COMMENT

On Bäcklund transformations and identities in bilinear form

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Abstract. For bilinear equations of the form \( P(D)f \cdot f = 0 \) we find all possibilities
for rewriting \( g^2 P(D)f \cdot f - f^2 P(D)g \cdot g = 0 \) in the form \( Q(D)f \cdot g = 0 \). This is the
first step in finding a Bäcklund transformation.

1. Introduction

In this comment we study Bäcklund transformations in bilinear form. This technique
was introduced by Hirota [1, 2]. Let us start with a brief sketch of the method, and
formulate the questions that we want to answer. We suppose that we are given an
equation in bilinear form

\[
P(D)f \cdot f = 0.
\]

Here \( P \) is a polynomial in, say, \( n \) variables, i.e.

\[
P(D_1, \ldots, D_n) = \sum c_\alpha D^\alpha
\]

where \( \alpha = (\alpha_1, \ldots, \alpha_n) \) and \( D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n} \), where

\[
D_1^{\alpha_1} \cdots D_n^{\alpha_n} f \cdot g = \frac{\partial^{\alpha_1}}{\partial y_1^{\alpha_1}} \ldots \frac{\partial^{\alpha_n}}{\partial y_n^{\alpha_n}} f(x_1 + y_1, \ldots, x_n + y_n)
\times g(x_1 - y_1, \ldots, x_n - y_n) \bigg|_{y_1 = \cdots = y_n = 0}.
\]

To find a Bäcklund transformation, we apply the following trick: consider the equation

\[
g^2 P(D)f \cdot f - f^2 P(D)g \cdot g = 0.
\]

Then we note that for a solution \( f, g \) to (2), the following holds:

\( f \) is a solution of (1) \iff \( g \) is a solution of (1).

Suppose that we could rewrite (2) in the following form:

\[
Q(D)[Q_1(D)(f \cdot g) \cdot Q_2(D)(f \cdot g)] = 0
\]

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\]
Then by suitably splitting $Q$ we derive two equations:

$$
P_1(D)f \cdot g = 0
$$

$$
P_2(D)f \cdot g = 0.
$$

The system (4) can be called a Bäcklund transformation for $P(D)f \cdot f = 0$. Namely, suppose $g$ is given; then a solution $f$ to equation (4) will also satisfy $P(D)f \cdot f = 0$.

The process of sensibly splitting $Q$ is strongly equation-dependent; it seems unclear in general how to perform it. Two other general questions remain:

1. Is there for any $P$ a solution $Q$?
2. Is this solution unique? If not, can one find all possibilities?

In this comment we answer these questions. The answer to the first is yes; the proof is already essentially in Hirota [1,2]. This solution is not at all unique. We give all possibilities in terms of a generating identity. The proof that these are all the possibilities is the most difficult part, and not fully included here. For complete proofs, see [3].

2. Algebraic background; partial solution

Let $J$ denote the space of multi-indices $(i_1, \ldots, i_n)$. In practice we will encounter $f^{(i)}$ and $g^{(j)}$ $(i, j \in J)$, which will substitute for the partial derivatives. We will deal with the polynomial algebra $A = \mathbb{R}[f^{(i)}, g^{(j)}]$, $(i, j \in J)$. In this algebra, we have derivations $\partial_r$ $(r = 1 \cdots n)$ which act in the obvious way. In particular

$$
\partial_r(f^{(i)}) = f^{(i+1,r)} \text{ and } \partial_r(g^{(j)}) = g^{(j+1,r)}
$$

where $1_r = (0, \ldots, 0, 1, 0, \ldots, 0)$, the $r$th basis vector.

Corresponding to $\partial_r$, we introduce the linear map $D_r$ (the Hirota derivative) as a map $A \otimes A \to A \otimes A$, by

$$
D_r(a \otimes b) = \partial_r(a) \otimes b - a \otimes \partial_r(b) \quad (r = 1, \ldots, n)
$$

The derivative $\partial_r$ is also extended to $A \otimes A$ by

$$
\partial_r(a \otimes b) = \partial_r(a) \otimes b + a \otimes \partial_r(b).
$$

Obviously, we can define a Hirota derivative corresponding to any derivation of $A$.

Lemma 1. Let $D = D_r$ and $\partial = \partial_r$ and $A$ be as above. Then

$$
\exp(cD)(a \otimes b) = \exp(c\partial)a \otimes \exp(-c\partial)b.
$$

The equality is meant as formal power series, and follows by computing homogeneous terms with reference to $c^i$, $i = 0, 1, 2, \ldots$.

Note that our definition of $D_r$ differs from the usual one in the following sense: the image is again in $A \otimes A$ and not in $A$. This is a major difference, as we will see shortly. To get the usual Hirota derivatives $D_r$ we have to project the image of $D_r$ to $A$: let $\pi : A \otimes A \to A$, $\pi(\sum a_i \otimes b_i) = \sum a_i b_i$, denote this projection. Then

$$
D_r(a \otimes b) := \pi D_r(a \otimes b)
$$

and more generally

$$
D^k_{\pi}(a \otimes b) := \pi D^k_r(a \otimes b), \ k = 1, 2, \ldots
$$

The exchange formula is central in this comment.
Theorem 1 (cf Hirota). Let \( \partial_{\alpha} = \sum \alpha_i \partial_i \), \( \partial_{\beta} = \sum \beta_i \partial_i \) and \( \partial_{\gamma} = \sum \gamma_i \partial_i \) and let \( D_{\alpha} \) etc. be the corresponding (projected) Hirota derivatives. Then we have for all \( a, b, c, d \in A \)

\[
\exp(D_{\alpha})(\exp(D_{\beta})(a \otimes b) \otimes \exp(D_{\gamma})(c \otimes d)) = \exp\left(\frac{1}{2}(D_{\beta} - D_{\gamma})\right)(\exp\left(\frac{1}{2}(D_{\beta} + D_{\gamma})\right) + D_{\alpha})(a \otimes d) \\
\otimes \exp\left(\frac{1}{2}(D_{\beta} + D_{\gamma} - D_{\alpha})(c \otimes b)\right)
\]

Proof. The proof is based on lemma 1: expressing all the \( D \) in their corresponding \( \partial \) and projecting on \( A \) gives the result immediately.

Note that this identity solves our first question by taking \( a = b = f \) and \( c = d = g \). Using the fact that \( \exp(\sum \epsilon_i D_i) = \prod \exp(\epsilon_i D_i) \), we see that for three multi-indices \( k, l, m \in J \), the coefficient of \( \alpha^k \beta^l \gamma^m \) in (5) expresses

\[
D^k(D^l(f \cdot f) \cdot D^m(g \cdot g))
\]

in terms of

\[
D^\cdot(D^\cdot(f \cdot g) \cdot D^\cdot(g \cdot f)).
\]

Moreover we note that such an expression is not unique. If we take \( a = c = f \) and \( b = d = g \), then again comparing the coefficients of \( \alpha^k \beta^l \gamma^m \) in (5), we see that \( D^k(D^l(f \cdot g) \cdot D^m(f \cdot g)) \) can be re-expressed. These observations solve a part of our problem; however, they do not solve the most difficult part. It is important to find all possibilities, hence all identities of the form

\[
D^k(D^l(f \cdot g) \cdot D^m(f \cdot g)) = \sum D^\cdot(D^\cdot(f \cdot g) \cdot D^\cdot(f \cdot g)).
\]

This problem will be solved in the next section. The answer is slightly surprising: equation (5) already contains all non-trivial identities!

3. Finding all identities

To study the identities of the form (6) more closely, we introduce two subspaces of \( A \otimes A \). The first one, denoted by \( B \), is the linear space spanned by the elements \( \{f(i) \otimes g(j), (i, j \in J)\} \). Clearly these elements form a basis. Note that \( \pi |_{B} \) is injective. This allows us to view \( B \) as a subspace of \( A \). We introduce

\[
e^{(\beta)}_\alpha := \partial^\beta D^\alpha (f \otimes g)
\]

which are again elements of \( B \). For these elements one can prove the following lemma.

Lemma 2. \( \{e^{(\beta)}_\alpha\} \) is a basis for \( B \).

Proof. The proof (by induction on the number of independent variables \( n \)) is based on the observation that \( D^\alpha (f \otimes g) \not\in \sum_{r=1}^n \text{Im}(\partial_r) \).
Identifying $B$ with a subspace of $A$, we can define $D_r$ (and $\partial_r$) : $B \otimes B \to B \otimes B$. In $B \otimes B$ we define the subspace $C$, spanned by the elements

$$D^k(D^l(f \cdot g) \otimes D^m(f \cdot g)) \quad (k, l, m \in J).$$

Thanks to lemma 2, these elements are linearly independent, i.e. they form a basis for $C$. This shows that finding identities of the form (6) is equivalent to finding $\ker(\pi) |_C$. Before turning to $\ker(\pi)$ we mention:

**Lemma 3.** Let $B$ and $C$ be as above. Then $B \otimes B = C \oplus \text{Im}(\partial)$, where $\text{Im}(\partial) := \sum_{r=1}^{n} \text{Im}(\partial_r)$.

Expressed in normal derivatives, $\ker(\pi) |_{B \otimes B}$ is easily described.

**Lemma 4.** $\ker(\pi) |_{B \otimes B}$ is generated by

1. $f^{(i)}g^{(j)} \otimes f^{(k)}g^{(l)} - f^{(k)}g^{(l)} \otimes f^{(i)}g^{(j)}$

2. $f^{(i)}g^{(j)} \otimes f^{(k)}g^{(l)} - f^{(i)}g^{(l)} \otimes f^{(k)}g^{(j)}$.

The elements mentioned under 1 give rise to trivial identities, namely the identities

$$D^k(D^l(f \cdot g) \otimes D^m(f \cdot g)) = (-1)^{|k|} D^k(D^m(f \cdot g) \otimes D^l(f \cdot g)).$$

To analyse the elements under 2, we introduce $d_r$ and $d_r^+$ : $B \to B$, defined by

$$d_r(f^{(i)}g^{(j)}) = f^{(i+1,r)}g^{(j)}, \quad d_r^+(f^{(i)}g^{(j)}) = f^{(i)}g^{(j+1,r)}.$$ 

The elements $f^{(i)}g^{(j)} \otimes f^{(k)}g^{(l)} - f^{(i)}g^{(l)} \otimes f^{(k)}g^{(j)}$ are precisely the homogeneous terms in

$$\exp \left( \sum_{r=1}^{n} (\alpha_r d_r + \beta_r d_r^+) \right) f \cdot g \otimes \exp \left( \sum_{r=1}^{n} (\gamma_r d_r + \delta_r d_r^+) \right) f \cdot g$$

$$- \exp \left( \sum_{r=1}^{n} (\alpha_r d_r + \delta_r d_r^+) \right) f \cdot g \otimes \exp \left( \sum_{r=1}^{n} (\gamma_r d_r + \beta_r d_r^+) \right) f \cdot g. \quad (7)$$

Using $\partial_r = d_r + d_r^+$ and $D_r = d_r - d_r^+$, we can rewrite (7) in terms of $\partial_r$ and $D_r$: (7) turns into

$$\exp \left( \frac{1}{4} \sum (\alpha_r + \beta_r + \gamma_r + \delta_r) \partial_r \right) \left\{ \exp \left( \frac{1}{4} \sum (\alpha_r + \beta_r - \gamma_r - \delta_r) D_r \right) \right.$$  

$$\left( \exp \left( \frac{1}{2} \sum (\alpha_r - \beta_r) D_r \right) (f \cdot g) \otimes \exp \left( \frac{1}{2} \sum (\gamma_r - \delta_r) D_r \right) (f \cdot g) \right)$$

$$- \exp \left( \frac{1}{4} \sum (\alpha_r + \delta_r - \gamma_r - \beta_r) D_r \right) \left( \exp \left( \frac{1}{2} \sum (\alpha_r - \delta_r) D_r \right) (f \cdot g) \right.$$  

$$\otimes \exp \left( \frac{1}{2} \sum (\gamma_r - \beta_r) D_r \right) (f \cdot g) \right\}. \quad (8)$$
On Bäcklund transformations

Changing variables

$$
\begin{align*}
\tilde{\delta}_r &= \frac{1}{4}(\alpha_r + \beta_r + \gamma_r + \delta_r) \\
\tilde{\alpha}_r &= \frac{1}{4}(\alpha_r + \beta_r - \gamma_r - \delta_r) \\
\tilde{\beta}_r &= \frac{1}{2}(\alpha_r - \beta_r) \\
\tilde{\gamma}_r &= \frac{1}{2}(\gamma_r - \delta_r)
\end{align*}
$$

expression (8) transforms into

$$
\exp\left(\sum \tilde{\delta}_r \partial_r\right) \left\{ \exp\left(\sum \tilde{\alpha}_r D_r\right) \left(\exp\left(\sum \tilde{\beta}_r D_r\right) (f \cdot g) \otimes \exp\left(\sum \tilde{\gamma}_r D_r\right) (f \cdot g) \right) \\
- \exp\left(\frac{1}{2} \sum (\tilde{\beta}_r - \tilde{\gamma}_r) D_r\right) \left(\exp\left(\frac{1}{2} \sum (2\tilde{\alpha}_r + \tilde{\beta}_r + \tilde{\gamma}_r) D_r\right) (f \cdot g) \right) \\
\otimes \exp\left(\frac{1}{2} \sum (-2\tilde{\alpha}_r + \tilde{\beta}_r + \tilde{\gamma}_r) D_r\right) (f \cdot g) \right\}.
$$

Since (9) is an invertible transformation, the homogeneous terms in (10) span the same space as in (7). For $\ker(\pi)|_C$ we only need to consider the coefficients in which $\tilde{\delta}$ does not appear (see lemma 3). Hence we are left with the expression between braces, which is identical to theorem 1 for $a = c = f$ and $b = d = g$. So these (and the trivial ones) are all identities.

I would like to thank Professors Conte and Martini for drawing my attention to this problem.

References

[3] Post G 1990 Bäcklund transformations and identities in bilinear form Memorandum 856 University of Twente, The Netherlands