2] gives that

$$\lim_{h_j \to 0} \max_{h \subset T_h} h_K = 0$$

and therefore, using (73) and (74) we obtain that

$$\lim_{h_j \to 0} \frac{h_{\alpha,h_j}^2}{\|e_{h_j}\|_{\text{curl}}^2} \leq \lim_{h_j \to 0} \frac{\max_{K \in T_h} h_K^{2q} \sum_{K \in T_h} \|e_{h_j}\|_{L_2(K)}^2}{\|e_{h_j}\|_{L_2(\Omega)}^2} = \lim_{h_j \to 0} \max_{h \subset T_h} h_K^{2q} = 0,$$

which proves the theorem. □

Appendix B

Proof of Lemma 5. Let $\hat{K}$ denote the unit simplex, which is used as the reference tetrahedron. Then by the trace theorem there is a positive constant $C_\hat{K}$ such that

$$\|\hat{v}\|_{L_2(\partial \hat{K})}^2 \leq C_\hat{K}\|\hat{v}\|_{H^1(\hat{K})} = C_\hat{K}(\|\hat{v}\|_{L_2(\hat{K})} + \|
abla \hat{v}\|_{L_2(\hat{K})})$$

hence for some $C > 0$ the inequality (43) holds, where $h_\hat{K} = \sqrt{2}$. Since the family of the meshes is non-degenerated, there is a constant $C$ such that (43) is valid for all subdomains $\tilde{K}$ with $h_\tilde{K} = h_\hat{K} = \sqrt{2}$. The proof of this statement is rather technical, one has to use the fact that the determinant of the Jacobian corresponding to the change of variables between $K$ and $\hat{K}$ has a positive upper and lower bound.

Then we try to find constants $s_1$ and $s_2$ such that

$$\|v\|_{L_2(\partial K)}^2 \leq \frac{1}{h_K^{s_1}}\|v\|_{L_2(K)}^2 + \frac{h_K^{s_2}}{h_K^{s_2}}\|
abla v\|_{L_2(K)}^2$$

holds for any subdomain $K$. Any subdomain $K$ can be obtained via a simple transformation $D_K : \hat{K} \to K$, where $D_K^{-1}$ is defined as

$$D_K^{-1} = \frac{\sqrt{2}}{h_K} I,$$

where $I$ denotes the identity and $\text{diam} \hat{K} = \sqrt{2}$. We transform the function $v$ accordingly such that $\hat{v}(x) = v(D_K x)$. The face $l$ of $\hat{v}$ corresponds to the face $l$ of
Then a simple change of variables in the integrals gives that
\[ \|v\|^2_{L^2(\partial K)} = \frac{h^2}{2} \|\tilde{v}\|^2_{L^2(\partial \hat{K})}, \]
\[ \|v\|^2_{L^2(K)} = \frac{h^3}{\sqrt{8}} \|\tilde{v}\|^2_{L^2(\hat{K})}, \]
\[ \|\nabla v\|^2_{L^2(K)} = \sqrt{2} \frac{h^3}{h^2} \|\tilde{v}\|^2_{L^2(\hat{K})}. \]
Comparing these with (79) we obtain that \( s_1 = -1 \) and \( s_2 = 1 \) are appropriate as stated in the lemma. □

References


Department of Applied Mathematics, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands and, Department of Applied Analysis and Computational Mathematics, ELTE P.O. Box 120, 1518 Budapest, Hungary

E-mail address: izsakf@cs.elte.hu

Department of Applied Mathematics, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands

E-mail address: j.j.w.vandervegt@math.utwente.nl