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A note on a conjecture concerning
tree-partitioning 3-regular graphs

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Abstract
If $G$ is a 4-connected maximal planar graph, then $G$ is hamiltonian (by a theorem of Whitney), implying that its dual graph $G^*$ is a cyclically 4-edge connected 3-regular planar graph admitting a partition of the vertex set into two parts, each inducing a tree in $G^*$, a so-called tree-partition. It is a natural question whether each cyclically 4-edge connected 3-regular graph admits such a tree-partition. This was conjectured by Jaeger, and recently independently by the first author. The main result of this note shows that each connected 3-regular graph on $n$ vertices admits a partition of the vertex set into two sets such that precisely $\frac{1}{2}n+2$ edges have end vertices in each set. This is a necessary condition for having a tree-partition. We also show that not all cyclically 3-edge connected 3-regular (planar) graphs admit a tree-partition, and present the smallest counterexamples.

Keywords: tree partition, 3-regular graph, cyclically 4-edge connected, planar graph, dual graph.

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1 Preliminaries
For any terminology or notation not defined here we refer to [1].
All graphs considered are finite and without loops but we allow multiple edges. In the latter case we speak about multigraphs. Given a (multi)graph $G$ and subsets $S$ and $T$ of $V(G)$, we shall use $G[S]$ to denote the subgraph of $G$ induced by $S$, and $e[S, T]$ to denote the number of edges (with their multiplicities in case of multiple edges) in $[S, T]$ joining vertices of $S$ with vertices of $T$. As usual $\lambda(K_1) = 0$ and if $G \neq K_1$, $\lambda(G)$ is equal to the minimum size of an edge cut of $G$ and $G$ is $k$-edge connected if $\lambda(G) \geq k$, where $k$ is an integer; $\lambda_e(G)$ is equal to the minimum size of an edge cut $[S, \overline{S}]$ such that both $G[S]$ and $G[\overline{S}]$ have at least one edge, if such an edge cut exists and $\lambda(G)$ otherwise. Analogously $\lambda_e(G)$ is equal to the minimum size of an edge cut $[S, \overline{S}]$ such that both $G[S]$ and $G[\overline{S}]$ contain a cycle, if such an edge cut exists and $\lambda_e(G)$ otherwise. $G$ is called essentially $k$-edge connected if $\lambda_e(G) \geq k$ and cyclically $k$-edge connected if $\lambda_e(G) \geq k$, where $k$ is an integer. Clearly, $\lambda_e(G) \geq \lambda_e(G) \geq \lambda(G)$ for any graph $G$.

A tree-partition of a graph $G = (V, E)$ is a pair $(T_1, T_2)$ such that $(V_1, V_2) := (V(T_1), V(T_2))$ is a partition of $V$ and both $T_1 = G[V_1]$ and $T_2 = G[V_2]$ are trees. We will also say that $(V_1, V_2)$ is a tree-partition of $G$ if $(V_1, V_2)$ is a partition of $V$ such that $T_i := G[V_i]$ is a tree $(i = 1, 2)$. Note that not every graph has a tree-partition (for example $K_6$).

The dual graph of a planar graph $G$ will be denoted by $G^*$. A natural class of graphs that admit a tree-partition is obtained from the class of 4-connected maximal planar graphs by taking their duals.

**Theorem 1**

Every 3-regular cyclically 4-edge connected planar graph has a tree-partition.

The proof follows from the following two well-known lemmas, and a theorem of Whitney [5].

**Lemma 2**

Let $G$ be a 3-regular cyclically 4-edge connected planar graph. Then $G^*$ is 4-connected.

**Lemma 3**

A planar graph is hamiltonian if and only if the dual graph has a tree-partition.

**Theorem 4**

A 4-connected maximal planar graph has a Hamilton cycle.

Now we can prove Theorem 1. Let $G$ be a 3-regular cyclically 4-edge connected planar graph. Then it is clear that the dual $G^*$ is maximal planar and (Lemma 2) $G^*$ is 4-connected. By Theorem 4 $G^*$ has a Hamilton cycle. Thus by Lemma 3 $G$ has a tree-partition.

It is a natural question whether each cyclically 4-edge connected 3-regular graph admits a tree-partition. This was recently conjectured by the first author, but has been conjectured before by Jaeger ([2]).
Conjecture 5
Every 3-regular cyclically 4-edge connected graph has a tree-partition.

In this note we give some partial results that may turn out to be helpful in proving or disproving the conjecture. We start with the following easy observations.

Proposition 6
If a 3-regular graph has a tree-partition \((V_1, V_2)\), then \(G[V_i]\) is connected \((i = 1, 2)\), \(e[V_1, V_2] = \frac{1}{2}n + 2\) and \(|V_1| = |V_2| = \frac{1}{2}n\).

Proof. Let \(G = (V, E)\) be a 3-regular graph with a tree-partition \((V_1, V_2)\). Then \(G[V_i]\) is a tree, so \(G[V_i]\) is connected \((i = 1, 2)\). Define \(n := |V|\) and \(T_i := G[V_i]\), \(n_i := |V_i|\), \(x_i := \text{number of vertices in } T_i\) with degree 1 in \(T_i\), \(y_i := \text{number of vertices in } T_i\) with degree 2 in \(T_i\), \(z_i := \text{number of vertices in } T_i\) with degree 3 in \(T_i\) for \(i = 1, 2\). Then \(n_1 + n_2 = n\), \(|E| = \frac{3}{2}n\) and \(|E(T_i)| = n_i - 1\) for \(i = 1, 2\). Because \(G[V_i]\) is a tree \((i = 1, 2)\), all edges of \(G\) not in \(T_1\) or \(T_2\) are between \(V_1\) and \(V_2\). So \(e[V_1, V_2] = |E| - (n_1 - 1) - (n_2 - 1) = \frac{1}{2}n + 2\). It is easily seen that the number of edges between \(T_1\) and \(T_2\) is equal to \(2x_1 + y_1 = 2x_2 + y_2\). Furthermore, since \(n_i \geq 2\), \(x_i = 2 + z_i\) for \(i = 1, 2\), so \(n_1 = x_1 + y_1 + z_1 = x_1 + y_1 + x_1 - 2 = 2x_1 + y_1 - 2 = 2x_2 + y_2 - 2 = x_2 + y_2 + x_2 - 2 = x_2 + y_2 + z_2 = n_2\).

Proposition 7
Every connected graph on at least two vertices has a partition \((V_1, V_2)\) of \(V\) such that \(G[V_i]\) is connected \((i = 1, 2)\).

Proof. Let \([V_1, V_2]\) be a minimal edge cut. Then \(G[V_i]\) is connected \((i = 1, 2)\).

In the next section we deduce that all connected 3-regular (multi)graphs satisfy the following necessary condition for the existence of a tree-partition, supporting Conjecture 5.

Theorem 8
Every connected 3-regular multigraph \(G\) on \(n\) vertices admits a partition \((V_1, V_2)\) of \(V(G)\) such that \(e[V_1, V_2] = \frac{1}{2}n + 2\).

For convenience we call such a partition a fair partition, and we omit the brackets in the notation.

2 Proof of Theorem 8

We need some auxiliary definitions and results.

We call a \(K_3\) with precisely one edge \(e\) duplicated (of multiplicity 2) a cone, the vertex of \(K_3\) not incident with \(e\) the end vertex of the cone, and the other two vertices of \(K_3\) the internal vertices of the cone. For any integer \(k \geq 2\), a \(k\)-cone is defined as a 3-regular multigraph obtained from \(k\) disjoint cones and a tree with \(k\) end vertices and all internal vertices of degree 3, by identifying each end vertex of the tree with an end vertex of one of the cones.
Proposition 9
Any $k$-cone ($k > 2$) can be obtained from a $(k - 1)$-cone by subdividing an edge $e$ not belonging to a cone with one new vertex, and making the new vertex adjacent to an end vertex of a newly added cone.

**Proof.** Let $G$ be a $k$-cone ($k > 2$) and let $C$ be a cone of $G$ with end vertex $v$. Clearly $G - V(C)$ contains one vertex $u$ of degree 2, and contracting one of the edges incident with $u$ yields a $(k - 1)$-cone $H$. Hence $G$ can be obtained from $H$ by the reverse procedure. \qed

Lemma 10
All $k$-cones ($k \geq 2$) have a fair partition.

**Proof.** We use induction on $k$. For the 2-cone $G$ one readily checks that the partition of $V(G)$ in $S, T$, where $S$ and $T$ both contain exactly one end vertex of a cone and the two other vertices of the other cone, is a fair partition. Assume all $j$-cones with $2 \leq j < k$ have a fair partition, and consider a $k$-cone $G$. Let $C$ be a cone of $G$ and let $G$ be obtained from a $(k - 1)$-cone $H$ by adding $C$ to $e = uv$ in the sense of Proposition 9. Suppose $S, T$ is a fair partition of $H$, which exists by the induction hypothesis. Without loss of generality we may assume that at least one of the vertices $u$ and $v$ belongs to $S$. Then adding the two internal vertices of $C$ to $T$ and the other two new vertices to $S$ yields a fair partition of $G$. \qed

Lemma 11
Let $G$ be a connected 3-regular multigraph on $n \geq 4$ vertices. If $G$ is not a $k$-cone, then $G$ contains an edge which is not a bridge and does not belong to a cone of $G$.

**Proof.** Clearly any connected 3-regular multigraph contains an edge that is not a bridge. Suppose all nonbridges of $G$ are cone edges. Then $G$ has at least one bridge, because otherwise $G$ cannot have any cones (the edge of $G - E(C)$ incident with the end vertex of a cone $C$ of $G$ is a bridge of $G$), a contradiction. Now, since all noncone edges of $G$ are bridges of $G$, the noncone edges of $G$ induce a tree in $G$ in which all internal vertices have degree 3 and all end vertices correspond to end vertices of cones of $G$. Then $G$ is a $k$-cone. \qed

Theorem 12
Every connected 3-regular multigraph $G$ admits a fair partition.

**Proof.** By induction on $n = |V(G)|$. The statement of the theorem is clearly true for $n = 2$. Suppose $n \geq 4$, and assume every connected 3-regular multigraph on less than $n$ vertices has a fair partition. If $G$ is a $k$-cone for some $k \geq 2$, we are done by Lemma 10. Next assume $G$ is not a $k$-cone and let $e$ be a nonbridge of $G$ which does not belong to a cone of $G$ (such an edge exists by Lemma 11). Let $C$ be the smallest cycle of $G$ containing $e$. 

We first deal with the case that $|V(C)| = 2$. Since $e = xy$ is not in a cone, the two neighbors $u$ and $v$ of respectively $x$ and $y$ in $V(G) \setminus V(C)$ do not coincide. Consider the connected 3-regular multigraph $H$ obtained from $G \setminus V(C)$ by adding an edge $uv$ (possibly $uv \in E(G)$). Clearly $|V(H)| = |V(G)| - 2$, hence $H$ has a fair partition $S, T$. If $u$ and $v$ belong to different parts, say $u \in S$ and $v \in T$, then clearly $S \cup \{x\}, T \cup \{y\}$ is a fair partition of $G$. Hence assume without loss of generality that $\{u, v\} \subset S$, implying $|V(H)| \neq 2$. Since $H$ is connected and $T \neq \emptyset$, without loss of generality assume there is a path $P$ in $G[S]$ from $u$ to some $p$ (possibly $p = u$) such that $v \notin V(P)$, at least one neighbor of $p$ is in $T$, and all neighbors of other vertices of $P$ (if any) are in $S$. If precisely one neighbor of $p$ is in $T$, then $(S \setminus \{p\}) \cup \{x, y\}, T \cup \{p\}$ is a fair partition of $G$. Suppose now that two neighbors of $p$ are in $T$. If $|V(P)| \geq 3$, then let $q$ denote the neighbor of $p$ in $P$, and note that $S \setminus \{p, q\}, T \cup \{p, q\}$ is a fair partition of $H$ which can be used to define a fair partition of $G$ as in the previous argumentation. If $|V(P)| = 2$ ($P = up$), then $(S \setminus \{u, p\}) \cup \{y\}, T \cup \{u, p, x\}$ is a fair partition of $G$. If $V(P) = 1$, we consider the two neighbors $q_1, q_2$ of $u = p$ in $T$ (possibly $q_1 = q_2$), and the neighbors $r_1, r_2$ of $q_1$ in $V(G) \setminus \{u\}$ (possibly $r_1 = r_2$). If $q_1 = q_2$ and $r_1 = r_2$, then in the case $r_1 \in T$, $S \cup \{q_1\}, (T \setminus \{q_1\}) \cup \{x, y\}$ is a fair partition of $G$, and in the case $r_1 \in S$, $(S \setminus \{u\}) \cup \{y, q_1\}, (T \setminus \{q_1\}) \cup \{u, x\}$ is; otherwise, if $r_1, r_2 \in T$, then $S \cup \{x, y, q_1\}, (T \setminus \{q_1\}) \cup \{u, y\}$ is a fair partition of $G$; if $r_1, r_2 \in S$, then $(S \setminus \{u\}) \cup \{x, q_1\}, (T \setminus \{q_1\}) \cup \{u, y\}$ is; in all other subcases, $S \cup \{q_1\}, (T \setminus \{q_1\}) \cup \{x, y\}$ is a fair partition of $G$.

For the remaining cases we may assume that, apart from the cones, there is no $C_2$ in $G$. Let $e = xy$ and $N(x) \setminus \{y\} = \{a, b\}, N(y) \setminus \{x\} = \{c, d\}$ (note that $a \neq b, c \neq d$, but possibly $a = c$ etc.). Consider the connected 3-regular multigraph $H$ obtained from $G \setminus \{x, y\}$ by adding the edges $ab$ and $cd$ (possibly $ab \in E(G)$ or $cd \in E(G)$). Clearly $H$ has a fair partition $S, T$. If not all of $\{a, b, c, d\}$ belong to the same part, it is quite easy to check that $S \cup \{x\}, T \cup \{y\}$ is a fair partition of $G$. We are left with the case that $\{a, b, c, d\} \subset S$ (or $T$). Since $H$ is connected and $T \neq \emptyset$, without loss of generality assume there is a path $P$ in $G[S]$ from $a$ to some $p$ (possibly $a = p$) such that $V(P) \cap \{b, c, d\} \setminus \{a\} = \emptyset$, at least one neighbor of $p$ is in $T$, and all neighbors of the other vertices of $P$ (if any) are in $S$. If precisely one neighbor of $p$ is in $T$, then $(S \setminus \{p\}) \cup \{x, y\}, T \cup \{p\}$ is a fair partition of $G$. Suppose now that two neighbors of $p$ are in $T$. If $|V(P)| \geq 2$, then let $q$ denote the neighbor of $p$ in $P$, and note that $S \setminus \{p, q\}, T \cup \{p, q\}$ is a fair partition of $H$ which can be used to define a fair partition of $G$ as in the previous argumentation. If $|V(P)| = 1$, then $S \setminus \{a\}, T \cup \{a, x, y\}$ is a fair partition of $G$. \qed

3 Final remarks

Suppose $G$ is a connected 3-regular graph.

It is clear that if $\lambda_c(G) = \lambda(G) = 1$, then $G$ does not admit a tree-partition, since each of the two components of $G - e$, where $e$ is a bridge of $G$, contains a cycle, and hence $e$ should be in both trees.
Figure 1 shows two graphs with $\lambda_c = \lambda = 2$. The left one has a tree-partition, indicated by the black and white vertices; the right one does not admit a tree-partition: this is easily seen using the following general observation.

![Figure 1: The left graph has a tree-partition, the right one has none.](image1)

**Proposition 13**

If $\{e, f\}$ is a minimum edge cut of $G$, and $S, T$ is a tree-partition, then $e \in G[S]$ and $f \in G[T]$ or vice versa.

**Proof.** Both components of $G - \{e, f\}$ contain a cycle, so both $G[S]$ and $G[T]$ must contain at least one vertex of each of these components. \[\Box\]

Using Lemma 3 one can construct (planar) examples with $\lambda_c = \lambda = 3$ not admitting a tree-partition, starting with nonhamiltonian maximal planar graphs with connectivity 3. For example, deleting the five black vertices from the left graph in Figure 2 yields six components, hence $G$ is not hamiltonian, and hence the right graph, which is the dual of the left one, does not admit a tree-partition.

![Figure 2: The right graph is the dual of the left graph and admits no tree-partition.](image2)

A nonplanar example with $\lambda_c = 3$ not admitting a tree-partition is the graph obtained from a $K_{3,3}$ by replacing each vertex $v$ by a triangle $v_1 v_2 v_3 v_1$ and joining each of the $v_i$'s
with one of the vertices of each of the three triangles corresponding to the neighbors of $v$ in $K_{3,3}$ (the so-called inflation of $K_{3,3}$).

The two examples with $\lambda_c = 3$ were the smallest examples found by an exhaustive search by computer, using a self-written program in Mathematica and relying heavily on algorithms of the software packages Combinatorica and GENREG, which were found on the Internet. The Combinatorica package due to Skiena [4] is a Mathematica package for Combinatorics and Graph Theory with over 230 functions that can be downloaded from http://www.mathsource.com/cgi-bin/MathSource/Enhancements/DiscreteMath/0200-170. A description of this package is [4]. The C-program GENREG due to Meringer [3] can generate all connected $k$-regular graphs on $n$ vertices quite fast. The format of the generated graphs can be adapted in such a way that it can be used directly as input for Combinatorica. GENREG can be downloaded from http://www.mathe2.uni-bayreuth.de/markus/reggraphs.html.

References