Toughness and Triangle-Free Graphs

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In this paper, we prove that there exist triangle-free graphs with arbitrarily large toughness, thereby settling a longstanding open question. We also explore the problem of whether there exists a $t$-tough, $n/(t+1)$-regular, triangle-free graph on $n$ vertices for various values of $t$, and provide a relatively complete answer for small values of $t$. * 1995 Academic Press, Inc.

1. Introduction

In this paper we consider only undirected graphs without loops or multiple edges. Our terminology and notation will be standard except as indicated. A good reference for any undefined terms is Bondy \& Murty [5].

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We begin with a few definitions and some notation. We will use $\delta(G)$, $\alpha(G)$, $\kappa(G)$, and $\omega(G)$ to denote the minimum vertex degree, independence number, vertex connectivity, and number of components of a graph $G$. Given $v \in V(G)$, we will use $N_G(v)$ (or just $N(v)$ if $G$ is understood) to denote the set of neighbors of $v$ in $G$. If $X \subseteq V(G)$, we will use $\langle X \rangle$ to denote the subgraph induced by $X$. The number of vertices of the graph under consideration will be denoted by $n$.

The notion of toughness in graphs was introduced in Chvátal [7]. Let $t$ be a positive real number. A graph $G$ is said to be $t$-tough if $t \cdot \omega(G - X) \leq |X|$ for all $X \subseteq V(G)$ with $\omega(G - X) > 1$. The toughness of $G$, denoted $\tau(G)$, is the maximum value of $t$ such that $G$ is $t$-tough (taking $\tau(K_n) = \infty$ for all $n \geq 1$). The interest in toughness stems primarily from an apparent, though not yet fully understood, relation between the toughness of a graph and its cycle structure. It is well-known that every hamiltonian graph is necessarily 1-tough. On the other hand, in [7] it is conjectured that there exists a positive constant $t_0$ such that every $t_0$-tough graph is hamiltonian. This conjecture remains open and appears difficult. However, it is known (Enomoto, Jackson, Katerinis & Saito [11]) that any such $t_0$ must be at least 2 since, for any $\epsilon > 0$, there exists a $(2 - \epsilon)$-tough graph which does not contain even a 2-factor.

A graph on $n$ vertices is called pancyclic if it contains an $l$-cycle for every $l$ such that $3 \leq l \leq n$. In [7] it is also conjectured that there exists a positive constant $t_1$ such that every $t_1$-tough graph is pancyclic. More recently, in Jackson & Katerinis [13] it is asked if there is a positive constant $t_2$ such that every $t_2$-tough graph contains a triangle. In the next section (Section 2) we will prove the following result, which simultaneously settles both of these questions in the negative.

**Theorem 1.** There exist arbitrarily tough, triangle-free graphs.

There is a connection between Theorem 1 and a number of other results on triangle-free graphs. It has been shown [10, 14, 15, 16] that there exist triangle-free graphs with arbitrarily large chromatic number. Let $\chi(G)$ denote the chromatic number of a graph $G$. It is easy to see that if $G$ is a graph on $n$ vertices, then

$$\chi(G) \geq \frac{n}{\alpha(G)} \geq \tau(G) + 1. \quad (1)$$

In [12], Erdős used a clever probabilistic argument to show that there exist graphs with arbitrarily high girth (length of a shortest cycle) and arbitrarily high chromatic number. He also showed that these graphs have arbitrarily high $n/\alpha$-ratio. By (1) we see that Theorem 1 represents a strengthening of these previous results for the case of triangle-free graphs.
It is easy to verify that every $t$-tough graph $G$ with $\delta(G) > n/(t+1)$ contains a triangle. In Section 3 we consider the problem of whether there exist $t$-tough, $n/(t+1)$-regular, triangle-free graphs for various values of $t > 1$. After observing that no such graph exists for $1 < t < 3/2$, we indicate how to construct such graphs for every $t \in [3/2, 2)$ of the form $t = 2 - 1/k$, where $k \geq 2$, and for every $t \in [2, 3)$, of the form $t = 3 - 4/(k+1)$, where $k \geq 3$. Although we have not yet proven that such graphs exist for any $t \geq 3$, we give the construction of a family of graphs that is conjectured to contain $t$-tough, $n/(t+1)$-regular, triangle-free graphs for arbitrarily large $t$. We also discuss a number of interesting properties of this family, as well as similarly constructed families of graphs. Finally, in Section 4 we give a proof of one of the results in Section 3.

2. The Existence of Arbitrarily Tough, Triangle-Free Graphs

Let $G$ be a graph with vertices $v_1, v_2, \ldots, v_n$, and let $l \geq 1$ be an integer. We begin by defining the graph $G_l$, which is constructed by layering $G$ $l$ times. For each $k$, $1 \leq k \leq l$, the $k$th layer of $G_l$ will induce a complete bipartite graph with bipartition sets $\{u_{k,1}, u_{k,2}, \ldots, u_{k,n}\}$ and $\{w_{k,1}, w_{k,2}, \ldots, w_{k,n}\}$. (See Fig. 1, in which the layers of $G_l$ are schematically illustrated.) We will denote the set of vertices $\{u_{k,j} \mid 1 \leq k \leq l, 1 \leq j \leq n\}$ as $\text{Top}$, and the remaining set of vertices as $\text{Bottom}$. For $1 \leq j \leq n$, the set of vertices $\{u_{1,j}, u_{2,j}, \ldots, u_{l,j}\}$ (respectively, $\{w_{1,j}, w_{2,j}, \ldots, w_{l,j}\}$) will be called the $j$th top (respectively $j$th bottom)

![Diagram](image-url)
column of \(G_j\). The \(n\)th top column of \(G_j\) is circled in Figure 1. The edges joining vertices in different layers of \(G_j\) are determined by the edges of \(G\). If \(e_i, e_j \in E(G)\), then each vertex in the \(i\)th top (respectively, \(i\)th bottom) column is joined to every vertex in the \(j\)th top (respectively, \(j\)th bottom) column which belongs to a different layer of \(G_j\) (i.e., all the top (respectively, bottom) vertices in the same layer of \(G_j\) will be independent). We note that if \(G\) is triangle-free, than \(G_j\) will also be triangle free for all \(t \geq 1\).

Before stating and proving the main results of this section, it will be useful to prove some preliminary lemmas.

**Lemma 2.** Let \(G\) be a 1-tough graph on \(n \geq 2\) vertices, and let \(A, B \subseteq V(G)\) with \(|A| + |B| \geq n + 1\). Then some vertex in \(A\) is adjacent to a vertex in \(B\).

**Proof.** We may assume \(A \cap B\) is an independent set in \(G\), since otherwise we are done. Let \(s = |A \cap B| \geq 1\), \(a = |A - B|\), \(b = |B - A|\), and \(c = |V(G) - (A \cup B)|\). By assumption, \(|A| + |B| = (a + s) + (b + s) \geq n + 1\), and so \(a + b + s \geq n - s + 1\). But \(a + b + c + s = n\), and thus \(c \leq s - 1\).

If a vertex in \(A \cap B\) has a neighbor in \(A \cup B\), we would be done, and thus we may assume \(N(A \cap B) \subseteq V(G) - (A \cup B)\) and \(c \geq 1\). Setting \(X = V(G) - (A \cup B)\), we have that \(\omega(G - X) \geq |A - B| = s \geq 2\), while \(|X| = c \leq s - 1\). This contradicts the assumption that \(G\) is 1-tough.

**Lemma 3.** Let \(G\) be a connected graph on \(n \geq 2\) vertices. Suppose we obtain \(G_i\) by layering \(G\) \(l\) times. Then \(G_i\) is 1-tough.

**Proof.** Let \(X \subseteq V(G_i)\) such that \(\omega(G_i - X) > 1\) and \(\tau(G_i) = |X|/\omega(G_i - X)\). Let \(V_i\) denote the vertices in the \(i\)th layer of \(G_i\), \(X_i = X \cap V_i\), and \(\omega_i = \omega(<V_i - X_i>)\). If \(X_i = \emptyset\) for some \(i\), then immediately \(G_i - X\) is connected, contradicting \(\omega(G_i - X) > 1\). Hence we may assume \(|X_i| \geq 1\) for all \(i\). Since \(<V_i> = K_{n_i, n}\) is 1-tough, we have \(\omega_i \leq |X_i|\), for all \(i\). But then

\[
|X| = \sum_{i=1}^{l} |X_i| \geq \sum_{i=1}^{l} \omega_i \geq \omega(G_i - X),
\]

and thus \(G_i\) is 1-tough.

We are now in a position to prove the key result in this section.

**Theorem 4.** Let \(G\) be a 1-tough graph on \(n \geq 2\) vertices. Form \(G_i\) by layering \(G\) \(l\) times. Let \(n_i = |V(G_i)|\) and \(\alpha_i = \alpha(G_i)\). Then \(\tau(G_i) \geq \sqrt{n_i/2\alpha_i}\).

**Proof.** Set \(t = \tau(G_i)\) and assume \(t < \sqrt{n_i/2\alpha_i}\). Let \(X \subseteq V(G_i)\) such that \(\omega(G_i - X) > 1\) and \(t = |X|/\omega(G_i - X)\). By Lemma 3 we can assume \(t \geq 1\). Let \(V_k\) denote the vertices in the \(k\)th layer of \(G_i\) and let \(X_k = X \cap V_k\). We
assume that $X_1$ has the minimum number of vertices among the $X_k$. Since 
\[ \sum_{k=1}^{i} |X_k| = |X| = t \cdot \omega(G_i - X), \]
we have
\[ \frac{|X_i|}{\omega(G_i - X)} \leq \frac{t}{i}. \tag{2} \]

Claim 1. $|X_i|$ satisfies $1 \leq |X_i| < n/t \leq n$.

Proof of Claim 1. If $|X_i| = 0$, then obviously $\omega(G_i - X) = 1$, a contradicition. So we have $|X_i| \geq 1$. If $|X_i| \geq n/t$, then by (2) we have
\[ t \geq \frac{|X_i|}{\omega(G_i - X)} \geq \frac{n/t}{\omega(G_i)} = \frac{n}{t \omega_i}. \]
Thus $t \geq n/(n/2 \omega_i)$, and hence $t \geq n/2 \omega_i$, contradicting the assumption. Since $t \geq 1$, we have $n/t \leq n$.

This proves Claim 1. \[ \square \]

Since $|X_i| < n$, clearly $V_1 - X_1$ contains vertices from both Top and Bottom, and the vertices in $V_1 - X_1$ all belong to a single component of $G_i - X$. Henceforth, we will denote this component as $H$.

Let us now partition the layer numbers \{1, 2, ..., l\} into two sets Small and Big as follows: For $1 \leq j \leq l$, $j \in$ Small (respectively, $j \in$ Big) if $|X_j| \leq n - 1$ (respectively, $|X_j| \geq n$). Note that $1 \in$ Small by Claim 1.

Claim 2. $\cup_{j \in \text{Small}} (V_j - X_j)$ (i.e., the vertices which remain in the small layers when $X$ is removed) all belong to the component $H$.

Proof of Claim 2. If $j \in \text{Small} - \{1\}$, then certainly all vertices in $V_j - X_j$ will belong to the same component of $G_j - X$, since $|V_j - X_j| = 2n - |X_j| \geq n + 1$, and so $V_j - X_j$ contains vertices from both Top and Bottom. Thus it suffices to show there is an edge between $V_1 - X_1$ and $V_j - X_j$.

Since $|V_j - X_j| \geq n + 1$ and $|V_1 - X_1| \geq n + 1$ we have
\[ |(V_j - X_j) \cap \text{Top}| + |(V_1 - X_1) \cap \text{Top}| + |(V_j - X_j) \cap \text{Bottom}| + |(V_1 - X_1) \cap \text{Bottom}| \geq n + 2, \]
so either $|(V_j - X_j) \cap \text{Top}| + |(V_1 - X_1) \cap \text{Top}| \geq n + 1$ or $|(V_j - X_j) \cap \text{Bottom}| + |(V_1 - X_1) \cap \text{Bottom}| \geq n + 1$. Let us assume the former and define $A = (V_j - X_j) \cap \text{Top}$ and $B = (V_1 - X_1) \cap \text{Top}$. Thus we have $|A| + |B| \geq n + 1$. Since $G$ is 1-tough, it follows by Lemma 2 (thinking of $A$, $B$ as subsets of $V(G)$) that some vertex in $A$ is adjacent to some vertex in $B$.

This proves Claim 2. \[ \square \]
We now know that all vertices in $\bigcup_{j \in \text{Small}} (V_j - X_j)$ belong to the component $H$ of $G_t - X$.

Let us now turn to a consideration of the layers whose indices are in $\text{Big}$, recalling that $j \in \text{Big}$ implies $|X_j| \geq n$. We know

$$t = \frac{|X_1|}{\omega(G_t - X)} \geq \frac{|X_1| + |\bigcup_{j \in \text{Big}} X_j|}{\omega(G_t - X)} \geq \frac{|X_1| + |\text{Big}| \cdot n}{\omega(G_t - X)}.$$

However, if $k \in \text{Big}$, then by Lemma 2 the maximum number of vertices (and hence components) in $V_k - X_k$ that lie outside $H$ is $|X_1|$. Hence $\omega(G_t - X) \leq 1 + |\text{Big}| \cdot |X_1|$, which gives $t \geq |X_1| + |\text{Big}| \cdot n/(1 + |\text{Big}| \cdot |X_1|)$. Since $n > t \cdot |X_1|$ by Claim 1, we conclude

$$|\text{Big}| \leq \frac{t - |X_1|}{n - t \cdot |X_1|} \cdot (3)$$

If $|X_1| \geq t$, then by (3) we have $|\text{Big}| \leq 0$, so we have only the component $H$ in $G_t - X$, contradicting $\omega(G_t - X) > 1$. So we can assume $|X_1| < t$.

If $n > t^2$ (equivalently $n/(t - |X_1|) > t - |X_1|$), then by (3) we find

$$|\text{Big}| \leq \frac{1}{t} \cdot \frac{t - |X_1|}{n/t - |X_1|} < \frac{1}{t} \leq 1,$$

implying the contradiction $|\text{Big}| = 0$. But if $n \leq t^2$, then since $x_i < l\pi(G)$ and $t < \sqrt{n/t / 2x_i}$, we obtain $n \leq t^2 < n/t \cdot 2x_i \leq 2ln/2\pi(G) = n/\pi(G)$, a contradiction since $\pi(G) \geq 1$.

This completes the proof of Theorem 4. $\square$

Before proving our main result, we need one more lemma.

Lemma 5. Let $G$ be a graph with $n$ vertices and independence number $x$. Let $G_t$ be constructed by layering $G$ $t$ times. Then $\pi(G_t) \leq 2n + (t - 2) \pi$. $\pi$.

Proof. Let $I$ denote a maximum independent set in $G_t$. Denote by $l_T$ (respectively, $l_B$) the number of layers of $G_t$ which contain one or more vertices of $I \cap \text{Top}$ (respectively, $I \cap \text{Bottom}$). Clearly these $l_T$ and $l_B$ layers are all distinct, and so $l_T + l_B \leq l$. It is easy to see that if any top column contains two or more vertices of $I$, then we may assume it contains $l_T$ vertices of $I$ (one in each of the $l_T$ layers described above), and of course the set of all such top columns corresponds to an independent set of vertices in $G$ (assuming the $j$th top column corresponds to $v_j \in V(G)$, etc.). Thus we have $|I \cap \text{Top}| \leq n + (l_T - 1) \pi$. But of course an analogous inequality holds for $\text{Bottom}$. Combining these inequalities yields $|I| \leq 2n + (l - 2) \pi$, as asserted. $\square$
Corollary 6. Let \( G \) be a 1-tough graph on \( n \geq 2 \) vertices with independence number \( \alpha \). Form \( G_i \) by layering \( G \) \( l \leq 2((n/\alpha) + 1) \) times. Then \( \tau(G_i) \geq (1/2) \sqrt{l} \).

Proof. By Lemma 5 we have \( \alpha(G_i) \leq 2n + (l - 2) \cdot \alpha \). Since \( l \leq 2((n/\alpha) + 1) \), this gives \( \alpha(G_i) \leq 4n \). Now use Theorem 4 and \( |V(G_i)| = n_i = 2ln \) to obtain \( \tau(G_i) \geq \frac{1}{2} \sqrt{l} \).

It is now an easy matter to prove our main result.

Theorem 1. There exist arbitrarily tough, triangle-free graphs.

Proof. Define a sequence of triangle-free graphs \( H_1, H_2, H_3, \ldots \) as follows: \( H_1 \) is the 4-cycle \( C_4 \), and for \( j \geq 1 \) \( H_{j+1} \) is obtained by layering \( H_j \) \( 2j + 4 \) times. Clearly each \( H_j \) is 1-tough by Lemma 3. Using Lemma 5, it is straightforward to derive (formally by induction on \( j \)) that \( |V(H_j)| = (1/2)(j + 1)! \cdot 4^j \), \( \alpha(H_j) \leq 4 |V(H_{j-1})| = (1/2) j! \cdot 4^j \), and so \( |V(H_j)|/\alpha(H_j) \geq j + 1 \). So by Corollary 6, when we layer \( H_j \) \( l \leq 2j + 4 \) times, the resulting graph has toughness at least \( (1/2) \sqrt{l} \). In particular, \( \tau(H_{j+1}) \geq (1/2) \sqrt{2j + 4} \), and so \( \tau(H_j) \) can be made arbitrarily large by simply taking \( j \) sufficiently large.

This completes the proof of Theorem 1.

3. The Existence of \( t \)-Tough, \( n/(t+1) \)-Regular, Triangle-Free Graphs

Though we have established the existence of triangle-free graphs with arbitrarily large toughness, there remains the problem of determining a tight lower bound on \( \delta(G) \), in terms of \( n \) and \( \tau(G) \), which would imply that \( G \) contains a triangle. It is easy to verify that if any vertex in a \( t \)-tough graph \( G \) has degree larger then \( n/(t+1) \), then \( G \) must contain a triangle. Thus it seems interesting to inquire about the existence of \( t \)-tough, \( n/(t+1) \)-regular, triangle-free graphs for various values of \( t \). Our goal in this section is to provide a relatively complete solution for this problem when \( 1 \leq t < 3 \), and to summarize what we know when \( t \geq 3 \).

Let us first consider the case \( 1 \leq t < 3/2 \). We begin by describing two important prior results.

Theorem 7 (Bauer, Broersma, Van den Heuvel, and Veldman [3]). If \( G \) is a \( t \)-tough graph on \( n \) vertices with \( \delta(G) \geq n/(t+1) \), then \( G \) is Hamiltonian.
**Theorem 8** (Amar, Flandrin, Fournier, and Germa [2]). *If G is a hamiltonian graph on n vertices with \( \delta(G) > (2/5)n \), then G is pancyclic or bipartite.*

Together, these two results imply the following theorem (which supports Bondy's Pancyclic Metaconjecture (Chvátal [8]) relative to Theorem 7 for \( 1 \leq t < 3/2 \)).

**Theorem 9.** Let \( 1 \leq t < 3/2 \). If G is a t-tough graph on n vertices with \( \delta(G) \geq n/(t + 1) \), then G is pancyclic, unless \( t = 1 \) and G is isomorphic to \( K_{n-2, n-2} \).

In particular, note that Theorem 9 implies that there does not exist a t-tough, \( n/(t+1) \)-regular, triangle-free graph for any t such that \( 1 < t < 3/2 \), and exactly one such graph for \( t = 1 \). For \( t \geq 3/2 \), however, such graphs do exist as we now proceed to show.

Consider first the case \( 3/2 \leq t < 2 \). For infinitely many such t (namely, every t of the form \( 2 - 1/k \), for \( k \geq 2 \)), we will construct a t-tough, \( n/(t+1) \)-regular, triangle-free graph on n vertices. First define a preliminary family of graphs \( \Gamma_2, \Gamma_3, \ldots \) as follows: Given \( k \geq 2 \), begin with a \((3k-1)\)-cycle \( (x_0, x_1, \ldots, x_{3k-2}, x_0) \), and for each vertex \( x_i \), add the chords between \( x_i \) and \( x_{i+4}, x_{i+7}, \ldots, x_{i+3k-5} \) (indices are modulo \( 3k-1 \)). Call the resulting graph \( \Gamma_k \). We illustrate \( \Gamma_4 \) in Fig. 2. It is easy to verify that \( \Gamma_k \) is k-regular and triangle-free.

Next define \( G_k = \Gamma_k[K_3] \), the lexicographic product of \( \Gamma_k \) and \( K_3 \). It is immediate that \( G_k \) is \( 3k - \) regular and triangle-free. Observe also that if we

![Fig. 2. The graph \( \Gamma_4 \).](image-url)
delete from $G_k$ the three copies of $K_3$ corresponding to the vertices $x_{3k-4}$, $x_{3k-3}$, $x_{3k-2}$ in $\Gamma_k$, the graph which results is isomorphic to $G_{k-1}$. This observation allows for a straightforward proof of the following lemma. The full proof is given in Section 4.

**Lemma 10.** The toughness of $G_k$ ($k \geq 2$) satisfies $\tau(G_k) = 2 - 1/k$.

Hence for $k \geq 2$, $|V(G_k)|/(\tau(G_k) + 1) = 3(3k - 1)/(2 - 1/k + 1) = 3k$, the degree of regularity of $G_k$. Thus $G_k$ is a $t$-tough, $3(3k - 1)/(t + 1)$-regular, triangle-free graph on $3(3k - 1)$ vertices with $t = 2 - 1/k$.

Consider next the case where $2 \leq t < 3$. For each $t$ of the form $t = 3 - 4/(k + 1)$, where $k \geq 3$, we will show that there exists a $t$-tough, $n/(t + 1)$-regular, triangle-free graph on $n$ vertices. Indeed, consider the graph $F_k$ obtained by layering $C_k$ $k$ times. It is immediate that $|V(F_k)| = 8k$ and that $F_k$ is $(2k + 2)$-regular and triangle-free. The toughness of $F_k$ is given in the following lemma. A simple but tedious proof can be found in Bauer, van den Heuvel, and Schmeichel [4].

**Lemma 11.** The toughness of $F_k$ ($k \geq 3$) satisfies $\tau(F_k) = 3 - 4/(k + 1)$.

Hence for $k \geq 3$, $|V(F_k)|/(\tau(F_k) + 1) = 8k/(3 - 4/(k + 1) + 1) = 2k + 2$, the degree of regularity of $F_k$. Thus $F_k$ is a $t$-tough, $8k/(t + 1)$-regular, triangle-free graph on $8k$ vertices with $t = 3 - 4/(k + 1)$.

Finally, we consider the existence of a $t$-tough, $n/(t + 1)$-regular graph without triangles for $t \geq 3$. While we have not yet established the existence of such a graph, we conjecture the following.

**Conjecture 12.** For arbitrarily large $t$, there exists a $t$-tough, $n/(t + 1)$-regular, triangle-free graph on $n$ vertices.

Before discussing Conjecture 12 further, we wish to state another, closely related conjecture.

**Conjecture 13.** Let $G$ be a $t$-tough graph on $n$ vertices with $\delta(G) > n/(t + 1)$. Then $G$ is pancyclic.

As noted above, under the hypothesis of Conjecture 13, $G$ must contain both a Hamilton cycle and a triangle. We also note that Conjecture 13 is true for $t \leq 3/2$ by Theorems 7 and 8. Conjectures 12 and 13 together would provide a satisfying relation for the toughness, minimum vertex degree, and cycle structure of a graph.

Let us return now to Conjecture 12. We believe the family of graphs $H_1$, $H_2$, $H_3$, ... defined in the proof of Theorem 1 provide examples of the type of graphs called for in Conjecture 12. The difficulty in proving this, of course, is in the need to establish $\tau(H_j)$ exactly (we only crudely estimated $\tau(H_j)$ in the proof of Theorem 1). On the other hand, we have obtained
some interesting general properties of layered graphs (like the family of graphs \( H_i \)) which we now present.

For each pair of integers \( n, r \), with \( n \geq 3 \), define the class of graphs \( \mathcal{F}(n, r) \) to be the set of all \( n \)-vertex graphs \( G \) which satisfy the following properties:

(a) \( G \) is triangle-free;

(b) \( G \) is \( r \)-regular;

(c) diameter \( (G) = 2 \);

(d) \( \kappa(G) = r \);

(e) \( \chi(G) = r \);

(e') every maximum independent set \( I \) has the form \( I = N(v) \) for some \( v \in V(G) \).

In particular, note that \( C_4 \in \mathcal{F}(4, 2) \). In a moment we will prove the following result.

**Theorem 14.** Let \( G \in \mathcal{F}(n, r) \) with \( n \geq 3 \). Suppose we obtain \( G_i \) by layering \( G \) \( l \geq 2n - 1 \) times. Then \( G_i \in \mathcal{F}(2nl, n + (l - 1) r) \).

However, we wish to conjecture an even stronger result. A graph \( G \in \mathcal{F}(n, r) \) is said to be in \( \mathcal{F}(n, r) \) if \( G \) also satisfies:

(f) \( \tau(G) = (n - \chi(G))/\chi(G) \);

(f') every set \( X \subseteq V(G) \) such that \( \omega(G - X) > 1 \) and \( \tau(G) = |X|/\omega(G - X) \) has the form \( X = V(G) - N(v) \) for some \( v \in V(G) \).

In particular, note that \( C_4 \in \mathcal{F}(4, 2) \). Moreover, every graph in \( \mathcal{F}(n, r) \) is by definition an example of a \( t \)-tough, \( n/(t + 1) \)-regular, triangle-free graph for \( t = (n - r)/r \). We put forth the following conjecture.

**Conjecture 15.** Let \( G \in \mathcal{F}(n, r) \). Suppose we obtain \( G_i \) by layering \( G \) \( l > (n/\chi(G)) + 1 \) times. Then \( G_i \in \mathcal{F}(2nl, n + (l - 1) r) \).

Note that any \( t \)-tough, \( n/(t + 1) \)-regular, triangle-free graph must satisfy properties (a), (b), (e) and (f). The properties (c), (d), (e') and (f') are not essential for the results of this paper, but they may be useful in determining the toughness of layered graphs.

Before proceeding with the proof of Theorem 14 we state and prove the following lemma due to S. Brandt [6].

**Lemma 16.** Let \( G \) be a triangle-free graph. Then \( \chi(G) + \kappa(G) \geq 2 \delta(G) \).

**Proof.** This is obvious if \( \kappa(G) = \delta(G) \) since \( \chi(G) \geq \delta(G) \). If \( \kappa(G) < \delta(G) \), let \( X \) be a separating set of cardinality \( \kappa(G) \) and let \( G_1, G_2 \) be 2 of the components of \( G - X \). Since every vertex outside \( X \) has a neighbor outside \( X \).
every component of $G - X$ has an edge. Consider the endvertices of edge $e_W$ in $G$. Since they have no common neighbor in $X$, one of them must have at most $(1/2) \kappa(G)$ neighbors in $X$. Hence $\kappa(G_1) \geq \delta(G) - (1/2) \kappa(G)$, implying $\alpha(G) \geq \alpha(G_1) + \alpha(G_2) \geq 2 \delta(G) - \kappa(G)$. 

Proof of Theorem 14. The proofs that $G_i$ satisfies properties (a) through (c) with $r$ replaced by $n + (l - 1) r$ are straightforward and left to the reader. We also note that property (d) follows easily from property (e) and Lemma 16. Thus we prove only that $G_i$ satisfies properties (e) and (e') with $r$ replaced by $n + (l - 1) r$. We use the notation of Section 2.

Proofs of (e) and (e'). For any $v \in V(G_i)$, we note that $N_{G_i}(v)$ is an independent set with cardinality $n + (l - 1) r$. Thus to show that $\alpha(G_i) = n + (l - 1) r$, it suffices to show that $\alpha(G_i) \leq n + (l - 1) r$. Let $I$ denote any maximum independent set in $G_i$, so that $|I| \geq n + (l - 1) \alpha(G_i)$. Let $l_T$ (respectively, $l_B$), denote the number of layers of $G_i$ which contain vertices of $I \cap \text{Top}$ (respectively, $I \cap \text{Bottom}$). Clearly $l_T + l_B \leq l$. We may assume each top column contains $0$, $1$, or $l_T$ vertices of $I$, since if some top column contained $k$ vertices of $I$, $2 \leq k < l_T$, we could obtain a larger independent set in $G_i$ by adding to $I$ more vertices from that top column. By symmetry, we may assume each bottom column contains $0$, $1$, or $l_B$ vertices of $I$. Since $l \geq 2n - 1 > n + 1 \geq (n - \alpha(G)) + 1$, we have $|I| = \alpha(G_i) \geq n + (l - 1) \alpha(G) > 2n$. This means $|I \cap \text{Top}| \geq n + 1$ or $|I \cap \text{Bottom}| \geq n + 1$.

First assume $|I \cap \text{Top}| \geq n + 1$. Thus some top column contains at least 2 vertices of $I$. But then we may assume every layer contains at least one vertex of $I$, for if some layer contains no vertices of $I$, we could enlarge $I$ by adding a vertex from that layer which belongs to a top column already containing 2 or more vertices of $I$. Hence $l_T + l_B = l \geq 2n - 1$. If $|I \cap \text{Bottom}| \geq n + 1$, then in a similar way we obtain $l_T + l_B = l \geq 2n - 1$.

So we may assume, without loss of generality, that $l_T \geq n$. We now wish to show that we may also assume

$$|I \cap \text{Top}| \leq l_T \alpha(G). \quad (4)$$

Towards this goal, let us denote the set of vertices in $G$ corresponding to the top columns containing $l_T$ vertices (respectively, 1 vertex) from $I \cap \text{Top}$ as $\text{Heavy}$ (respectively, Sparse). Clearly $\text{Heavy}$ is an independent set $G$, and so $|\text{Heavy}| \leq \alpha(G)$. If Sparse = \emptyset, then $|I \cap \text{Top}| = l_T \cdot |\text{Heavy}| \leq l_T \alpha(G)$, which is (4). If Sparse \neq \emptyset, then $\text{Heavy}$ is not a maximal independent set in $G$ and so there exists a vertex $v_0 \in V(G) - (\text{Heavy} \cup N(\text{Heavy}))$. We then construct another independent set $I'$ in $G_i$ with $|I'| \geq |I|$ as follows: Delete from $I$ all vertices in the top columns which correspond to vertices in Sparse (this deletes at most $|\text{Sparse}| \leq n$ vertices from $I$), and add to $I$ the $l_T \geq n$ vertices in the top column corresponding to $v_0$ which belong to
a layer already containing a vertex of $I \cap \text{Top}$. By the construction of $I'$, we now have $\text{Sparse} = \emptyset$. Thus we may assume (4) holds.

In the proof of Lemma 5, we proved that

$$|I \cap \text{Bottom}| \leq n + (l_B - 1) \chi(G).$$

Combining (4) and (5), we obtain $\chi(G) = |I| \leq n + (l - 1) \chi(G) = n + (l - 1) r$, as desired.

If equality holds in (4) and (5), it is easy to verify that we must have $l_B = 1$, $|I \cap \text{Bottom}| = n$, and $|I \cap \text{Top}| = (l - 1) \chi(G)$. It is then a simple matter to verify that $I = N(v)$ for some $v \in V(G)$.

\section{Proof of Lemma 10}

We use the following observation, the proof of which is straightforward and left to the reader.

\begin{lemma}
Let $G$ be a graph and $X \subseteq V(G)$ such that $\omega(G - X) > 1$ and $\tau(G) = |X|/\omega(G - X)$. Suppose $a, b \in V(G)$ satisfy $N(a) - \{b\} = N(b) - \{a\}$. Then $\{a, b\} \cap X = \emptyset$ or $\{a, b\} \cap X = \{a, b\}$.
\end{lemma}

\begin{proof}[Proof of Lemma 10]
Let $V(G_k) = \{x_{0,1}, x_{0,2}, x_{0,3}, \ldots, x_{3k-2,1}, x_{3k-2,2}, x_{3k-2,3}\}$, where $x_{k,1}, x_{k,2}, x_{k,3}$ are the three vertices in the $\bar{K}_3$ corresponding to the vertex $x_k$ in $\bar{G}_k$. Define $\bar{x}_i = \{x_{i,1}, x_{i,2}, x_{i,3}\}$ ($0 \leq i \leq 3k - 2$).

Let $Y = V(G_k) - (\bar{x}_1 \cup \bar{x}_4 \cup \cdots \cup \bar{x}_{3k-2})$. Then $|Y| = 3(2k - 1)$ and $\omega(G_k - Y) = 3k$ so $\tau(G_k) \leq 3(2k - 1)/3k = 2 - 1/k$. In the remainder of the proof we will show that $\tau(G_k) = 2 - 1/k$.

Choose $X \subseteq V G_k$ with $\omega(G_k - X) > 1$ such that $\tau(G_k) = |X|/\omega(G_k - X)$. We proceed by induction on $k \geq 2$. Using Lemma 17 it is easy to verify that $\tau(G_3) = 3/2$, noting that each $\bar{K}_3$ lies completely in $X$ or is completely disjoint from $X$. Hence we suppose $k \geq 3$.

By Lemma 17 we can assume $\bar{x}_i \subseteq X$ or $\bar{x}_i \cap X = \emptyset$ for all $i$. Trivially, there exists an index $i$ such that $X \cap (\bar{x}_i \cup \bar{x}_{i+1} \cup \bar{x}_{i+2}) = \bar{x}_i$ or $\bar{x}_i \cup \bar{x}_{i+2}$. We can assume that $X \cap (\bar{x}_0 \cup \bar{x}_1 \cup \bar{x}_2) = \bar{x}_0$ or $\bar{x}_0 \cup \bar{x}_2$. Let $G' = G_k - (\bar{x}_0 \cup \bar{x}_1 \cup \bar{x}_2)$, so that $G' \cong G_{k-1}$ and let $X' = X - (\bar{x}_0 \cup \bar{x}_1 \cup \bar{x}_2)$. Since by induction we are assuming that $\tau(G_{k-1}) = 2 - 1/(k - 1) = (2k - 3)/(k - 1)$, it follows that if $\omega(G' - X') > 1$, then $\omega(G' - X') \leq ((k - 1)/(2k - 3)) |X'|$.

We consider two cases.

\textbf{Case 1.} $X \cap (\bar{x}_0 \cup \bar{x}_1 \cup \bar{x}_2) = \bar{x}_0$. Let $B = N(\bar{x}_1) \cap V(G')$ and $C = N(\bar{x}_2) \cap V(G')$. Note that $B \cap C = \emptyset$, $|B| = 3(k - 2)$ and $|C| = 3(k - 1)$. 

Suppose first that \( \langle \bar{x}_1 \cup \bar{x}_2 \rangle \) forms a separate component of \( G_k - X \). Then \( B \cup C \subseteq X' \), and so \( |X'| \geq |B| + |C| = 3(2k - 3) \). Since \( |V(G')| = 3(3k - 4) \), \( \omega(G' - X') \leq 3(k - 1) \). But then \( |X'| = |X'| + 3 \geq 3(2k - 2) \), while \( \omega(G_k - X) = \omega(G' - X') + 1 \leq 3(k - 1) + 1 \). Since obviously \( (3k - 2)(2k - 1) \leq 3k(2k - 2) \) for \( k \geq 2 \), we have

\[
\omega(G_k - X) \leq \frac{3(k - 1) + 1}{3(2k - 2)} |X'| \leq \frac{k}{2k - 1} |X|,
\]
as desired.

Suppose therefore that \( \langle \bar{x}_1 \cup \bar{x}_2 \rangle \) does not form a separate component of \( G_k - X \). Then \( \omega(G' - X') \geq \omega(G_k - X) > 1 \) and so

\[
\omega(G_k - X) \leq \omega(G' - X') \leq \frac{k - 1}{2k - 3} |X'|.
\]
Since \( X \cap (\bar{x}_1 \cup \bar{x}_2) = \emptyset \), we have \( |X| \leq 3(3k - 3) \leq 3(2k - 1)(k - 1) \). Thus we have \( (2k - 1)(k - 1)(|X| - 3) \leq (2k - 3) k |X| \), hence

\[
\omega(G_k - X) \leq \frac{k - 1}{2k - 3} |X'| = \frac{k - 1}{2k - 3} (|X| - 3) \leq \frac{k}{2k - 1} |X|,
\]
as desired.

**Case 2.** \( X \cap (\bar{x}_0 \cup \bar{x}_1 \cup \bar{x}_2) = \bar{x}_0 \cup \bar{x}_2 \). First suppose that \( \langle \bar{x}_1 \rangle \) forms three separate components of \( G_k - X \). So we have \( |X'| \geq 3 \). If \( \omega(G' - X') = 1 \), then \( \omega(G' - X') \leq ((k - 1)/(2k - 3)) |X'| \) for \( k \geq 3 \). And if \( \omega(G' - X') > 1 \), then \( \omega(G' - X') \leq ((k - 1)/(2k - 3)) |X'| \) by induction hypothesis. So we always have

\[
\omega(G_k - X) = \omega(G' - X') + 3 \leq \frac{k - 1}{2k - 3} |X'| + 3
\]

\[
= \frac{k}{2k - 1} |X'| + \frac{3}{(2k - 1)(2k - 3)} |X'| + 3.
\]

If \( |X'| \leq 3(2k - 3) \), then

\[
\omega(G_k - X) \leq \frac{k}{2k - 1} |X'| + \frac{3}{2k - 1} + \frac{3}{2k - 1} + \frac{2k - 1}{2k - 1}
\]

\[
= \frac{k}{2k - 1} (|X'| + 6) = \frac{k}{2k - 1} |X|,
\]
as desired. And if \( |X'| \geq 3(2k - 2) \), then \( \omega(G' - X') \leq 3(k - 2) \) and so \( |X| = |X'| + 6 \geq 6k \), while \( \omega(G_k - X) = \omega(G' - X') + 3 \leq 3(k - 1) \). Thus

\[
\omega(G_k - X) \leq \frac{k - 1}{2k} \cdot |X| < \frac{k}{2k - 1} |X|,
\]

a contradiction.

Thus we may assume \( \langle \bar{x}_1 \rangle \) does not form a separate component of \( G_k - X \). We can solve this case exactly as we did at the end of Case 1.

This completes the proof of Lemma 10.

Finally, we note that N. Alon [1] has recently proven that for every \( t \) and \( g \) there exists a \( t \)-tough graph with girth greater than \( g \).

Acknowledgments

We thank Stephan Brandt for suggesting the use of Lemma 16.

Note added in proof. Recently Stephan Brandt has settled Conjecture 12 in the affirmative.

References

6. S. Brandt, Personal communication.