SEMII IMPLICIT SECOND ORDER DISCONTINUOUS
GALERKIN CONVECTION FOR ALE CALCULATIONS

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Abstract. Element results are in general discontinuous across element boundaries. In
the ALE method and related moving element methods convection of these data with respect
to the element grid is required. The Discontinuous Galerkin Method provides an obvious
choice for discretization of this convective process.
In order to assure stability and accuracy at large step sizes (large values of the Courant
number), the Discontinuous Galerkin method is extended to second order. This is not
sufficient to obtain an attractive stability region. Therefore the equations are enriched with
selective implicit terms. This results in a remarkably stable convection scheme without the
use of any explicit limiting.

Results are shown of a standard pure advection test problem, the Molenkamp test and
of an extrusion simulation, which resembles convection with source terms
1 INTRODUCTION

In the Arbitrary Lagrangian Eulerian (ALE) Method the finite element mesh is allowed to move independently from the material. In this way it is possible to retain a good mesh quality even at large material distortions [1, 2, 3]. The outline of the method is shown in figures 1 and 2. The material moves with a velocity \( v_m \), the grid velocity is denoted as \( v_g \). A convective velocity \( v_c = v_m - v_g \) can be defined. The benefit of the good mesh quality is obtained at the cost of having to convect (usually discontinuous) finite element result data with respect to the mesh.

An often applied procedure is to first construct a continuous field based on nodal averaging and then apply the convection equation to this field, to obtain either the new value or the increment. It appears, that the construction of the continuous field causes a considerable amount of diffusion which on the one hand enhances the stability, but on the other hand deteriorates the accuracy [3]. The choice for the discontinuous Galerkin method is obvious because it does not require a continuous field for the discretization of the convection. Moreover, the result of the convection will again be discontinuous, which is in sync with the finite element theory.

The Discontinuous Galerkin method is well suited for solution of the convection equation on an element by element basis. Assembly of a system matrix and solution of con-
vection for all elements simultaneously is still possible (sometimes advantageous).

2 ONE DIMENSIONAL CONvection

Convection using the Discontinuous Galerkin Method and the extensions as proposed in this paper will be demonstrated on the one dimensional convection equation. Generalization to the multi-dimensional case is shown in section 3.

The discontinuous Galerkin method was introduced by Lesaint and Raviart [4] and is often used in viscoelastic computations [5, 6]. It is assumed that the convective displacement $u_c = v_c \Delta t$ is known based on the results of the previous Lagrangian calculation. In that case the convective increment $\Delta f$ of any element variable $f$ can be written as:

$$\Delta f = f(x - u_c) = f(x)$$  \hspace{1cm} (1)

The increment $\Delta f$ is written by means of Taylor expansion as:

$$\Delta f = -u_c \frac{df}{dx} + O(u_c^2)$$  \hspace{1cm} (2)
2.1 first order discretization

Based on equation (2) a Discontinuous Galerkin (DG) discretization can be derived for the increment $\Delta f$ in the $i^{th}$ element.

$$\int_{x_i}^{x_{i+1}} \bar{f} \Delta f \, dx = - \int_{x_i}^{x_{i+1}} \bar{f} u_c \frac{df}{dx} \, dx - \bar{f} u_c (f - f^{up}) \bigg|_{x_i}$$

Since the boundary condition is not satisfied at the inflow ($x_i$) of the element, the discrepancy between the value $f$ inside the element and the value $f^{up}$ in the upwind element is also weighted giving rise to the boundary term. After partial integration it follows:

$$\int_{x_i}^{x_{i+1}} \bar{f} \Delta f \, dx = \int_{x_i}^{x_{i+1}} \frac{df}{dx} u_c f \, dx + \bar{f} u_c f^{up} \bigg|_{x_i} - \bar{f} u_c f \bigg|_{x_{i+1}}$$

The value of $\Delta f$ is solved for on an element by element base. When $f$ and $\Delta f$ are discretized with constant values per element the integral in the right hand side disappears. The result is the finite volume equation which states that the increment $\Delta f$ is the balance of influx and outflux of $f$.

When discretized using a linear interpolation per element as in figure 3 this scheme is very unstable for practical values of the Courant number $Cr$. The Courant number is defined in one dimension as $Cr = u_c / L_e$. The scheme may be stabilized by formulating an implicit scheme. Also limiters are often used [7]. However we prefer not to use limiters and follow [8] and write an implicit form.

$$\int_{x_i}^{x_{i+1}} \bar{f} \Delta f \, dx = \int_{x_i}^{x_{i+1}} \frac{df}{dx} u_c (f + \alpha \Delta f) \, dx + \bar{f} u_c f^{up} \bigg|_{x_i} - \bar{f} u_c (f + \alpha \Delta f) \bigg|_{x_{i+1}}$$
All terms with $\Delta f$ are collected on the left hand side and $\Delta f$ is solved for. In [8] $\alpha = 1$ is used. The resulting convection scheme is extremely stable. There are no oscillations for Courant numbers up to 1 on a uniform grid. However the accuracy, especially in terms of the phase velocity is extremely bad (figure 4). This is clearly caused by the unbalanced ‘implicitness’ of the influx and the outflux. This is very easily remedied by adding a corresponding higher order term to the influx from the upwind element.

$$
\int_{x_i}^{x_{i+1}} \bar{f} \Delta f \, dx = \int_{x_i}^{x_{i+1}} \frac{df}{dx} u_c(f + \alpha \Delta f) \, dx
$$

$$
+ \bar{f} u_c \left( f_{up} - \alpha u_c \frac{df_{up}}{dx} \right) \bigg|_{x_i} - \bar{f} u_c (f + \alpha \Delta f) \bigg|_{x_{i+1}}
$$

Applied to one dimensional convection, this scheme is stable and accurate for high Courant numbers. As an example convection of a Gaussian bump on a uniform grid with Courant number $Cr = 5/6$ is shown in figure 5.

### 2.2 second order discretization

Inspired by the success of adding a higher order term to the implicit first order method of equation (5) we are tempted to explore what a second order method can bring. An interesting extension to a scheme of indefinitely high order is presented by [9]. In our
method the Taylor expansion (2) is extended with a second order term.

\[ \Delta f = -u_c \frac{df}{dx} + \frac{1}{2} u_c^2 \frac{d^2 f}{dx^2} + O(u_c^3) \]  

Again this is discretized by the DG method and after partial integration the following equation is found:

\[ \int_{x_i}^{x_{i+1}} \bar{f} \Delta f \, dx = \int_{x_i}^{x_{i+1}} \frac{df}{dx} u_c \left( f - \frac{1}{2} u_c \frac{df}{dx} \right) \, dx \]

\[ + \bar{f} u_c \left( f_{up} - \frac{1}{2} u_c \frac{df_{up}}{dx} \right) \Big|_{x_i} - \bar{f} u_c \left( f - \frac{1}{2} u_c \frac{df}{dx} \right) \Big|_{x_{i+1}} \]  

Disappointingly this scheme is only stable for small Courant numbers (\(Cr < 0.3\)) which is not of much help. Therefore it is again extended by adding implicit terms. We learned our lesson from the implicit first order scheme and resist the temptation of also making the outflux implicit.

\[ \int_{x_i}^{x_{i+1}} \bar{f} \Delta f \, dx = \int_{x_i}^{x_{i+1}} \frac{df}{dx} u_c \left( f + \alpha \Delta f - \frac{1}{2} u_c \frac{d(f + \beta \Delta f)}{dx} \right) \, dx \]

\[ + \bar{f} u_c \left( f_{up} - \frac{1}{2} u_c \frac{df_{up}}{dx} \right) \Big|_{x_i} - \bar{f} u_c \left( f - \frac{1}{2} u_c \frac{df}{dx} \right) \Big|_{x_{i+1}} \]
After some experimenting a good choice for $\alpha$ and $\beta$ is found as: $\alpha = 1/24; \quad \beta = 2/3$. The result is a very stable convection scheme ($Cr \leq 1$) with little diffusion and an excellent phase velocity accuracy. This is shown in figure 6 where a Gaussian bump is convected with a Courant number of 5/6. Another illustration is given in figure 7 where the uniform mesh is randomly perturbed such that the ratio of the shortest and longest element is 5/7 and such that the Courant number with respect to the shortest element is exactly 1.

3 MULTI DIMENSIONAL CONVECTION

In two or three dimensions equation (7) is written as:

$$\Delta f = -\mathbf{u}_c \cdot \nabla f + \frac{1}{2} \mathbf{u}_c \cdot \nabla \nabla f$$

This is written in the weak form while weighting the discontinuity over the inflow boundary.

$$\int_V \bar{f} \Delta f \, dV = - \int_V \bar{f} \mathbf{u}_c \cdot \nabla f \, dV + \sum_{N_e} \int_{\Gamma_{in}} \bar{f} \mathbf{u}_c \cdot \mathbf{n} (f - f^{up}) \, dA$$

$$+ \frac{1}{2} \int_V \bar{f} \mathbf{u}_c \cdot (\nabla \nabla f) \cdot \mathbf{u}_c \, dV - \sum_{N_e} \frac{1}{2} \int_{\Gamma_{in}} \bar{f} (\mathbf{u}_c \cdot \mathbf{n}) \{ \mathbf{u}_c \cdot (\nabla f - \nabla f^{up}) \} \, dA$$

(11)
After partial integration the terms with \( \alpha \) and \( \beta \) are added like in equation (9) and all terms containing \( \Delta f \) are collected in the left hand side.

\[
\int_V \bar{f} \Delta f \, dV - \alpha \int_V ((\nabla \bar{f}) \cdot \mathbf{u}_c + \bar{f}(\nabla \cdot \mathbf{u}_c)) \Delta f \, dV + \\
\frac{1}{2} \beta \int_V ((\nabla \bar{f}) \cdot (\mathbf{u}_c \mathbf{u}_c) + \bar{f}(\nabla \cdot \mathbf{u}_c \mathbf{u}_c + \mathbf{u}_c \cdot (\nabla \mathbf{u}_c))) \cdot \nabla \Delta f \, dV = \\
\int_V ((\nabla \bar{f}) \cdot \mathbf{u}_c + \bar{f}(\nabla \cdot \mathbf{u}_c)) f \, dV - \sum_{N_e} \int_{\Gamma_{in}} \bar{f} \mathbf{u}_c \cdot \mathbf{n} f^{up} \, dA - \sum_{N_e} \int_{\Gamma_{out}} \bar{f} \mathbf{u}_c \cdot \mathbf{n} f \, dA - \\
\frac{1}{2} \int_V ((\nabla \bar{f}) \cdot (\mathbf{u}_c \mathbf{u}_c) + \bar{f}(\nabla \cdot \mathbf{u}_c \mathbf{u}_c + \mathbf{u}_c \cdot (\nabla \mathbf{u}_c))) \cdot \nabla f \, dV + \\
\sum_{N_e} \frac{1}{2} \int_{\Gamma_{in}} \bar{f} (\mathbf{u}_c \cdot \mathbf{n}) (\mathbf{u}_c \cdot \nabla f^{up}) \, dA + \sum_{N_e} \frac{1}{2} \int_{\Gamma_{out}} \bar{f} (\mathbf{u}_c \cdot \mathbf{n}) (\mathbf{u}_c \cdot \nabla f) \, dA
\] (12)

The leading second order term in the right hand side,

\[
\int_V \nabla \bar{f} \cdot (\mathbf{u}_c \mathbf{u}_c) \cdot \nabla f \, dV
\]

can be interpreted as a diffusion with the direction dependent diffusion coefficient tensor \( \mathbf{u}_c \mathbf{u}_c \).
The Discontinuous Galerkin method is implemented for 6-node quadratic triangular elements with 3 integration points. This means that the interpolation of element variables is linear and discontinuous over element boundaries. Inspection of equation (12) shows that the boundary fluxes are of degree 5, whereas the area integrals involve polynomials of degree 4. For reasons of conservativity the boundary flux integrals are evaluated as accurately as possible. Three integration points per edge are used. For evaluation of the area integrals these points are used also plus one extra integration point in the center. The integration point locations and weight factors are summarized in table 1.

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<th>int. point</th>
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<th>$L_2$</th>
<th>$L_3$</th>
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<th>$W_{\text{boundary}}$</th>
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Table 1: Integration point locations and weight factors

5 SIMULATIONS

5.1 Molenkamp test [10]

The Molenkamp test is a more or less standard test for pure advection. The $f$-field is discretized using triangular elements with a linear distribution within each element. A square of size 2x2 is divided into 1022 elements. The initial value of the field to be
Figure 9: Molenkamp test using the semi implicit second order DG method Eq. (12), final profile.

convected is a discontinuous least squares approximation of a Gaussian bump (figure 8).

\[ f = 0.01 r^2 \]
\[ r = \sqrt{(x + \frac{1}{2})^2 + y^2} \]  \hspace{1cm} (13)

Figure 10: Evolution of minimum and maximum \( f \) value during molenkamp test with different number of steps.
The Gaussian bump is rotated about the center of the domain by one full revolution. In figure 9 the final distribution is shown when the rotation is done in 240 and in 640 steps. In the simulation with 240 steps the slight disturbance in the corners (where the velocity is highest) is clearly visible. Also the peak value of \( f \) has grown by approximately 20\%. 240 Steps is too few for one revolution. A simulation with 320 steps would show no more disturbances. The results of the simulation with 640 steps are almost exact.

In figure 10 the evolution of the maximum and minimum values of the nodal averaged \( f \) is given. It appears that on this mesh 640 steps will give no more growth of the peak height. For all simulations the minimum value of the nodal averaged \( f \) remains less than 1% below zero level.

### 5.2 ALE CALCULATIONS

![Figure 11: Axissymmetric extrusion, mesh and eventual distribution of equivalent plastic strain.](image)

In the Molenkamp test the convective displacements are such that the individual el-

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lements rotate but do not change shape. In a realistic ALE simulation the convective displacements will cause changes in the shapes of the elements. The applicability of the method is demonstrated on a simulation of an axial symmetric extrusion. At the end of each Lagrangian step the nodal point coordinates are restored to their original positions. The convective displacement increment equals the material displacement increment.

In figure 11 the eventual distribution of equivalent plastic strain is shown. For plotting purposes from the discontinuous element data average nodal data are calculated. Note that the in the outflow region the equivalent plastic strain only shows very little cross wind diffusion, which seems to be primarily connected to the coarsening of the grid in downstream direction.

6 CONCLUSIONS

In ALE simulations the grid is allowed to move independently from the material. This gives rise to the necessity for convection of the element data. The element data to be convected are discontinuous across element boundaries. The obvious choice for discretisation of the convection is the Discontinuous Galerkin method.

A discretization is shown based on a second order Taylor expansion of the convection equation. The discretized equations are enriched with selective implicit terms. The resulting scheme has excellent stability properties. This is demonstrated on a standard pure advection simulation and on a simulation of an extrusion process.

The extent of the stability region is currently investigated, as well as the role of the implicitness parameters $\alpha$ and $\beta$. 
REFERENCES


