Invariant Option Pricing & Minimax Duality of American and Bermudan Options

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A conceptual definition of an option

An *Option* is a pair $O = (T, O)$ consisting of an *Expiry* $T$ and a *Payoff* $O$ (in base currency) paid at $T$. The expiry $T$ is a stopping time (bounded by $m > 0$), and the payoff $O$ is a random variable known at time $T$, i.e., measurable w.r. to $\sigma$-algebra $\mathcal{F}_T$ of events at or before $T$:

$$\mathcal{F}_T := \{ \Lambda \in \mathcal{F} : \Lambda \cap \{ T \leq t \} \in \mathcal{F}_t \}.$$

- For a European option, the expiry $T$ is deterministic.
- American and Bermudan options: $T$ is the optimal exercise time.
- Barrier options: $T$ is the first passage time to the barrier.
- Credit derivatives with recovery : $T$ is the default time.
A conceptual definition of an option • • •

• The definition so far depends only on the filtration \((\mathcal{F}_t)_{t\geq 0}\). To talk about the option price, a probability measure \(\mathbb{P}\) and an integrability condition are required.

• Think of a *Numeraire* \(\beta\) as a claim which pays no dividend and has a positive price \(\beta_t > 0\) at all times \(t \leq m\), e.g., a zero-dividend stock, or the \(m\)-maturity zero-coupon bond.

• To each numeraire \(\beta\), there is associated a *Numeraire Measure* \(\mathbb{P}^\beta\), characterized by the property that if \(B\) is any other numeraire, then the relative price process \((B_t/\beta_t)\) is a (right-continuous) \(\mathbb{P}^\beta\)-martingale.

• The required integrability condition on option \(O = (T, O)\) is this: \(O/\beta_T\) is \(\mathbb{P}^\beta\)-integrable for some numeraire \(\beta\).
Numeraire invariance of the option definition

The important aspect of this integrability condition is that if it holds for some numeraire, then it holds for all numeraires:

\[
\frac{B_0}{\beta_0} \mathbb{E}^B \left[ \frac{O}{B_T} \right] = \mathbb{E}^\beta \left[ \frac{O}{B_T} \frac{B_m}{\beta_m} \right] \quad \text{(change of numeraire)}
\]

\[
= \mathbb{E}^\beta \left[ \mathbb{E}^\beta \left[ \frac{O}{B_T} \frac{B_m}{\beta_m} \bigg| \mathcal{F}_T \right] \right] \quad \text{(iterating expectation)}
\]

\[
= \mathbb{E}^\beta \left[ \frac{O}{B_T} \mathbb{E}^\beta \left[ \frac{B_m}{\beta_m} \bigg| \mathcal{F}_T \right] \right] \quad \text{(by } \mathcal{F}_T \text{ measurability of } \frac{O}{B_T})
\]

\[
= \mathbb{E}^\beta \left[ \frac{O}{B_T} \frac{B_T}{\beta_T} \right] \quad \text{(optional sampling theorem)}
\]

\[
= \mathbb{E}^\beta \left[ O/\beta_T \right] < \infty.
\]
Price process $\mathcal{O}_t$ of an option

Consider investing the option payoff $O$ at expiry $T$ in a numeraire $\beta$ and holding this position until the terminal date $m$. We end up with a claim that pays $\beta_m O / \beta_T$ at time $m$. At a time and state before expiry (i.e., $t < T(\omega)$), the option has not yet been invested in the numeraire, so this claim is identical with the option itself. We are thus forced to define the option price $\mathcal{O}_t$ to be the price of this claim, i.e., $\beta_t \mathbb{E}^\beta [O / \beta_T | \mathcal{F}_t]$.

After expiry, the option has ceased to exist and has no price. In this case it is convenient to define the option price to be zero. We thus arrive at the following definition of an option price:

$$
\mathcal{O}_t := 1_{t \leq T} \beta_t \mathbb{E}^\beta [O / \beta_T | \mathcal{F}_t].
$$

This definition is independent of choice of numeraire $\beta$. 
Price Transitivity Law

Using the optional sampling theorem, the option price at any stopping time $\tau$ (bounded by $m$) is given by

$$O_\tau = 1_{\tau \leq T} \beta_\tau \mathbb{E}^\beta \left[ \frac{O}{\beta_T} \mid \mathcal{F}_\tau \right] \text{ a.s.}$$

The pair $(\tau, O_\tau)$ is a $\tau$-expiry option with payoff $O_\tau$. Let $S$ be another stopping time. What can we say about the time $S$ price $(\tau, O_\tau)_S$ of this option? When $S \leq \tau \leq T$, we simply have

$$(\tau, O_\tau)_S = O_S.$$ 

That is, pricing to time $\tau$ and then pricing to time $S$ is the same as pricing directly to time $S$. 

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Indistinguishable options

Two options $O$ and $O'$ *Indistinguishable* if the price processes $(O_t)$ and $(O'_t)$ are indistinguishable, i.e., a.s. $O_t = O'_t$ all $t$.

**Theorem**: Options $O = (T, O)$ and $O' = (T', O)$ are indistinguishable if and only if $O = O'$ a.s. and 

$$\{T \neq T'\} \subset \{O = 0\}$$ a.s.

So, $T$ and $T'$ need not be the same, but they can differ only at zero payments.

**Notation**: We write $O \leq O'$ if a.s. $O_t \leq O'_t$ all $t$. 


Nonnegative options and nonnegative arbitrage

An option $O = (T, O)$ is Nonnegative if $O \geq 0$ a.s. It is Positive if $O > 0$ a.s.

(In practice, most options are nonnegative, but are not positive either, for they can have zero payoff in some states (i.e., $\mathbb{P}[O = 0] > 0$), e.g., call and put options and swaptions.)

**Nonnegative Arbitrage.** Once the price of a nonnegative option becomes zero, it stays zero thereafter, i.e., for almost all paths $\omega$,

- if $O_t(\omega) = 0$, then $O_s(\omega) = 0$ for all $s \geq t$.
- if $O(\omega) > 0$, then $O_t(\omega) > 0$ for all $t \leq T(\omega)$. 
Semipositive options

A nonnegative option $O$ is *semipositive* if its price is positive before expiry $T$, i.e., at each time $t$, $O_t > 0$ a.s. on $\{t < T\}$.

This then implies a.s. $O_t > 0$ on $\{t < T\}$ all $t$, i.e., for almost all paths $\omega$, $O_t(\omega) > 0$ at all times $t < T(\omega)$.

A *nonnegative option is indistinguishable from a semipositive option*: Let $O = (T, O)$ be a nonnegative option. Then, there exists a unique semipositive option that is indistinguishable from $O$, namely the option $(T^0, O)$, where

$$T^0 := \inf\{t > 0 : O_t = 0\}.$$ 

(It turns out that $O$ is automatically $\mathcal{F}_{T^0}$-measurable - in fact, $O = O_{T^0}$ a.s.)
Payoff processes and trigger options

A progressively measurable process $Z = (Z_t)$ is a Payoff Process if a.s. $|Z_t| \leq \beta_t$ all $t$ for some numeraire $\beta$.

Example: $Z_t = (\beta_t - K)^+, K > 0$.

**Proposition:** Let $Z$ be a payoff process. Then for any numeraire $\beta$, the $\beta$-deflated process $(Z_t/\beta_t)$ is of $\mathbb{P}^\beta$-class D. In particular, for any stopping time $T$, the pair $(T, Z_T)$ is an option.

We call the $T$-expiry option $(T, Z_T)$ as a $Z$-Trigger Option.

**Examples of trigger options:** American and Bermudan options, barrier options, credit derivatives with recovery process.
Dominated and trigger option convergence

**Dominated option convergence:** Let \((O^n)_{n=1}^\infty, O^n = (T_n, O^n) \) by a sequence of options such that \(T^n \downarrow T \) a.s., \(O^n \rightarrow O \) a.s. to some random variables \(T\) and \(O\). Assume there exists a numeraire \(\beta\) such that \(|O^n_t| \leq \beta_t\) a.s. for all \(t\) and \(n\). Then, \(O = (T, O)\) is an option, and a.s. \(O^n_t \rightarrow O_t\), all \(t\).

When the dominating numeraire \(\beta\) can be chosen continuous, then the condition \(T^n \downarrow T\) can be weakened to \(T^n \rightarrow T\).

**Trigger option convergence:** Let \((Z_t)\) be a right continuous payoff process and \(T_n \downarrow T\). The theorem implies that a.s. \((T_n, Z_{T_n})_t \rightarrow (T, Z_T)_t\), all \(t\).
Doob-Meyer decomposition of superclaims

A *Superclaim* is a right-continuous payoff process \((V_t)\) such that \(V_t \geq (s, V_s)_t\) a.s. for all \(t \leq s\). A *Supernumeraire* is a positive superclaim. As we saw, the Snell envelope is a superclaim.

**Proposition:** A right-continuous payoff process \((V_t)\) is a superclaim if and only if the process \((V_t/\beta_t)\) is a right-continuous \(\mathbb{P}^\beta\)-supermartingale for some (hence all) numeraire \(\beta\).

**Doob-Meyer decomposition:** Let \((V_t)\) be a supernumeraire and \(\beta\) be a numeraire. Then there exist a unique numeraire \(B\) with \(B_0 = V_0\) and a decreasing predictable process \((A_t)\) such that a.s. \(V_t = \beta_t A_t + B_t\) all \(t\).
The Snell envelope

The Snell Envelope process of a right continuous payoff process $Z = (Z_t)$ is defined by

$$V_t := \sup_{T \geq t} (T, Z_T)_t,$$

where supremum is taken over the set all stopping times $T$ satisfying $t \leq T \leq m$.

**Theorem.** $(V_t)$ is a right continuous payoff process, and

$$(s, V_s)_t = \sup_{T \geq s} (T, Z_T)_t. \quad (t \leq s)$$

**Corollary.** $(V_t)$ is a superclaim. Indeed, for $t \leq s$,

$$V_t := \sup_{T \geq t} (T, Z_T)_t \geq \sup_{T \geq s} (T, Z_T)_t = (s, V_s)_t.$$
The american option in the continuous case

Simple examples show that the supremum $\sup_{T \geq 0} (T, Z_T)_0$ in definition of Snell envelope is not necessarily attained at any stopping time $T$. In continuous case we show the supremum is attained (and suspect this to be the case when jumps of $Z$ are totally inaccessible). Set

$$T^*_t := \inf \{ s \in [t, m] : Z_s = V_s \}.$$ 

**Theorem.** Let $Z$ be a continuous payoff process that is dominated by a continuous numeraire. Then $Z^*_t = V^*_t$ and

$$V_t = (T^*_t, Z^*_t)_t.$$ 

Further for all times $t$ and $s$, we have

$$(T^*_s, Z^*_s)_t = 1_{t \leq T^*_s} \sup_{T \geq \max(t,s)} (T, Z_T)_t.$$ 

In particular, $A_t = 1_{t \leq T^*_0} V_t$, all $t$, where $A := (T^*_0, Z^*_0)$. 

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Multiplicative minimax duality

Let \((Z_t)\) be a positive right-continuous payoff process. The Snell envelope \((V_t)\) is then a supernumeraire, and Doob-Meyer decomposition implies there are many Domineering Numeraire, i.e., numeraires \(B\) such that \(B_0 = V_0\) and \(B_t \geq V_t\) for all \(t\).

Proposition: Let \(B\) be a domineering numeraire. Then

\[
\sup_{t \geq 0} \left( \frac{Z_t}{B_t} \right) = 1.
\]

Multiplicative minimax duality formula: a.s., all \(t\),

\[
V_t = \inf_{\beta \in \mathcal{C}^+} \beta_t \mathbb{E}^{\beta} \left[ \sup_{s \geq t} \left( \frac{Z_s}{\beta_s} \right) \mid \mathcal{F}_t \right].
\]
In particular, \( V_0 = \inf_\beta \beta_0 \mathbb{E}^\beta [\sup_{t \geq 0} (Z_t / \beta_t)] \). The infimum is attained at any domineering numeraire. So, \( \beta_0 \mathbb{E}^\beta [\sup_{t \geq 0} (Z_t / \beta_t)] \) is an upper bound for the American option price \( V_0 \), for any numeraire \( \beta \). Such an upper bound can be computed by Monte-Carlo simulation. An suitable numeraire must first be chosen. These results extend to nonnegative payoff processes (\( Z_t \)). An “additive version” of minimax duality was previously derived by Rogers (2001) and Haugh & Kogan (2001), and further studied in Andersen & Broadie (2001), Joshi & Theis (2002), and Kolodko & Schoenmakers (2003).
Multiplicative Doob-Meyer decomposition

A *Local Numeraire* is an adapted, right continuous, positive process \((B_t)\) such that \((B_t/\beta_t)\) is a \(\mathbb{P}^\beta\)-local martingale for some (hence all) numeraire \(\beta\).

**Theorem:** Let \((V_t)\) be a supernumeraire. Then there exists a unique decomposition \(V_t = D_t B_t\), where \((D_t)\) is a decreasing predictable process and \((B_t)\) is a local numeraire with \(B_0 = V_0\).

A supernumeraire \((V_t)\) *Multiplicative*, if the local numeraire \((B_t)\) in the multiplicative decomposition \(V_t = D_t B_t\) is actually the price process of a numeraire \(B\). We then refer to this numeraire \(B\) as the *Rollover Numeraire* associated to \((V_t)\). It is clearly a domineering numeraire, and as such relevant to minimax duality.
An application of multiplicative decomposition

**Corollary:** Assume that discount factors satisfy \((s, 1)_t \leq 1\) for all \(t \leq s\). Then there exists a unique increasing, predictable, local numeraire \((B_t)\) with \(B_0 = 1\).

The assumption is equivalent to the identically one process being a supernumeraire. Applying the multiplicative Doob-Meyer decomposition, we obtain a decomposition \(1 = D_t B_t\). The resulting local numeraire \((B_t)\) is increasing, as it equals \((1/D_t)\).

This increasing local numeraire \((B_t)\) can be interpreted as the “continuous money market account”. If it is further assumed to be absolutely continuous, then, taking its logarithmic derivative, we obtain the “instantaneous interest-rate process” \(r_t\), which is nonnegative and satisfies \(B_t = \exp(\int_0^t r_s ds)\).
Option streams

A (Finite) Option Stream (or simply a Stream) is a finite sequence \( O^* = (O^n)_{n=1}^k \) of options \( O^n = (T_n, O^n) \) with increasing expiries \( T_1 \leq \cdots \leq T_k \). Stream \( O^* \) is called:

- **Positive** if \( O^n > 0 \) a.s., all \( n \). Ditto, **Nonnegative**.

- **Decreasing** if \( O^n_{T_n} \leq O^n \) a.s., all \( n \), or equivalently, if a.s.,
  \[
  1_{t \leq T_n} O^{n+1}_t \leq O^n_t \quad \text{all } t, n.
  \]
  It also equivalent to the condition that for some (hence all) numeraire \( \beta \), the discrete deflated envelope process \( (O^n/\beta_{T_n})_{n=1}^k \) is a supermartingale under the finite filtration \( (\mathcal{F}_{T_n})_{n=1}^k \).

- a **Trigger Stream** if \( O^n(\omega) = O^{n+1}(\omega) \) whenever
  \[
  T_n(\omega) = T_{n+1}(\omega).
  \]
  (Equivalently, \( 1_{T_i = T_j} O^i = 1_{T_i = T_j} O^j \), all \( i, j \)).
Bermudan option stream

Henceforth, we take as given a payoff process $Z$.

e.g., $Z_t = (K - S_t)^+$. 

Let $\tau = (\tau_1, \cdots, \tau_k)$ be a Tenor, i.e., sequence of increasing stopping times $\tau_1 \leq \cdots \leq \tau_k$. Define the trigger stream

$$\mathcal{E} := (\mathcal{E}^n_{n=1}^{k}, \mathcal{E}^n := (\tau_n, Z_{\tau_n}).$$

We view $\mathcal{E}$ as “the underlying european option stream.”

(In practice, the expiries $\tau_n$ are deterministic, such as annually or daily.)

Define the Associated Bermudan Stream $\mathcal{B} := (\mathcal{B}^n_{n=1}^{k}$ by $\mathcal{B}^n := (T_{n}^{*}, Z_{T_{n}^{*}})$, where the Optimal Exercise Tenor $T^{*} := (T_{1}^{*}, \cdots, T_{k}^{*})$ is defined by either of the following two equivalent definitions.
The optimal exercise tenor

**Backward inductive definition:** Set $T_k^* := \tau_k$. For $n < k$, define inductively (setting $B^{n+1} := (T_{n+1}, Z_{T_{n+1}})$),

$$T_n^* := 1_{Z_{\tau_n} \geq B^{n+1}_{\tau_n}} \tau_n + 1_{Z_{\tau_n} < B^{n+1}_{\tau_n}} T_{n+1}^*.$$

**Snell envelope definition:** Define

$$T_n^* := \min\{t \in \{\tau_n, \cdots, \tau_k\} : Z_{\tau_n} = \sup_{T \in \{\tau_n, \cdots, \tau_k\}} (T, Z_T)_{\tau_n}\}.$$

The well-known equivalence is easily proved by induction. Recall, the bermudan stream is defined as

$$B^* := (B^n)_{n=1}^k, \quad B^n := (T_n^*, Z_{T_n^*}).$$
Bermudan pricing formula and consequences

**Bermudan option pricing formula:**

\[ B^n_t = 1_{t \leq T^*_n} \sup_{t \leq T \in \{\tau_n, \ldots, \tau_k\}} (T, Z_T)_t. \]

**Corollaries:**

**Time-0 Pricing:** \[ B^n_0 = \sup_{T \in \{\tau_n, \ldots, \tau_k\}} (T, Z_T)_0. \]

**Snell envelope formula:** \[ B^n_{\tau_n} = \sup_{T \in \{\tau_n, \ldots, \tau_k\}} (T, Z_T)_{\tau_n}. \]

**Principle of Dynamic Programming:** \[ B^n_{\tau_n} = \max(Z_{\tau_n}, B^{n+1}_{\tau_n}). \]

**Bermudan stream \( B \) is decreasing:** \[ B^n_{T^*_n} \geq B^{n+1}_{T^*_n}. \]
Rolling an option over a positive option

Let $O = (T, O)$ be an option and $O' = (T', O')$ be a positive option with $T \leq T'$. Define Rollover Option $O \uparrow O'$ by

$$O \uparrow O' := (T', \frac{O O'}{O'_T}).$$

We interpret $O \uparrow O'$ as the option obtained by investing at time $T$ the payoff $O$ in option $O'$.

The rollover operator $\uparrow$ is associative on positive options:

$$(O^1 \uparrow O^2) \uparrow O^3 = O^1 \uparrow (O^2 \uparrow O^3).$$

We denote this by $O^1 \uparrow O^2 \uparrow O^3$, etc.
Rolling an option over a nonnegative option

Let \( O = (T, O) \) be an option and \( O' = (T', O') \) be a nonnegative option with \( T \leq T' \). Define Rollover Option \( O \uparrow O' \) by

\[
O \uparrow O' := (1_{O'_{T=0}} T + 1_{O'_{T>T}} T', 1_{O'_{T=0}} O + \frac{O}{O'_{T}} O').
\]

Pricing is given by

\[
(O \uparrow O')_t = O_t + 1_{t>T} \frac{O}{O'_{T}} O'_t.
\]

The rollover operator is not associative in general. A few properties of \( \uparrow \) are:

\[
(O \uparrow O')_{S} = O_{S}, \quad S \leq T.
\]

\( O \uparrow O = O; \quad O \uparrow (O \uparrow O') = O \uparrow O', \quad \text{etc.} \)

\[
O \uparrow (O' + O'') \leq O \uparrow O' + (O \uparrow O'').
\]

\[
\lim_{\varepsilon \to 0} (O \uparrow (O' + \varepsilon O''))_t = (\varepsilon \uparrow O')_t. \quad \text{(by dominated option convergence)}
\]
Minimax duality of nonnegative payoff processes

Henceforth, we assume $Z \geq 0$ (i.e., a.s. $Z_t \geq 0$, all $t$.) Let $O^+$ (resp. $O^{++}$) denote the set of all nonnegative (resp. positive) options $(T, O)$ with $T \geq \tau_k$.

**Multiplicative minimax duality formula:**

\[
B_1^0 = \inf_{O \in O^{++}} \left( \max_{n=1, \ldots, k} (\mathcal{E}^n \uparrow O) \right)_0 = \min_{O \in O^+} \left( \max_{n=1, \ldots, k} (\mathcal{E}^n \uparrow O) \right)_0.
\]

\[
= \min_{O \in O^{++}} \left( \max_{n=1, \ldots, k} (\mathcal{E}^n \uparrow O) \right)_0 \text{ if } Z > 0.
\]

(The “≤” part is easy.) More generally, for any $1 \leq i \leq k$ and stopping time $\tau \leq \tau_i$,

\[
B^i_\tau = \inf_{O \in O^{++}} \left( \max_{n=i, \ldots, k} (\mathcal{E}^n \uparrow O) \right)_\tau = \min_{O \in O^+} \left( \max_{n=i, \ldots, k} (\mathcal{E}^n \uparrow O) \right)_\tau.
\]
Domineering options and minimax duality

A *Domineering* option is a nonnegative option $\mathcal{O}$ such that $\mathcal{O}_{\tau_n} \geq Z_{\tau_n}$ and $\mathcal{O}_0 = \mathcal{B}_0^1$.

A nonnegative option $\mathcal{O}$ is domineering if and only if

$$\mathcal{O} = \max_{n=1,\ldots,k} (\mathcal{E}^n \uparrow \mathcal{O}).$$

**Corollary.** A domineering option $\mathcal{O}$ satisfies

$$\mathcal{B}_0^1 = (\max_{n=1,\ldots,k} (\mathcal{E}^n \uparrow \mathcal{O}))_0.$$

So, in the minimax formula $\mathcal{B}_0^1 = \min_{\mathcal{O} \in \mathcal{O}^+} (\max_{n=1,\ldots,k} (\mathcal{E}^n \uparrow \mathcal{O}))_0$, the minimum is attained at every domineering option $\mathcal{O}$.

Our next goal is to construct a domineering option denoted $\mathcal{B}^\cdot\uparrow$. 
Rollover of nonnegative option stream

Rollover option $\mathcal{O}^{\downarrow \cdot}$ of a positive stream $\mathcal{O}^{\cdot} = (\mathcal{O}^n)_{n=1}^{k}$ is

$$\mathcal{O}^{\downarrow \cdot} := \mathcal{O}^1 \downarrow \cdots \downarrow \mathcal{O}^k = (T_k, O^k \prod_{n=1}^{k-1} \frac{O^n}{O^{n+1}}),$$

For a nonnegative option streams we define the Left and Right Associative Rollover Options respectively by

$$\mathcal{O}^{\cdot \downarrow} = (\cdots ((\mathcal{O}^1 \downarrow \mathcal{O}^2) \downarrow \mathcal{O}^3) \cdots \downarrow \mathcal{O}^{k-1}) \downarrow \mathcal{O}^k.$$  

$$\mathcal{O}^{\cdot \cdot} = \mathcal{O}^1 \downarrow (\mathcal{O}^2 \downarrow (\mathcal{O}^3 \cdots \downarrow (\mathcal{O}^{k-1} \downarrow \mathcal{O}^k) \cdots ))).$$

Of course, for positive streams, $\mathcal{O}^{\cdot \downarrow} = \mathcal{O}^{\cdot \cdot}$. 
Stream rollover associativity and pricing formula

We call a nonnegative stream $O \cdot$ is Associative if $O \cdot \uparrow = O \cdot \downarrow$ and this holds for all “substreams” of $O \cdot$ too.

**Proposition.** Decreasing and increasing streams are associative. Semipositive trigger streams are associative.

**Rollover pricing formula:** For a nonnegative stream, a.s. all $t$,

$$O_t \cdot \uparrow = O^1_t + \sum_{n=2}^k 1_{T_{n-1} < t} \prod_{i=1}^{n-1} \frac{O^i_{T_i}}{O^i_{T_{i+1}}} O^n_t.$$

(A similar but more complex formula can be written for $O^\cdot \downarrow$. )
Some consequences of rollover pricing formula

\[ \mathcal{O}_0 \uparrow = \mathcal{O}_0 \downarrow = \mathcal{O}_0^1. \]

Denote \( \mathcal{O} \uparrow \) also by \( \mathcal{O} \uparrow^k \). Ditto for \( n \leq k \). Then

\[ \mathcal{O}_S \uparrow = \mathcal{O}_S \uparrow^n, \quad S \leq T_n. \]

(Ditto \( \mathcal{O}_S \downarrow \)). For positive streams we have,

\[ \mathcal{O}_T \uparrow^n = \mathcal{O} \downarrow \prod_{j=1}^{n-1} \frac{O_j}{O_{j+1}^T}; \]

\[ \mathcal{O}_T \downarrow \mathcal{O} \uparrow^k = \mathcal{O} \downarrow \cdots \downarrow \mathcal{O}_k. \]

**Proposition.** If \( \mathcal{O} \downarrow \) is a decreasing nonnegative stream then \( \mathcal{O}_k \leq \mathcal{O} \uparrow \), and \( \mathcal{O} \downarrow \mathcal{O} \downarrow \mathcal{O} \downarrow \cdots \downarrow \mathcal{O}_k \) for all \( n \).
The bermudan rollover option $B^r$

Since $B^r$ is decreasing, $B^r_{\tau_n} \geq (B^n \uparrow B^r)_{\tau_n} = B^n_{\tau_n} \geq Z_{\tau_n}$.

As $B^r_0 = B^1_0$, it follows $B^r$ is a domineering option. Hence,

$$B^r = \max_{n=1,\ldots,k} (E^n \uparrow B^r);$$

and

$$B^1_0 = (\max_{n=1,\ldots,k} (E^n \uparrow B^r))_0.$$

The minimax duality formula $B^1_0 = \min_{O \in O^+} (\max_{n=1,\ldots,k} (E^n \uparrow O))_0$ now easily follows.

The bermudan stream $B^r$ (is associative and) satisfies the “stream rollover strategy” formula:

$$B^r = (\tau_1, B^1_{\tau_1}) \uparrow \cdots \uparrow (\tau_k, B^1_{\tau_k}).$$
Regenerative tenors and streams

A tenor $T^* = (T_n)_{n=1}^k$ is Regenerative at tenor $\tau^* = (\tau_n)_{n=1}^k$ if $\tau_k = T_k$, and for $n < k$, $\tau_n \leq T_n$ and $1_{\tau_n < T_n} T_n = 1_{\tau_n < T_n} T_{n+1}$.

Last condition can also be written as $T_n = 1_{\tau_n = T_n} \tau_n + 1_{\tau_n < T_n} T_{n+1}$, or, if preferred,

$1_{\tau_n < T_n} = 1_{\tau_n < T_n = T_{n+1}}$.

Then $T_n \in \{\tau_n, \cdots, \tau_k\}$, all $n$. By induction one shows

$$1_{t > \tau_n} (T_n, Z_n)_t = \sum_{i=n+1}^{k-1} 1_{\tau_{i-1} < t \leq \tau_i \land T_n} (T_i, Z_i)_t.$$  

Define the associated Regenerative Stream $Z_{T^*}$ by

$$Z_{T^*} := (Z_{T^*})_{n=1}^k, \quad Z_{T^*}^n := (T_n, Z_{T_n}).$$

Example: The optimal exercise tenor $T^*_*$, and the bermudan stream $B^* = Z_{T^*_*}$ are regenerative.
Primal-dual bounds for regenerative streams

For any regenerative stream, we have *primal-dual bounds*:

\[(T_1, Z_{T_1})_0 \leq B_0^1 \leq (\max_{n=1,\ldots,k} (\mathcal{E}^n \rightharpoonup Z_{\tau_n}^{\tau_n})))_0.\]

Various examples of regenerative streams in the literature aim at a tight (additive) primal-dual bounds (or a tight “duality gap”).

A regenerative stream satisfies

\[Z_{T_\tau_n}^{\tau_n} \leq \max_{n=1,\ldots,k} (\mathcal{E}^n \rightharpoonup Z_{\tau_n}^{\tau_n}).\]

If \(Z_{\tau_n} \leq (Z_{T_\tau_n}^{\tau_n})_{\tau_n}\) for all \(n\), this implies \(Z_{T_\tau_n}^{\tau_n}\) is domineering, hence equality holds and \((T_1, Z_{T_1})_0 = B_0^1 = (\max_{n=1,\ldots,k} (\mathcal{E}^n \rightharpoonup Z_{\tau_n}^{\tau_n})))_0.\)
Stream rollover strategies

A Stream Rollover Strategy is a pair \((\tau, O^\cdot)\), where \(\tau\) is a tenor and \(O^\cdot\) is a stream such that its expiry tenor is regenerative at \(\tau\), and 
\[
1_{\tau_n<T_n}O^n = 1_{\tau_n<T_n}O^{n+1} \text{ for all } n < k.
\]

Examples are \((\tau, Z_T^\cdot)\), where \(Z_T^\cdot\) is a regenerative trigger stream as above, e.g., \((\tau, B^\cdot)\)

The “strategy” is to rollover \(O^\cdot\) at the tenor \(\tau\).

**Theorem.** Let \((\tau, O^\cdot)\) be a stream rollover strategy such that \(O^\cdot\) is nonnegative and associative. Then

\[
O^{\cdot\uparrow} = (\tau_1, O^{1}_{\tau_1}) \uparrow \cdots \uparrow (\tau_k, O^{1}_{\tau_k}).
\]