Degenerate polynomial patches of degree 4 and 5 used for geometrically smooth interpolation in $\mathbb{R}^3$

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Abstract

The problem of interpolating scattered 3D data by a geometrically smooth surface is considered. A completely local method is proposed, based on employing degenerate triangular Bernstein–Bézier patches. An analysis of these patches is given and some numerical experiments with quartic and quintic patches are presented.

Keywords: Local interpolation in $\mathbb{R}^3$; Geometric smoothness; Bernstein–Bézier representation; Degenerate polynomial triangular patches

1. Introduction

In the past years, there has been a wide interest in the problem of reconstructing surfaces from discrete points scattered in the three dimensional Euclidean space. The 3D-reconstruction problem has important applications in
many scientific disciplines, such as medicine, engineering and computer aided geometric design. In this paper we investigate one subproblem encountered in the general 3D-reconstruction, which can be formulated as follows.

Find a geometrically smooth \( (GC^1) \) surface interpolating a set of points in \( \mathbb{R}^3 \), of which a triangulation is given, together with associated normal vectors.

Constructing a triangulation in \( \mathbb{R}^3 \) from a set of points is not an easy task. Here we assume that a triangulation is already given. Note that the triangulation represents significantly more information than the individual points alone, since it defines the “topology” of the reconstructed surface. For information on methods for constructing such a triangulation we refer the reader to (Schumaker, 1990; Veltkamp, 1992) and references therein. Every triangulation of a set of points defines a piecewise linear interpolating surface. Hence, the 3D-reconstruction we are considering can also be viewed as “smoothing” of this piecewise linear surface.

By geometric smoothness \( (GC^1) \) we mean here the oriented tangent plane continuity, in accordance with the commonly accepted terminology (Boehm, 1990; DeRose, 1985).

There exist different approaches to the 3D-reconstruction which are not based on triangulations. Such methods will not be considered here and we do not include any consideration of or references to the functional, i.e., non-parametric case.

Here, we restrict ourselves to local methods, which seem to be inevitable in the context of general 3D-reconstruction problems. Local methods give us an easier control over the shape of the reconstructed surface. Global methods usually impose conditions on the size of the problem (they lead to large systems) and on the form of the surfaces (e.g. star-like (Schumaker, 1990)). Various methods of local interpolation have been surveyed and explained in (Peters, 1990a). Following the terminology in (Peters, 1990a) we refer to the single patch approach (Herron, 1987; Peters, 1988a; Peters, 1989; Peters, 1990b) the blending approach (Gregory, 1974; Gregory and Hahn, 1987; Nielson, 1987; Shirman and Séquin, 1990) and the splitting approach (Cottin and van Damme, 1990; Farin, 1983; Peters, 1988b; Peters, 1990d; Piper, 1987; Shirman and Séquin, 1987). This classification does not include another approach which uses subdivision (Dyn et al., 1990).

Essential to our investigations is the concept of locality. We call a method completely local if the resulting surface consists of triangular patches with each patch depending only on data associated with their corresponding triangle from the given triangulation. Note that a number of local methods are not completely local since a triangular patch may depend on data of neighbouring patches.

In case the normals associated with positional values are not available, they have to be chosen or estimated from positional data. It is known that the choice of normals affects heavily the quality of the surface. Therefore, an accurate
estimation of normals is an issue of significant importance. This estimation, however, cannot be made on the basis of the data associated with a single triangle since at least the information from all triangles surrounding the point at which the normal is being estimated has to be taken into account. Thus, it only makes sense to consider a completely local method once the normals are determined. Normal estimation is a problem of considerable difficulty, which we do not consider here. Instead, we refer the reader to (Sloan, 1992).

Another crucial property of a method is given by the kind of functions representing the surface patches. In this respect it is common to make a distinction between polynomial and non-polynomial methods. Considering existing methods we make the following observations.

- The single patch and splitting approaches lead to polynomial methods which are not completely local (they are local in a weaker sense).
  Blending methods are in general completely local.

- The existing blending methods are not polynomial.
  We have studied the blending methods described in (Gregory, 1974; Herron, 1985; Nielson, 1987). In these cases the patches are constructed only from “boundary information” leading to completely local interpolants. Polynomial patch constructions (including splitting methods) rely on $G^1$ conditions between neighbouring patches, thus giving rise to interrelations between polynomial coefficients (also called control points) (Degeen, 1990; DeRose, 1990; Farin, 1982; Farin, 1983; Hosaka and Kimura, 1984; Liu and Hoschek, 1989; Piper, 1987) and thereby violating the complete locality.

Often, the geometric smoothness is defined by means of a regular reparametrization (DeRose, 1985; Hahn, 1989). This implies that the unit normal vector varies continuously over the surface, which is a parametrization independent property. Blending methods employ non-regular parametrizations and they are $G^1$ in the sense that they lead to surfaces with continuously varying unit normal, but are not $G^1$ in the sense of the existence of a regular reparametrization.

The fact that blending methods are completely local can be considered a big advantage over other methods. Note, that they can be viewed as a generalization of the univariate Hermite cubic interpolation, which is also completely local. Unfortunately, existing blending methods are not polynomial and polynomial methods are preferred for obvious reasons. Therefore, our objective in this paper is to find and study a method, which is both completely local and polynomial.

The paper is organized as follows. In Section 2 we introduce the notation and present the basic ideas. Then, in Section 3, we show that requiring a method to be completely local and polynomial at the same time imposes strong restrictions. Such a method requires the use of special polynomial patches called degenerate polynomial patches with coalescent control points. We describe the cases of quartic (Section 4) and quintic (Section 5) patches in more detail and present some examples (Section 6). A short announcement of our results has appeared in (Pfluger and Neamtu, 1991). Finally, we mention that the
idea of employing singular parametrizations has independently attracted Peters (Peters, 1990c) in connection with the so called vertex enclosure problem. The use of degenerate patches has also been suggested by Du and Schmitt (see (Du and Schmitt, 1991) and references therein).

2. A completely local interpolating scheme

In this section we explain the basic ideas underlying our approach. Let us first establish the following notation:

\[ \mathbf{V} := \{ V_l | V_l \in \mathbb{R}^3, l = 1, \ldots, v \} \]

will denote a set of distinct vectors, called vertices,

\[ \Delta := \{ \Delta_l = \{ V_{i1}, V_{i2}, V_{i3} \} \subset \mathbf{V}, l = 1, \ldots, t \} \]

denotes a triangulation of \( \mathbf{V} \), and

\[ \mathbf{N} := \{ N_l | N_l \in \mathbb{R}^3 \backslash \{0\}, l = 1, \ldots, v \} \]

will stand for a set of vectors, called normals, one associated with each vertex. To emphasize the fact that the length of a normal is immaterial we consider in general not the given vector \( N_l \) but the equivalence class of vectors having the same direction and orientation as \( N_l \). To indicate that two vectors \( N^1, N^2 \) belong to the same equivalence class we employ the notation \( N^1 \simeq N^2 \), i.e., \( N^1 = cN^2 \) for \( c > 0 \).

The symbol

\[ S := \left\{ \lambda = (\lambda_1, \lambda_2, \lambda_3), \sum_{i=1}^{3} \lambda_i = 1, \lambda_i \geq 0, i = 1, 2, 3 \right\} \]

denotes the standard 2-simplex, which will be thought of as a triangle in \( \mathbb{R}^3 \) with vertices \( X_1 = (1, 0, 0), X_2 = (0, 1, 0), X_3 = (0, 0, 1) \); \( e_1, e_2, e_3 \) are the edges of \( S \) (see Fig. 1). The variables \( \lambda = (\lambda_1, \lambda_2, \lambda_3) \) are the so called barycentric coordinates. Instead of the indices 1, 2, 3 also \( i, j, k \) will be used, with the convention that \( i, j, k \) represent cyclic permutations of 1, 2, 3 , i.e.,

\[ (i, j, k) \in I := \{ (1, 2, 3), (2, 3, 1), (3, 1, 2) \} \].

![Fig. 1. Notation.](image-url)
With this notation we consider the following interpolation problem: Find a \( G^C_1 \) surface \( P \) interpolating \( V \) and \( N \). \( P \) will be represented by a collection of triangular patches \( P_i, i = 1, \ldots, t \). Each patch is an image of the standard simplex \( S \) under a map \( M_i : S \rightarrow \mathbb{R}^3 \), i.e., \( P_i = M_i(S) \). The patches will be determined such that \( P_i \) interpolates \( V \) and \( N \) restricted to \( A_i \), and such that all patches combine to a \( G^C_1 \) surface.

With every map \( M_i \) we also consider a normal map \( N[M_i] \) such that \( N[M_i](\lambda) \) is a nonzero normal vector of \( P_i \) at \( \lambda \in S \). A set of vectors \( Q_i = \{ N[M_i](\lambda), \lambda \in S \} \) will be called a normal patch of \( P_i \). The \( G^C_1 \) property of \( P_i \) is equivalent to the existence of a continuous normal patch to \( P_i \). We impose the following requirements on \( M_i \).

(I) **Smoothness** \( P_i = M_i(S) \) is \( G^C_1 \).

(II) **Interpolation**

\[
V_i = M_i(x_i), \quad N_i \approx N[M_i](x_i), \quad j = 1, 2, 3.
\]

(III) **Complete locality** \( P_i \) depends only on the data from \( A_i \), i.e., on \( V_i, V_h, V_h \) and \( N_i, N_h, N_h \).

Observe, that (I)–(III) alone, are not sufficient to obtain an overall \( G^C_1 \) surface \( P \), since no conditions on the smooth join between abutting patches have been imposed. However, imposing continuity conditions would violate locality (III). The only way to guarantee that \( P \) is \( G^C_1 \) is to require that \( M_i \) satisfy the following still stronger locality requirement.

(IV) **Independence** The restrictions of \( P_i \) and \( Q_i \) to an edge \( e_j \) depend uniquely on the data associated with that edge, i.e., they are independent of \( V_i, N_i \) and \( l \).

If \( \{ M_i \}_{i=1}^t \) is a collection of maps, such that each map satisfies (I)–(IV), then the corresponding surface \( P : = \{ P_i = M_i(S), i = 1, \ldots, t \} \) is a solution to the interpolation problem. A method based on this principle is completely local. A consequence of (III) is that the maps \( M_i \) do not interact and thus can be constructed separately. Hence, we omit the index \( l \) for the remainder of the paper.

The curves \( b_j : = M(e_j) \) and \( n_j : = N[M](e_j), \quad j = 1, 2, 3 \) are referred to as boundary curves and boundary normal curves, respectively. The boundary curves \( b_j \) and boundary normal curves \( n_j \) cannot be chosen independently to each other. If \( b_j \) is \( G^C_1 \), then the following must hold.

(V) **Compatibility** \( t_j(x_k + s(x_l - x_k)) \bot n_j(x_k + s(x_l - x_k)), \quad s \in [0, 1], \) where \( t_j \) is the unit tangent vector along \( b_j \).

The following blending technique is a natural way to construct a completely local map \( M \) in two stages, A1 and A2.
A1: Determine \( G^C_1 \) boundary curves \( b_j \) and continuous boundary normal curves \( n_j \) \((j = 1, 2, 3)\), satisfying (II), (IV), (V).

A2: Determine \( M \), satisfying (I), (III) and such that,

\[
\begin{align*}
b_j &= M(e_j), \\
n_j &\approx N(M)(e_j), \quad j = 1, 2, 3.
\end{align*}
\] (2.1)

Thus, smoothness constraints across boundaries between pairs of patches are separated and, as it will be seen later, they are linear. This idea is first due to Sabin (Sabin, 1968) and has subsequently been used in (Peters, 1990b).

The condition (2.1) requires that \( M \) interpolate not only the direction but also the orientation of \( n_j \). This will be referred to as strong interpolation (or simply interpolation). In contrast, weak interpolation requires that the curves \( n_j \) interpolate only the direction but not necessarily the orientation of the normals \( N_i, N_k \). For weak interpolation the symbol \( \sim \) will be used instead of \( \approx \). Note, that at first it seems that (2.1) cannot be replaced by weak interpolation, since this would guarantee, in the terminology of (DeRose, 1985), only a weak geometric smoothness between adjacent patches, which would not necessarily exclude cusps. It is well known, that the weak \( G^C_1 \) is easier to achieve, however. Lemma 2.1 below allows us to weaken the interpolation conditions in our situation and still obtain \( G^C_1 \) smoothness. Thus, we shall replace A1 and A2 by the following simpler construction, B1 and B2.

B1: Determine \( G^C_1 \) boundary curves \( b_j \) and continuous boundary normal curves \( n_j \) \((j = 1, 2, 3)\), satisfying (IV), (V) and the following interpolation conditions:

\[
\begin{align*}
b_j(X_i) &= V_i, \\
b_j(X_k) &= V_k,
\end{align*}
\] (2.2)

and

\[
\begin{align*}
n_j(X_i) &\approx N_i, \\
n_j(X_k) &\approx N_k.
\end{align*}
\] (2.2)

B2: Determine \( M \), satisfying (I), (III) and such that,

\[
\begin{align*}
b_j &= M(e_j), \\
n_j &\approx N(M)(e_j), \\
N_j &\approx N(M)(X_j), \quad j = 1, 2, 3.
\end{align*}
\] (2.3)

To show that B1 and B2 are equivalent to A1 and A2 we need the next lemma.

Lemma 2.1. Let \( M^1, M^2 \) be two \( G^C_1 \) surface patches which join weakly \( G^C_1 \) along their common boundary. Then they join \( G^C_1 \) iff they join \( G^C_1 \) at one point.

Proof. Let the common boundary be corresponding to the edge \( e_t \) of \( S \). i.e.,

\[
M^1(X_2 + s(X_3 - X_2)) = M^2(X_2 + s(X_3 - X_2)), \quad s \in [0, 1].
\]

Let \( M^1 \) and
$N^2$ be normal patches to $M^1$ and $M^2$, respectively. Since $N^1$ and $N^2$ are continuous on $e_1$ and $N^1 \sim N^2$, there is a continuous scalar function $f$ such that

$$N^1(x_2 + s(x_3 - x_2)) = f(s)N^2(x_2 + s(x_3 - x_2)), \quad s \in [0, 1].$$

Because $N^1$ and $N^2$ are normal patches, $f$ cannot vanish for $s \in [0, 1]$. Moreover, by the assumption that the two patches join $GC^1$ at one point of the boundary, $f$ is positive at that point. Therefore, by continuity, $f$ must be positive everywhere on $[0, 1]$, which is equivalent to saying that $M^1$ and $M^2$ join $GC^1$. 

From Lemma 2.1 it follows that the construction $B_1, B_2$ is equivalent to $A_1, A_2$. Thus, it also gives rise to a composite $GC^1$ surface $P$, since on account of (2.3), any two abutting patches join weakly $GC^1$ and in two points, namely in the two common vertices, they join (strongly) $GC^1$.

Finally, we make some remarks on the interpolation conditions (2.2). It might seem that the orientation of the boundary normal is immaterial and it should not be necessary to require that $n_j$ strongly interpolate $N_i, N_k$. However, as is shown in Lemma 2.2, it is not possible to construct the map $M$ (step $B_2$ of the algorithm) if the boundary normal does not satisfy (2.2) (or, alternatively, $n_j(x_i) \approx -N_i, n_j(x_k) \approx -N_k$). Without loss of generality in the following lemma we consider the edge $e_1$ only.

**Lemma 2.2.** If $M$ interpolates either $N_2, N_3$ or $-N_2, -N_3$, weakly interpolates $n_1$, and if either $n_1(x_2) \approx N_2, n_1(x_3) \approx -N_3$ or $n_1(x_2) \approx -N_2, n_1(x_3) \approx N_3$, then $M$ is not $GC^1$ at some point in $e_1$.

**Proof.** The proof is similar to the proof of Lemma 2.1. We show by contradiction that if $M$ interpolates either $N_2, N_3$ or $-N_2, -N_3$, weakly interpolates $n_1$ and $n_1(x_2) \approx N_2, n_1(x_3) \approx -N_3$, then $M$ is not $GC^1$ at some point inside $e_1$. Namely, $GC^1$ of $M$ along $e_1$ implies that $N(M)$ is continuous and non-zero along $e_1$. Since $n_1$ is continuous and $M$ interpolates weakly $n_1$, we have

$$N(M)(x_2 + s(x_3 - x_2)) = f(s)n_1(x_2 + s(x_3 - x_2)), \quad s \in [0, 1]$$

where $f$ is some scalar continuous function. Since $M$ interpolates $N_2, N_3$ or $-N_2, -N_3$ and $n_1$ interpolates $N_2, -N_3$ or $-N_2, N_3$, it follows that $f(0)f(1) < 0$. Hence, by continuity of $f$, $f$ is zero at some point in $(0, 1)$. This contradicts that $N(M)$ is nonzero in $e_1$, i.e., also that $M$ is $GC^1$. 

**3. Degenerate polynomial patches**

In this section we shall assume that $M$ is a polynomial $p_n$ of total degree $\leq n$. With every regular patch $p_n$ we can associate a normal patch $q_m$, which is a polynomial of total degree $m \leq 2n - 2$, given e.g. by
where $D_{\alpha_1} \times D_{\alpha_2}$ is the cross product operator and $\alpha_1, \alpha_2$ are two independent directions. The ordered pair $(\alpha_1, \alpha_2)$ is assumed to be consistent with the orientation of $p_n$, given by the counter-clockwise ordering of the vertices of $S$. We shall employ directional derivatives in specific directions depending on the edges. Namely,

$$D_i := \frac{\partial}{\partial \lambda_i} - \frac{\partial}{\partial \lambda_k},$$

using the directions $X_k - X_i$, $(i,j,k) \in I$. The normal patch can thus be calculated e.g. by

$$q_m = (D_k \times D_i)p_n.$$ 

We shall employ the Bernstein–Bézier representation for both $p_n$ and $q_m$, and denote respectively $P := \{P_r\}_{|r|=n}$ and $Q := \{Q_r\}_{|r|=m}$, the control points of $p_n$ and $q_m$ (Boehm et al., 1984; Farin, 1986). In particular, we have

$$p_n(\lambda) = \sum_{|\gamma| \leq n} P_{r} \frac{n!}{p!} \lambda^r, \quad q_m(\lambda) = \sum_{|\gamma| \leq m} Q_{r} \frac{m!}{p!} \lambda^r,$$

where $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{Z}_+^3, |\gamma| = \gamma_1 + \gamma_2 + \gamma_3$ and $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in S$.

The completely local method described by B1 and B2 in the case where the map $M$ is represented by a polynomial is now considered in more detail.

**Choice of boundary curves.** Without loss of generality consider the boundary curve $b_1$. The objective is to determine the control points $P_{0,\gamma_2,\gamma_3}, \gamma_2, \gamma_3 \geq 0, \gamma_2 + \gamma_3 = n$, such that the resulting curve interpolates the vertices and normals. The former is readily fulfilled by setting $P_{0,\gamma_2,\gamma_3} = V_2, P_{0,0,\gamma_3} = V_3$. For the latter the tangent vectors at both ends of $b_1$ have to be orthogonal to $N_2, N_3$, respectively. In general, there exist many boundary curves satisfying these conditions.

**Choice of boundary normal curves.** Again consider the edge $e_1$ only. The objective is to determine $n_1$, represented in terms of control points $Q_{0,\gamma_2,\gamma_3}, \gamma_2, \gamma_3 \geq 0, \gamma_2 + \gamma_3 = n$, such that the resulting curve interpolates $N_2$ and $N_3$ (i.e., $Q_{0,n,0} \approx N_2, Q_{0,0,n} \approx N_3$) and is compatible with $b_1$. The compatibility condition stated in (V) leads for the edge $e_1$ to the equation

$$D_1(p_n(\lambda)) \cdot q_m(\lambda) = 0, \quad \lambda_1 = 0.$$

Hence, the tangent vector to $b_1$ at each point of the boundary curve is perpendicular to the normal vector $n_1$. The left hand side of the equation (3.2) is a polynomial in $\lambda_2$ ($\lambda_3 = 1 - \lambda_2$) of degree $m + n - 1$ which must be identically
equal to zero. Setting all coefficients of this polynomial equal to zero leads to the following equivalent equations for the control points of $p_n$ and $q_m$:

$$Q^1 \cdot P_b^1 = 0,$$  \hspace{1cm} (3.3)

where $Q^1$ is an $(m + n) \times n$ matrix containing vectors from $\mathbb{R}^3$ as entries,

$$Q^1 := \begin{pmatrix}
\binom{m}{0} Q_{0,m,0} & 0 & \ldots & 0 \\
\binom{m}{1} Q_{0,m-1,1} & \binom{m}{0} Q_{0,m,0} & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & \binom{m}{m} Q_{0,0,m} & \ldots & \binom{m}{0} Q_{0,m,0} \\
\vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & \binom{m}{m} Q_{0,0,m}
\end{pmatrix},$$

and $P_b^1$ is an $n$-vector containing vectors as entries

$$P_b^1 := \begin{pmatrix}
\binom{n-1}{0} (p_{0,n,0} - p_{0,n-1,1}) \\
\vdots \\
\binom{n-1}{n-1} (p_{0,1,n-1} - p_{0,0,n})
\end{pmatrix}.$$

In the multiplication in (3.2) and (3.3) it is understood that the product of two vectors from $\mathbb{R}^3$ is the ordinary dot product. The system (3.3) will be referred to as the *local-edge system* for the edge $e_i$. The local-edge systems for the remaining boundary curves can be obtained from (3.3) by cyclic permutation of indices. Again, as in the case of boundary curves, there are usually many possible solutions of these equations.

**Determination of interior control points.** This stage $B_2$ of the algorithm amounts to determining the yet unknown *interior control points*, i.e., $P_i, y_i > 1, i = 1, 2, 3$. They have to be chosen such that the condition $n_1 \sim N(M)(e_1)$ is satisfied. In other words,

$$D_1 p_n(\lambda) \cdot q_m(\lambda) = 0, \quad \lambda_1 = 0,$$  \hspace{1cm} (3.4)

$$D_2 p_n(\lambda) \cdot q_m(\lambda) = 0, \quad \lambda_1 = 0.$$  \hspace{1cm} (3.5)

The equality (3.4) is automatically satisfied, since it is identical with (3.2). The equality (3.5) can in a similar manner be written as

$$Q^1 \cdot P_c^1 = 0,$$  \hspace{1cm} (3.6)
where
\[
P_c^1 := \begin{pmatrix}
{n-1 \choose 0} (P_{1,n-1,0} - P_{0,n-1,1}) \\
\vdots \\
{n-1 \choose n-1} (P_{1,0,n-1} - P_{0,0,n})
\end{pmatrix},
\]

Analogously, for the other two edges we have,
\[
\begin{align*}
Q_2 \cdot P_c^2 &= 0 \\
Q_3 \cdot P_c^3 &= 0,
\end{align*}
\]

which can be obtained by cyclic permutation of indices in (3.6). We call the system (3.6)-(3.8) the local-triangle system.

These equations are solved in the step B2 of the method. However, it is not difficult to see that in general this system has no solution. Note for instance, that the second to last equation in (3.6) and the second equation in (3.7) form a rank deficient system. A straightforward calculation gives the following necessary conditions for the existence of a solution.

**Lemma 3.1.** The system (3.6)-(3.8) is solvable only if,
\[
\begin{align*}
Q_{m-1,1,0} \cdot (P_{n-1,0,1} - P_{n,0,0}) &= Q_{m-1,0,1} \cdot (P_{n-1,1,0} - P_{n,0,0}) \\
Q_{0,m-1,1} \cdot (P_{1,n-1,0} - P_{0,n,0}) &= Q_{1,m-1,0} \cdot (P_{0,n-1,1} - P_{0,0,n}) \\
Q_{1,0,m-1} \cdot (P_{0,1,n-1} - P_{0,0,n}) &= Q_{0,1,m-1} \cdot (P_{1,0,n-1} - P_{0,0,n}).
\end{align*}
\]

This result asserts that, in general, there exists no completely local polynomial method since (3.9) imposes interrelations between control points belonging to different edges, which is inconsistent with (IV), unless these conditions are automatically satisfied by considering a special class of patches. There are two "symmetric" ways to define such a class, either by setting
\[
\begin{align*}
P_{n,0,0} &= P_{n-1,1,0} = P_{n-1,0,1} \\
P_{0,n,0} &= P_{0,n-1,1} = P_{1,n-1,0} \\
P_{0,0,n} &= P_{1,0,n-1} = P_{0,1,n-1}
\end{align*}
\]

or
\[
\begin{align*}
Q_{m-1,1,0} &= Q_{m-1,0,1} = 0 \\
Q_{0,m-1,1} &= Q_{1,m-1,0} = 0 \\
Q_{1,0,m-1} &= Q_{0,1,m-1} = 0.
\end{align*}
\]

We have decided to design a method for polynomial patches satisfying conditions (3.10) and we did not further consider the conditions (3.11). Therefore, we shall employ patches with three coalescent control points at vertices, called
Before we investigate the interpolation method using these patches, we analyse their smoothness properties. Since we are employing non-regular parametrizations at vertices, geometric continuity is not automatically satisfied.

**Degenerate patches.** Before we investigate the interpolation method using these patches, we analyse their smoothness properties. Since we are employing non-regular parametrizations at vertices, geometric continuity is not automatically satisfied.

**Geometric smoothness of degenerate patches.** Let us first consider the case of a degenerate polynomial curve $b_n$ defined as

$$b_n(\lambda) := \sum_{|\gamma|=n} B_\gamma \frac{n! \lambda^\gamma}{\gamma!}, \quad B_\gamma \in \mathbb{R}^3,$$

where $\lambda = (\lambda_1, \lambda_2), \gamma = (\gamma_1, \gamma_2) \in \mathbb{Z}_+^2$. We assume $b_n$ to be degenerate at the point $(1,0)$ in the sense that

$$B_{n,0} = B_{n-1,1} \neq B_{n-2,2}.$$

The unit tangent vector of $b_n$ at $(1,0)$ will be (see Fig. 2)

$$t = \frac{B_{n-2,2} - B_{n,0}}{\|B_{n-2,2} - B_{n,0}\|}.$$

This follows from the fact that,

$$t = \lim_{\lambda_1 \to 0} \frac{(B_{n-2,2} - B_{n,0})\lambda_1^{n-2}\lambda_2 + O(\lambda_2^2)}{\|(B_{n-2,2} - B_{n,0})\lambda_1^{n-2}\lambda_2 + O(\lambda_2^2)\|}.$$

Next, we analyse a similar problem for triangular patches. Let us consider the polynomial patch $p_n$ given by (3.1), which is degenerate at $X_1$ in the sense that

$$P_{n,0,0} = P_{n-1,1,0} = P_{n-1,0,1} \neq P_{n-2,0,2} - \gamma_3 \gamma_3, \quad \gamma_2 + \gamma_3 = 2$$

and such that $(P_{n-2,2,0} - P_{n,0,0}), (P_{n-2,1,1} - P_{n,0,0}), (P_{n-2,0,2} - P_{n,0,0})$ are pairwise linearly independent. In the following lemma the conditions for $GC^1$ smoothness of such degenerate patch are stated.

**Lemma 3.2.** The degenerate polynomial $p_n$ is $GC^1$ at $X_1$ iff the following conditions are satisfied (see Fig. 3):

(a) $P_{n,0,0}, P_{n-2,2,0}, P_{n-2,1,1}, P_{n-2,0,2}$ are coplanar.
Fig. 3. Conditions for $GC^1$ in $P_{n,0,0}$: $P_{n,0,0}, P_{n-2,2,0}, P_{n-2,1,1}, P_{n-2,2,1}$ span a plane and $P_{n-2,1,1}$ must lie in the dashed region.

(b) either $P_{n-2,1,1} - P_{n,0,0}$ or $-(P_{n-2,1,1} - P_{n,0,0})$ is contained in the cone spanned by $P_{n-2,0,2} - P_{n,0,0}$ and $P_{n-2,0,2} - P_{n,0,0}$. In the former case the unit normal vector of $p_n$ in $X = X_1$ is

$$r = \frac{(P_{n-2,2,0} - P_{n,0,0}) \times (P_{n-2,0,2} - P_{n,0,0})}{\| (P_{n-2,2,0} - P_{n,0,0}) \times (P_{n-2,0,2} - P_{n,0,0}) \|},$$

while in the latter, the unit normal vector is $-r$.

The condition (a) implies weak $GC^1$ smoothness only, while (b) guarantees also the continuity of the orientation of the normal at $X_1$.

**Proof.** Let us consider the tangent plane of the patch $p_n$ at the point $X = (1 - \epsilon, s\epsilon, (1 - s)\epsilon) \in S, \epsilon > 0, s \in [0, 1]$, which is regular. This plane is spanned by the two vectors

$$W_1(s, \epsilon) = ((P_{n-2,2,0} - P_{n,0,0})s\epsilon + (P_{n-2,1,1} - P_{n,0,0}))(1 - s)\epsilon + O(\epsilon^2),$$
$$W_2(s, \epsilon) = ((P_{n-2,0,2} - P_{n,0,0})(1 - s)\epsilon + (P_{n-2,1,1} - P_{n,0,0}))s\epsilon + O(\epsilon^2).$$

Thus, the unit normal vector of this plane is

$$\frac{W_1(s, \epsilon) \times W_2(s, \epsilon)}{\| W_1(s, \epsilon) \times W_2(s, \epsilon) \|}.$$

This expression has a limit for $\epsilon \to 0$ which is independent of $s$ only if the following equality holds:

$$\frac{(P_{n-2,2,0} - P_{n,0,0}) \times (P_{n-2,1,1} - P_{n,0,0})}{\| (P_{n-2,2,0} - P_{n,0,0}) \times (P_{n-2,1,1} - P_{n,0,0}) \|} = \frac{(P_{n-2,1,1} - P_{n,0,0}) \times (P_{n-2,0,2} - P_{n,0,0})}{\| (P_{n-2,1,1} - P_{n,0,0}) \times (P_{n-2,0,2} - P_{n,0,0}) \|}.$$

However, this is obviously equivalent to the conditions (a) and (b) of the lemma. $\square$
4. The quartic method

Obviously, a degenerate cubic polynomial has not enough freedom to satisfy the interpolation conditions, so the smallest possible choice for the degree of the polynomial is \( n - 4 \) and the lowest degree possible for the boundary normal curve is one. In this section we analyse the special case of a quartic patch with linear boundary normal curves. Nine control points of this patch are determined by interpolating the vertices:

\[
\begin{align*}
P_{4,0,0} &= P_{3,1,0} = P_{3,0,1} = V_1 \\
P_{0,4,0} &= P_{0,3,1} = P_{1,3,0} = V_2 \\
P_{0,0,4} &= P_{1,0,3} = P_{0,1,3} = V_3.
\end{align*}
\]

The unknown control points are thus \( P_{2,1,1}, P_{1,2,1}, P_{1,1,2}, P_{0,2,2}, P_{2,0,2}, P_{2,2,0}. \)

Choice of degenerate quartic boundary curves. Consider again the edge \( e_1 \). The boundary curve \( b_1 \) is of the form

\[
b_1(\lambda) = V_2(\lambda^4_2 + 4\lambda^3_2\lambda_3) + P_{0,2,2}\lambda_2^2\lambda_3^2 + V_3(4\lambda^3_2\lambda_2 + \lambda^4_2), \quad \lambda_1 = 0.
\]

This means that \( b_1 \) is a planar curve situated in the plane \( \omega \), spanned by the points \( V_2, P_{0,2,2}, V_3 \).

We have chosen \( \omega \) to be the plane \( \{ V = V_2 + a(V_3 - V_2) + b(\alpha N_2 + \beta N_3), a, b \in \mathbb{R} \} \), with \( \alpha = 1/\|N_2\| \) and \( \beta = 1/\|N_3\| \). There are other possible choices of the parameters \( \alpha, \beta \), for instance such that \( \omega \) bisects the angle of the planes passing through the points \( V_2, V_3, V_2 + N_2 \) and \( V_2, V_3, V_3 + N_3 \). If it is not possible to find \( \alpha, \beta \) such that \( V_3 - V_2 \) and \( \alpha N_2 + \beta N_3 \) span a plane, then there does not exist a degenerate quartic patch interpolating \( V_i \) and \( N_i \). Once the plane \( \omega \) has been chosen, the only free parameter involved in the boundary curve \( P_{0,2,2} \) is determined such that \( P_{0,2,2} \in \omega \) and such that the tangent vectors to \( b_1 \) at the vertices \( V_2 \) and \( V_3 \) are orthogonal to \( N_2 \) and \( N_3 \), respectively (see Fig. 4). Therefore, the following equations must hold:

\[
\begin{align*}
N_2 \cdot (P_{0,2,2} - V_2) &= 0 \quad (4.1) \\
N_3 \cdot (V_3 - P_{0,2,2}) &= 0. \quad (4.2)
\end{align*}
\]

Determination of linear boundary normal curves. Consider again the edge \( e_1 \). The objective here is to find a normal boundary curve \( n_1 \), which is a linear polynomial. However, in general, for a degenerate quartic patch the corresponding normal patch is not linear. Nevertheless, there exists a normal patch which is linear along a specific boundary curve. Therefore, we consider for each boundary curve a linear boundary normal curve, which do not in general give rise to one linear normal patch. Thus, the normal boundary curve \( n_1 \) has the form

\[
n_1(\lambda) = Q_{0,1,0}^1 \lambda_2 + Q_{0,0,1}^1 \lambda_3, \quad Q_{0,1,0}, Q_{0,0,1}^1 \in \mathbb{R}^3.
\]
The superscripts refer to the edge $e_i$. Due to (2.2), the boundary normal $n_1$ must interpolate $N_2$ and $N_3$, i.e.,

$$Q_{0,1,0} = c_1^2 N_2, \quad Q_{0,0,1} = c_1^3 N_3,$$

where $c_1^2, c_1^3$ are positive constants. Specializing the compatibility conditions (3.2) to the quartic case gives,

$$(\lambda_2^2 \lambda_3 (P_{0,2,2} - V_2) + \lambda_2 \lambda_3^2 (V_3 - P_{0,2,2})) \cdot (\lambda_2 c_1^2 N_2 + \lambda_3 c_1^3 N_3) = 0,$$

for all $\lambda_2, \lambda_3 \in [0, 1], \lambda_2 + \lambda_3 = 1$. This holds if in addition to (4.1) and (4.2) the following condition is satisfied:

$$c_1^2 N_2 \cdot (V_3 - P_{0,2,2}) + c_1^3 N_3 \cdot (P_{0,2,2} - V_2) = 0.$$

The last equation can be simplified by eliminating $P_{0,2,2}$ from (4.1) and (4.2),

$$(c_1^2 N_2 + c_1^3 N_3) \cdot (V_2 - V_3) = 0.$$

This shows, that if

$$N_2, N_3 \perp (V_2 - V_3) \quad (4.3)$$

or

$$[N_2 \cdot (V_2 - V_3)] [N_3 \cdot (V_3 - V_2)] > 0, \quad (4.4)$$

then there always exists a solution such that $c_1^2$ and $c_1^3$ are positive. We might as well choose both $c_1^2$ and $c_1^3$ to be negative; it is only essential that these constants have the same sign. It can be easily seen, that there exists a solution to B1 iff the data satisfy either (4.3) or (4.4).

**Determination of interior control points.** Here the objective is to determine the still unknown control points $P_{2,1,1}, P_{1,2,1}, P_{1,1,2}$. After omitting all redundant equations, the local-triangle system (3.6)–(3.8) reads as
where

\[ r_{11} = N_1 \cdot V_1, \quad r_{12} = c_1^3 N_1 \cdot N_2 \cdot V_1, \quad r_{13} = c_1^3 N_3 \cdot N_1 \cdot V_1, \]
\[ r_{22} = N_2 \cdot V_2, \quad r_{23} = c_2^3 N_2 \cdot N_3 \cdot V_2, \quad r_{21} = c_2^3 N_1 \cdot N_2 \cdot V_2, \]
\[ r_{33} = N_3 \cdot V_3, \quad r_{31} = c_3^3 N_3 \cdot N_1 \cdot V_3, \quad r_{32} = c_3^3 N_2 \cdot N_3 \cdot V_3, \]

(4.5)

Note, that this system does not contain the control points \( P_{0,0,2}, P_{2,0,2}, P_{2,2,0} \). It is a matter of a simple calculation to show, that satisfying either (4.3) or (4.4) readily implies the existence of a solution to (4.5), in fact infinitely many solutions. This means that in this case the conditions on the weak geometric smoothness and weak interpolation can be fulfilled. Using Lemma 3.2 one can show that with a proper choice of the free parameters it is possible to satisfy the strong interpolation and the strong geometric smoothness in the case that either (4.3) or (4.4) are fulfilled. These conditions represent actually sufficient and necessary conditions for the existence of a solution to \( B_1 \) and \( B_2 \). Notice, that they are fairly restrictive and therefore the method based on degenerate quartics cannot work for arbitrary data. An important special case when it is possible to apply the quartic method is when the data come from a convex surface, since then obviously one of the requirements (4.3), (4.4) is met.

5. The quintic method

The construction of the interpolating degenerate quintic patches follows similar lines as in the quartic case. Here, there are more degrees of freedom, which can be chosen without a need to seriously restrict the data. In this section we describe a possible choice for the free parameters.

We shall assume, that the data meet the following requirements. Let \( N_d \) be the unit vector, perpendicular to the plane spanned by the three vertices \( V_1, V_2, V_3 \), with orientation consistent with the orientation of the patch, i.e.,
\[ N_d = (V_j - V_i) \times (V_k - V_i) / \| (V_j - V_i) \times (V_k - V_i) \|. \]
We require that
\[ N_i \cdot N_d > 0, \quad i = 1, 2, 3. \]  
(5.1)

This condition can also be interpreted by saying that all normals \( N_d, N_1, N_2, N_3 \) must point in the same halfspace determined by the plane containing the triangle \( \Delta \). Note, that this is indeed a very natural and weak restriction on the data, which is often assumed, explicitly or implicitly, since in general the normals of a surface tend to be nearly perpendicular to the plane of \( \Delta \).

Choice of degenerate quintic boundary curves. Consider the edge \( e_1 \) only. Since in the quintic case there are two control points to be chosen, \( P_{0,3,2} \) and \( P_{0,2,3} \),
it means that the boundary curve need not be necessarily planar. However, for simplicity, we choose the boundary curve to be planar, as in the quartic case, and we choose the plane \( \omega \) similarly. The points \( P_{0,3,2}, P_{0,2,3} \) must be chosen such that they define proper tangent vectors at the end-points of the curve, i.e., which are perpendicular to the given normals \( N_2, N_3 \) (see Fig. 5).

Moreover, we require that the following conditions be satisfied:

\[
(P_{0,3,2} - V_2) \cdot (V_3 - V_2) > 0, \quad (P_{0,2,3} - V_3) \cdot (V_2 - V_3) > 0, \quad (5.2)
\]

i.e., the vectors \((P_{0,3,2} - V_2), (P_{0,2,3} - V_3)\) point to the interior of the edge \( e_1 \).

The lengths \(|P_{0,3,2} - V_2|\) and \(|P_{0,2,3} - V_3|\) we have chosen heuristically as

\[
|P_{0,3,2} - V_2| = (a \cos(\alpha_1) + b(1 - \cos(\alpha_1)))|V_3 - V_2|,
\]

\[
|P_{0,2,3} - V_3| = (a \cos(\alpha_2) + b(1 - \cos(\alpha_2)))|V_3 - V_2|,
\]

for some weights \(a, b \in \mathbb{R}\), which must be such that the resulting boundary curve is regular inside \( e_1 \) (i.e., \( C^1 \) on \( e_1 \)) and possesses no self-intersections. In the experiments below we used the values \(a = 1/3, b \in [0, 3]\). There are various other possibilities to determine the control points \( P_{0,3,2}, P_{0,2,3} \).

**Choice of quadratic boundary normal curves.** It can be easily seen that for the quintic degenerate boundary curve, linear polynomials do not provide sufficient freedom to represent a boundary normal curve. Therefore, the boundary normal curve has to be in this case at least quadratic. We shall show, that under the assumptions (5.1) and (5.2), quadratic polynomials are indeed sufficient for this purpose. The local-edge system (3.3) for the edge \( e_1 \) reads

\[
4Q_{0,2,0}^1 \cdot (V_2 - P_{0,3,2}) = 0
\]

\[
8Q_{0,1,1}^1 \cdot (V_2 - P_{0,3,2}) + 6Q_{0,2,0}^1 \cdot (P_{0,3,2} - P_{0,2,3}) = 0
\]

\[
4Q_{0,0,2}^1 \cdot (V_2 - P_{0,3,2}) + 12Q_{0,1,1}^1 \cdot (P_{0,3,2} - P_{0,2,3}) + 4Q_{0,2,0}^1 \cdot (P_{0,2,3} - V_3) = 0
\]

\[
6Q_{0,0,2}^1 \cdot (P_{0,3,2} - P_{0,2,3}) + 8Q_{0,1,1}^1 \cdot (P_{0,2,3} - V_3) = 0
\]

\[
4Q_{0,0,2}^1 \cdot (P_{0,2,3} - V_3) = 0
\]

(5.3)

By construction of the boundary curve, the first and last from the above equations are readily satisfied. In analogy with the previous section, the objective is
to find $Q_{0,1,1}$ and real numbers $c_1, c_3^1$ such that the equations (5.3) are satisfied and such that

$$Q_{0,2,0}^1 = c_1^1 N_2$$

$$Q_{0,0,2}^1 = c_3^1 N_3,$$  \hspace{1cm} (5.4) \hspace{1cm} (5.5)

where $c_1, c_3 > 0$.

**Lemma 5.1.** Under the conditions (5.1), (5.2) there always exists a solution to (5.3)-(5.5) such that $c_1, c_3 > 0$.

**Proof.** A closer analysis of the system (5.3)-(5.5) shows, that there is always a solution for some real $c_1, c_3$. We omit the detailed proof of this, since it is elementary.

Next we show that the existence of a solution to (5.3)-(5.5) implies the existence of a solution for which $c_1, c_3 > 0$. Let $c_1, c_3, Q_{0,1,1}$ be a solution to (5.3)-(5.5) and suppose $c_2 c_3 < 0$. We claim that there is some $\lambda^0 = (0, \lambda_2^0, \lambda_3^0)\lambda_2^0 + \lambda_3^0 = 1, \lambda_2^0, \lambda_3^0 > 0$, such that the quadratic polynomial

$$q^\omega (\lambda) := Q_{0,2,0}^{\omega,1} \lambda_2^0 + Q_{0,0,2}^{\omega,1} \lambda_2 \lambda_3 + Q_{0,0,2}^{\omega,2} \lambda_3^2, \lambda_2 + \lambda_3 = 1,$$

vanishes at $\lambda^0$, where $Q_{0,2,0}^{\omega,1}, Q_{0,1,1}^{\omega,0}, Q_{0,0,2}^{\omega,1}$ denote orthogonal projections of the boundary normal curve $q$ and its control points $Q_{0,2,0}^{\omega,1}, Q_{0,1,1}^{\omega,0}, Q_{0,0,2}^{\omega,1}$ to $\omega$, respectively. To show this, let $(p_1(\lambda), p_2(\lambda))$ be coordinates of the boundary curve $p(\lambda)$ in a Cartesian coordinate system of the plane $\omega$. Then, since $p$ is constructed to be regular inside $e_1$, every continuous boundary normal curve $n^\omega$ to $p$ (in $\omega$) can be written as

$$n^\omega (\lambda) = (n_1^\omega (\lambda), n_2^\omega (\lambda)) = (-D_1 p_2 (\lambda), D_1 p_1 (\lambda)) f (\lambda), \lambda_2 + \lambda_3 > 0,$$

where $f$ is some continuous scalar function. Clearly, conditions (5.1), (5.2) imply that $(-D_1 p_2 (\lambda), D_1 p_1 (\lambda))$ interpolates either $N_2^\omega, N_3^\omega$ or $-N_2^\omega, -N_3^\omega$, depending on the choice of the Cartesian system. Therefore, in order that

$$n^\omega (\lambda) = q^\omega (\lambda), \lambda_2 + \lambda_3 > 0,$$

it is necessary that $f ((1,0)) f ((0,1)) < 0$. However, by continuity of $f$ this implies that $f (\lambda^0) = 0$ at some $\lambda^0$, i.e., $q^\omega (\lambda^0) = 0$.

Since $q^\omega$ is quadratic, it can be factorized as

$$q^\omega (\lambda) = l^\omega (\lambda) \left( \frac{\lambda_2}{\lambda_2^0} - \frac{\lambda_3}{\lambda_3^0} \right),$$

where $l^\omega$ is a linear (parametric) polynomial. However, then the coefficients of one of the quadratic polynomials $l^\omega (\lambda) (\lambda_2 + \lambda_3)$ or $-l^\omega (\lambda) (\lambda_2 + \lambda_3)$ represent a solution to the problem (5.3)-(5.5) satisfying $c_1, c_3 > 0$. \qed
Determination of the interior control points. The unknown control points are $P_{1,1,1}, P_{1,3,1}, P_{1,1,3}, P_{1,1,3}, P_{2,1,2}, P_{2,3,3}$. They are to be determined from the local-triangle system (3.6) - (3.8) (after eliminating redundant and trivially satisfied equations):

$$
4Q_{2,2,0} \cdot (P_{1,3,1} - P_{0,3,2}) = 0
$$

$$
8Q_{0,1,1} \cdot (P_{1,1,1} - P_{0,5,2}) + 6Q_{1,2,0} \cdot (P_{1,2,2} - P_{0,2,3}) = 0
$$

$$
6Q_{0,2,2} \cdot (P_{1,2,2} - P_{0,2,3}) + 8Q_{1,1,1} \cdot (P_{1,1,3} - V_{3}) = 0
$$

$$
8Q_{1,3,1} \cdot (P_{1,3,1} - P_{0,3,2}) + 6Q_{0,2,0} \cdot (P_{1,2,2} - P_{0,2,3}) + 4Q_{1,2,3} \cdot (P_{1,1,3} - V_{3}) = 0
$$

This system of 12 equations and 18 unknowns can be shown to have always a solution, in fact infinitely many. However, it is less clear whether there exists a solution which also preserves orientation of the normals and thus, which avoids cusping between adjacent patches. Although we do not have a rigorous proof of the existence of such solution, we believe, based on our numerical experiments, that this is always possible by making use of the available freedom in the underdetermined system (5.6). In our implementation, we find a solution which minimizes a suitably chosen objective function and then the constrained optimization is solved by the standard Lagrange multipliers technique (see Section 6).

6. Examples

Figs. 6–9 present some graphical results obtained with quartic and quintic degenerate patches. The data set (i.e., positional data and normals) for Figs. 6 and 7 is taken from the unit sphere as shown in Fig. 6(a). Fig. 6(b) shows a $G^{1}$ interpolant consisting of eight quartic patches.

The same surface is also reconstructed using the quintic method. Here, there is more freedom to choose a solution. We have chosen a quintic patch which is in a sense close to a patch of lower degree interpolating the vertices and normals. In our experiments we have determined the free parameters by minimizing the following quadratic objective function.
subject to (5.6), where the vectors having superscript 0 are reference control points. They are chosen either to be the control points corresponding to the linear patch interpolating the vertices (degree elevated up to the degree five), or to a cubic patch interpolating the vertices and normals (elevated up to degree five). With this strategy for the choice of the free parameters we aim to avoid unwanted oscillations and self-intersections of patches. Since the resulting surface is "close" to the piecewise linear or a piecewise cubic interpolating surface, we may expect its shape to be well behaved. A similar approach has also been used in the case of quartic patches.

Different quintic interpolants for three different choices of the reference control points are shown in Figs. 7(a–c).

Example in Figs. 8(a–c) illustrates the fact that degenerate patches offer more flexibility to represent geometrically smooth surfaces than regular ones. Namely, using two degenerate quintic patches it is possible to reconstruct a $GC^1$ surface which is topologically equivalent to the sphere. Note that in order to obtain such surface by using regular patches, at least four triangular patches would be needed.

Fig. 9 shows an example of the interpolation of a less trivial data set for the quintic method. Note, that the quartic method cannot be applied to the data from Fig. 9(a), since these do not satisfy the restrictions described in Section 4.
7. Discussion

In this paper we have studied a completely local polynomial interpolation method. Our analysis has led us to the conclusion that degenerate patches can be used for the construction of the interpolating surface. Such surface contains points which are non-regular. Therefore, the notion of geometric smoothness ($GC^1$) has been formulated in terms of continuously varying unit normals and not in terms of the existence of a regular reparametrization.

We have designed an interpolation method based on quartic and quintic patches. While the quartic method can be employed only for a restricted class of data, such as data originating from a convex surface or a surface with normals that do not oscillate "wildly", the quintic method works satisfactory.
Fig. 8. Reconstruction of a sphere-like surface using two quintic patches.
for a much larger class. We point out, that the restriction in the quartic case is not surprising, since even in the functional case there is no local method for the general quartic interpolation (Nadler, 1990).

Our experiments show that resulting interpolating surfaces are often quite sensitive to the way how one treats the available free parameters. We have followed a strategy for the choice of these parameters which leads to a plausible solution. However, we did not investigate the problem of making an optimal choice.

We did neither address in this paper the problem to guarantee the resulting interpolating patches to be $GC^1$ in the interior of $S$ and to exclude self-intersections. On the basis of our numerical experiments we believe, however, that minimizing an appropriate norm, such as (6.1), makes it very likely that the optimal solution will be $GC^1$ and have no self-intersections.

Finally, we wish to make a few remarks about curvature properties of surfaces composed of degenerate patches. Although our primary goal in this paper has been to deal with $GC^1$ interpolation, the behaviour of curvature is of importance in this context since it has influence on the visual appearance of the surface. It turns out that the Gaussian curvature of the patches tends to infinity at vertices, that is, at singular points. On the other hand, our graphical results do not suggest any artificial behaviour at vertices: the surfaces look smooth and do not seem to show any artifacts near singularities. We have made a first attempt to analyse this phenomenon and its consequences in (Pfluger and Neamtu, 1992). Observations we made there suggest that infinite curvatures of degenerate polynomial patches might not have a negative impact on their shape, at least from a practical point of view. This is also supported by results.
in (Hogervorst and van Damme, 1992), where certain higher degree degenerate patches have been employed to actually construct "almost $GC^2$" interpolating surfaces, i.e., surfaces which are $GC^2$ everywhere except at vertices.

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