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Long $D_3$-cycles in graphs with large minimum degree

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Abstract

It is shown that if $G$ is a 2-connected graph on $n$ vertices, with minimum degree $\delta$ such that $n \leq 4\delta - 6$, and with a maximum independent set of cardinality $\alpha$, then $G$ contains a cycle of length at least $\min\{n, n + 2\delta - 2\alpha - 2\}$ or $G \in \mathcal{F}$, where $\mathcal{F}$ denotes a well-known class of nonhamiltonian graphs of connectivity 2.

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We use [1] for terminology and notation not defined here and consider finite simple graphs only.

Let $G$ be a graph and $k$ a positive integer. We denote by $n(G)$ the number of vertices of $G$, by $\alpha(G)$ the number of vertices in a maximum independent set of $G$, and by $\omega(G)$ the number of components of $G$. The number of components of $G$ with at least $k$ vertices is denoted by $\omega_k(G)$. When no confusion can arise, we will often write $n, \alpha, V, \ldots$ instead of $n(G), \alpha(G), V(G), \ldots$

Let $C$ be a cycle of $G$ and $u, v \in V(C)$. We denote by $\overrightarrow{C}$ the cycle $C$ with a given orientation. By $u\overrightarrow{C}v$ we denote the set of consecutive vertices of $C$ from $u$ to $v$ in the direction specified by $\overrightarrow{C}$. We use $u^+$ to denote the successor of $u$ on $\overrightarrow{C}$ and $u^-$ to denote its predecessor. We call $C$ a $D_k$-cycle if $\omega_k(G - V(C)) = 0$. Similar notation is used for paths.

If $v \in V(G)$ and $A \subseteq V(G)$, then $\varepsilon(v, A)$ denotes the number of vertices in $A$ which are adjacent to $v$. If $H$ is a subgraph of $G$ and $v \in V(G)$ then $N_H(v) = \{x \in V(H) \mid xv \in E(G)\}$.

We now define a number of specific graphs and classes of graphs. For a positive integer $k$ we denote by $K_k$ the set of all graphs consisting of three disjoint complete graphs, where each of the components has order at least $k$. The class $\mathcal{G}$ will be the set of all spanning subgraphs of graphs that can be obtained as the join of $K_2$ and a graph in $K_1$. The class $\mathcal{H}$ is the set of all spanning subgraphs of graphs that can be obtained from the join of $K_1$ and a graph $H$.  

1
in \( K_2 \) by adding the edges of a triangle between three vertices from distinct components of \( H \). The class \( J \) is the set of all spanning subgraphs of graphs that can be obtained from a graph \( H \) in \( K_3 \) by adding the edges of two triangles between two disjoint triples of vertices, each containing one vertex of each component of \( H \). We set \( F = \mathcal{G} \cup \mathcal{H} \cup \mathcal{J} \). The class \( F \) is a well-known class of nonhamiltonian graphs. Note that all graphs in \( F \) have connectivity at most 2.

A generalization of the following result occurs in [3].

**Theorem 1 ([3])**

*If \( G \) is a 2-connected graph with \( n \leq 3\delta - 2 \), then \( G \) contains a cycle of length at least \( \min\{n, n + \delta - \alpha\} \).*

In a sense, Theorem 1 is best possible. The complete bipartite graph \( K_{p,q} \) \((p \geq 2, q \geq p + 1)\) contains a longest cycle of length \( 2p = n + \delta - \alpha \). Theorem 1 is also best possible in the sense that the upper bound \( 3\delta - 2 \) imposed on \( n \) is tight, as shown by the graph \( K_2 \cup 3K_p \) \((p \geq 2)\) with \( n = 3\delta - 1 \), which has a longest cycle of length \( 2p + 2 < \min\{n, n + \delta - \alpha\} = \min\{3p + 2, 4p\} \). Nevertheless it is possible to obtain a lower bound for the length of a longest cycle in graphs with \( n > 3\delta - 2 \) outside the class \( F \). We will prove the following result.

**Theorem 2**

*If \( G \) is a 2-connected graph with \( n \leq 4\delta - 6 \), then \( G \) contains a cycle of length at least \( \min\{n, n + 2\delta - 2\alpha - 2\} \) or \( G \notin F \).*

In a sense, Theorem 2 is best possible. The graph \( K_p \cup qK_2 \) \((p \geq 2, q \geq p + 1)\) has a longest cycle of length \( 3p = n + 2\delta - 2\alpha - 2 \). Again, the upper bound \( 4\delta - 6 \) imposed on \( n \) is tight, as shown by the graph \( K_3 \cup 4K_p \) \((p \geq 3)\) with \( n = 4\delta - 5 \), which has a longest cycle of length \( 3p + 3 < \min\{n, n + 2\delta - 2\alpha - 2\} = \min\{4p + 3, 6p - 3\} \).

Theorem 2 generalizes the following result from [5].

**Corollary 3 ([5])**

*If \( G \) is a 2-connected graph with \( n \leq 4\delta - 6 \) and \( \alpha \leq \delta - 1 \), then \( G \) is hamiltonian or \( G \notin F \).*

It should be noted that for nonhamiltonian graphs with \( n \leq 3\delta - 2 \) the lower bound for the length of a longest cycle in Theorem 1 is always strictly greater than the one in Theorem 2.

To prove Theorem 2, we use the following result.

**Theorem 4 ([5])**

*If \( G \) is a 2-connected graph with \( n \leq 4\delta - 6 \), then \( G \) contains a \( D_3 \)-cycle or \( G \notin F \).*

**Proof of Theorem 2**

Let \( G \) be a 2-connected graph with \( n \leq 4\delta - 6 \) and assume \( G \notin F \). By Theorem 4 we know that \( G \) contains a \( D_3 \)-cycle. Let \( C \) be a longest \( D_3 \)-cycle such that \( \omega_2(G - V(C)) \geq \omega_2(G - V(C')) \) for every longest \( D_3 \)-cycle \( C' \). Fix an orientation on \( C \). We distinguish two cases.
Case 1 \( C \) is not a \( D_2 \)-cycle.
Then \( G - V(C) \) contains at least one component of cardinality 2. Let \( x \) and \( y \) be the vertices of such a component and let \( u_1, \ldots, u_k \) be the vertices in \( N_C(x) \cup N_C(y) \), ordered cyclically around \( C \). For every segment \( u_i \overrightarrow{C} u_{i+1} \) (subscripts taken modulo \( k \)) we will mark two or three vertices according to the first feasible rule out of the following.

By the way \( C \) the choice of \( C \) the cycle \( C \) around \( N \) the cycle \( u \) has both \( x; y \) as neighbors we will call the segment \( u_i \overrightarrow{C} u_{i+1} \) a 1-segment. In that case we will mark two vertices according to the first feasible rule out of the following.

1. \( s_{i1} = u_i^+, s_{i2} \in N(s_{i1}) \setminus V(C) \).
2. \( s_{i1} = u_i^+, s_{i2} = u_i^{++} \).

If \( u_i \) has both \( x \) and \( y \) as neighbors we will call \( u_i \overrightarrow{C} u_{i+1} \) a 2-segment. In that case we will mark three vertices according to the first feasible rule out of the following.

3. \( s_{i1} = u_i^+, s_{i2} \in N(s_{i1}) \setminus V(C), s_{i3} \in N(s_{i2}) \setminus V(C) \).
4. \( s_{i1} = u_i^+, s_{i2} \in N(s_{i1}) \setminus V(C), s_{i3} = u_i^{++} \).
5. \( s_{i1} = u_i^+, s_{i2} = u_i^{++}, s_{i3} \in N(s_{i2}) \setminus V(C) \).
6. \( s_{i1} = u_i^+, s_{i2} = u_i^{++}, s_{i3} = u_i^{+++} \).

By the way \( C \) and the marked sets are chosen, \( S_i \subseteq u_i^+ \overrightarrow{C} u_{i+1} \cup (V(G) \setminus V(C)) \) for every marked set \( S_i \).

In the proof we will often use the following path to construct a cycle which contradicts the choice of \( C \). Let \( Q_{ij} \) be a longest \((u_i, u_j)\)-path which has all its internal vertices in \( \{x, y\} \). Note that if \( u_i \overrightarrow{C} u_{i+1} \) or \( u_j \overrightarrow{C} u_{j+1} \) is a 2-segment, then \( Q_{ij} \) always contains both \( x \) and \( y \).

Claim 1 \( S_i \cap S_j = \emptyset \) for all \( i, j = 1, \ldots, k \) with \( i \neq j \).

Proof Assuming the contrary, let \( v \in S_i \cap S_j \). Since \( S_i \subseteq u_i^+ \overrightarrow{C} u_{i+1} \cup (V(G) \setminus V(C)) \) for every marked set \( S_i \), we have \( v \in V(G) \setminus V(C) \). If there exists a \((u_i^+, u_j^+)\)-path \( P \) internally disjoint from \( C \), then clearly \( x, y \notin V(P) \) and the cycle \( u_i \overrightarrow{Q_{ij}} u_j \overrightarrow{C} u_i \) contradicts the choice of \( C \) (since it is a longer \( D_3 \)-cycle than \( C \)).

Now we are left with the case that \( S_i \) or \( S_j \) is of type 5. By symmetry we may assume that \( S_j \) is of type 5. If \( S_j \) is of type 1, 3 or 4, and \( v = s_{i2} = s_{j3} \), then the cycle \( u_i \overrightarrow{Q_{ij}} u_j \overrightarrow{C} u_i \overrightarrow{s_{i2}} u_i^{++} \overrightarrow{C} u_i \) contradicts the choice of \( C \). If \( S_i \) is of type 3 and \( v = s_{i3} = s_{j3} \), then the cycle \( u_i \overrightarrow{Q_{ij}} u_j \overrightarrow{C} u_i \overrightarrow{s_{i2}} s_{i3} u_j^{++} \overrightarrow{C} u_i \) contradicts the choice of \( C \). If \( S_i \) and \( S_j \) are both of type 5, then \( v = s_{i3} = s_{j3} \) and the cycle \( u_i \overrightarrow{Q_{ij}} u_j \overrightarrow{C} u_i \overrightarrow{s_{i3}} s_{j3} u_j^{++} \overrightarrow{C} u_i \) contradicts the choice of \( C \).

Claim 2 \( N(u_i^+) \subseteq V(C) \) for all \( i = 1, \ldots, k \).
Proof Assume that $u^+_i$ has a neighbor $v \in V(G) \setminus V(C)$. By showing that $v$ does not have neighbors in the marked sets $S_2, \ldots, S_k$, we will get a contradiction with the degree condition. Since $S_1 \cap S_j = \emptyset$ for all $j = 2, \ldots, k$ and since there does not exist a $(u^+_i, u^+_j)$-path internally disjoint from $C$ for all $j = 2, \ldots, k$, we are left with the following cases.

a. $S_j$ of type 2, $v s_j \in E(G)$.

b. $S_j$ of type 4, $v s_j \in E(G)$.

c. $S_j$ of type 5, $v s_j \in E(G)$.

d. $S_j$ of type 5, $v s_j \in E(G)$.

e. $S_j$ of type 6, $v s_j \in E(G)$.

Note that the case ‘$S_j$ of type 6 with $v s_j \in E(G)$’ does not occur, since then $S_j$ would have been a marked set of type 5.

In Case a the cycle $C' = u_1 Q_j u_j C u^+_i v s_j C u_1$ contradicts the choice of $C$, since $C'$ is a longer $D_3$-cycle. In Case b, c and d we can find similar cycles. In Case e the cycle $C' = u_1 Q_j u_j C u^+_i v s_j C u_1$ contradicts the choice of $C$, since $Q_j$ contains both $x$ and $y$ (since $S_j$ is a 2-segment), and thus $C'$ is a longer $D_3$-cycle.

Now let $p$ be the number of 2-segments. Then $C$ has $d(x) + d(y) - 2 - 2p$ 1-segments and $p$ is at most $d(x) - 1$. Since $S_i \cap S_j = \emptyset$ for all $i, j = 1, \ldots, k$ with $i \neq j$, we get

\[
|\bigcup_{i=2}^{k} S_i| \geq |\bigcup_{i=1}^{k} S_i| - 3 = 3p + 2(d(x) + d(y) - 2 - 2p) - 3 \\
\geq d(x) + 2d(y) - 6 \geq 3\delta - 6.
\]

Using the fact that $v$ does not have neighbors in $S_2, \ldots, S_k$ and $v \notin \bigcup_{i=2}^{k} S_i$ (since we may assume that $v = s_{12}$), and since $n \leq 4\delta - 6$ we get

\[
d(v) \leq n - 1 - |\bigcup_{i=2}^{k} S_i| \leq n - 3\delta + 5 \leq \delta - 1,
\]

a contradiction. □

Now consider the set consisting of a vertex from each component of $G - V(C)$ and the vertices $u^+_1, \ldots, u^+_k$. This set is an independent set. Since $C$ is a $D_3$-cycle, $G - V(C)$ has at least $(n - |V(C)|)/2$ components. Since $k \geq \delta - 1$, we get

\[
\alpha \geq \delta - 1 + \frac{n - |V(C)|}{2},
\]
Case 2 \(C\) is a \(D_2\)-cycle.

Assuming \(G\) is nonhamiltonian, let \(x \in V(G) \setminus V(C)\) and let \(u_1, \ldots, u_k\) be the vertices in \(N_C(x)\), ordered cyclically around \(C\). As in Case 1 we will mark two or three vertices for every segment \(u_i C u_{i+1}\). If \(u_i^{++} = u_{i+1}\) and \(N(u_i^+) \subseteq V(C)\) we define \(S_i = \{u_i^+, u_i^{++}\}\) and call this set a set of type 1. In all other cases we will mark three vertices according to the first feasible rule out of the following.

1. \(s_{i1} = u_i^+\), \(s_{i2} \in N(s_{i1}) \setminus V(C)\), \(s_{i3} = u_i^{++}\).
2. \(s_{i1} = u_i^+, s_{i2} = N(s_{i1}) \setminus V(C)\), \(s_{i3} = u_i^{++}\).
3. \(s_{i1} = u_i^+, s_{i2} = u_i^{++}, s_{i3} \in N(s_{i2}) \setminus V(C)\).
4. \(s_{i1} = u_i^+, s_{i2} = u_i^{++}, s_{i3} = u_i^{+++}\).

Claim 3 If \(S_i\) and \(S_j\) \((i \neq j)\) are not both of type 3, then \(S_i \cap S_j = \emptyset\)

The proof of Claim 3 is similar to the proof of Claim 1.

Claim 4 \(N(u_i^+) \subseteq V(C)\) for all \(i = 1, \ldots, k\).

Proof Assume that \(u_i^+\) has a neighbor \(v \in V(G) \setminus V(C)\). Showing that \(v\) does not have neighbors in the marked sets \(S_2, \ldots, S_k\) will not be enough, since there may be an overlap between marked sets of type 3. However, given the vertex \(v\), we can modify \(S_2, \ldots, S_k\) to a collection \(S'_2, \ldots, S'_k\) of marked sets, such that \(v\) does not have neighbors in \(S'_2, \ldots, S'_k\), \(|S'_i| = |S_i|\) for all \(i = 1, \ldots, k\) and \(S'_i \cap S'_j = \emptyset\) for all \(i, j = 1, \ldots, k\) with \(i \neq j\).

Consider a maximal collection of \(t\) marked sets of type 3, with \(t \geq 2\), which all have the same vertex \(z \in V(G) \setminus V(C)\) in common. Let \(S'_i = (S_i \cup \{u_i^{+++}\}) \setminus \{z\}\) for the first \(t - 1\) sets and \(S'_t = S_t\) for the remaining set. Let \(u^*\) be the vertex in \(N_C(x)\) corresponding to this remaining set. Note that \(u_i^{+++} \notin S_{i+1}\), otherwise \(S_i\) would have been a set of type 1. We also see that \(v u_i^{+++} \notin E(G)\), otherwise the cycle \(u_1 x u_i C u_i^{+++} v u_i^{++} C u_i^{++} z u_i^{++} C u_1\) would be a longer \(D_3\)-cycle than \(C\).

We repeat this procedure for all maximal collections of at least two marked sets of type 3, with the same vertex in \(V(G) \setminus V(C)\) in common. Finally, for all marked sets which have not been modified yet, we define \(S'_i = S_i\).

Using similar arguments as in the proof of Claim 2 we can prove that \(v\) does not have neighbors in \(S'_2, \ldots, S'_k\). If \(S_i\) is of type 4 and \(v s_{i3} \in E(G)\), the cycle \(u_1 x u_i C u_i^{+++} v s_{i3} C u_1\) would be a \(D_3\)-cycle of the same length as \(C\), but with more components of order 2 outside of it (one instead of zero). In all other cases we can construct longer \(D_3\)-cycles.
Let \( T_1, \ldots, T_m \) be the segments of \( C \) remaining after deleting \( S_2', \ldots, S_k' \) and let \( T = \bigcup_{i=1}^{m} T_i \). Since \( v \) does not have two consecutive vertices of \( C \) as its neighbors, we have \( \varepsilon(v, T_i) \leq \frac{1}{2}|T_i| + 1 \). Let \( p \) be the number of segments of type 1. Since \( x, v \not\in V(C) \), we get

\[
|T| \leq n - 2p - 3(d(x) - p - 1) - 2.
\]

Thus, using that \( n \leq 4\delta - 6 \), we have

\[
d(v) = \varepsilon(v, T) = \sum_{i=1}^{m} \varepsilon(v, T_i) \leq \frac{1}{2} \sum_{i=1}^{m} (|T_i| + 1) = \frac{1}{2}|T| + \frac{1}{2}m
\]

\[
\leq \frac{1}{2}(n - 2p - 3(d(x) - p - 1) - 2) + \frac{1}{2}(d(x) - p)
\]

\[
= \frac{1}{2}n - d(x) + \frac{1}{2} \leq \frac{1}{2}n - \delta + \frac{1}{2} \leq \delta - \frac{5}{2},
\]

a contradiction. \( \square \)

Using Claim 4, we know that the set consisting of all vertices of \( G - V(C) \), together with \( u_1^+, \ldots, u_k^+ \), is an independent set. Since \( k \geq \delta \) we get

\[
\alpha \geq \delta + n - |V(C)| \geq \delta - 1 + \frac{n - |V(C)|}{2},
\]

from which it follows that

\[
|V(C)| \geq n + 2\delta - 2\alpha - 2.
\]

The proof of Theorem 2 is quite long and can be considerably shortened if we use the following result independently obtained by Brandt and Jung.

**Theorem 5** ([2], [4])

If \( G \) is a 2-connected graph with \( n \leq 4\delta - 6 \), then every longest cycle of \( G \) is a \( D_3 \)-cycle or \( G \in \mathcal{F} \).

If we use Theorem 5, we can distinguish the cases '\( G \) has a longest cycle which is a \( D_3 \)-cycle and not a \( D_2 \)-cycle' and 'every longest cycle of \( G \) is a \( D_2 \)-cycle'. Instead of marking two or three vertices as in our proof of Theorem 2, we can now simply mark \( u_i^+ \) and \( u_i^{++} \), or \( u_i^+ \), \( u_i^{++} \) and \( u_i^{+++} \). The rest of the proof is similar to ours.

**References**


