Polynomial J-Spectral Factorization
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Abstract—Several algorithms are presented for the J-spectral factorization of a para-Hermitian polynomial matrix. The four algorithms that are discussed are based on diagonalization, successive factor extraction, interpolation, and the solution of an algebraic Riccati equation, respectively. The paper includes a special algorithm for the factorization of unimodular para-Hermitian polynomial matrices and deals with canonical, noncanonical, and nearly noncanonical factorizations.

I. INTRODUCTION

This paper reviews several methods for the J-spectral factorization of a polynomial para-Hermitian matrix \( Z \) with real coefficients. A polynomial matrix \( Z \) is said to be para-Hermitian if \( Z^* = Z \). The polynomial matrix \( Z^* \) is the adjoint of \( Z \), defined by \( Z^*(s) = Z^T(-s) \), with the superscript \( T \) denoting the transpose. We call \( Z = P^* JP \) (1)
a spectral factorization if \( J \) is a signature matrix and \( P \) a square matrix with real coefficients such that \( \det P \) is Hurwitz, that is, has all its roots in the closed left-half plane. The signature matrix \( J \) is of the form
\[ J = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & -I_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]
with \( I_1 \) and \( I_2 \) unit matrices of not necessarily the same dimensions. A factorization of the form \( Z = P^* JP \), with \( P \) square such that \( \det P \) is Hurwitz is called a spectral cofactorization. A spectral cofactorization may be obtained by transposing a spectral factorization of the transpose \( Z^T \) of \( Z \).

The best known polynomial spectral factorization problem is that of factoring a para-Hermitian polynomial matrix \( Z \) which is positive-definite on the imaginary axis, that is, \( Z(j\omega) > 0 \) for all \( \omega \in \mathbb{R} \). Within the systems and control area, this problem arises in multivariable Wiener filtering and in frequency domain versions of LQG theory. In this case the signature matrix \( J \) is the unit matrix, and the desired factorization is of the form \( Z = P^* P \). There are numerous references on this problem (see for instance [13], [2], [11]).

The more general polynomial spectral factorization problem, with \( Z \) indefinite on the imaginary axis, is encountered in LQG game theory and, notably, in \( H_\infty \)-optimization theory. J-spectral factorization is fundamental for the solution of the \( H_\infty \) standard problem (see, for instance, [9] and [1]). In the state space approach the rational J-spectral factorization problem that lies at the heart of the solution leads to a pair of algebraic Riccati equations (see [5]). In the "polynomial approach," representing rational matrices by polynomial matrix fractions results in two polynomial J-spectral factorizations [15]. Because the polynomial approach allows nonproper rational transfer matrices, it has advantages over the state space approach.

In the literature few, if any, practical algorithms for the J-spectral factorization of polynomial matrices may be found. This paper develops four factorization methods. They are based on
1) diagonalization,
2) successive factor extraction,
3) interpolation, and
4) solution of an algebraic Riccati equation.
Earlier versions of the algorithms were discussed in conference papers [20], [21].

After a review of the existence and uniqueness of factorizations and canonical and noncanonical factorizations in Section II, first the factorization of para-Hermitian unimodular polynomial matrices is discussed in Section III. Following this, a factorization algorithm based on diagonalization is described in Section IV. Section V presents an algorithm that uses successive factor extraction. After an intermezzo on the computation of the zeros of a polynomial matrix in Section VI, a factorization algorithm based on interpolation is explained in Section VII. The last method that is discussed is based on the solution of a suitable Riccati equation and is the subject of Section VIII. Section IX deals with nearly noncanonical factorizations.

We use various notions from polynomial matrix theory and its application to system theory. Suitable references are Kučera [13], Kailath [12], Callier and Desoer [3], and Vardulakis [22]. A mathematical text on polynomial matrices is Golberg et al. [7].

II. EXISTENCE AND UNIQUENESS

No necessary and sufficient existence conditions appear to be known for J-spectral factorization. The following sufficient condition is well known from the work of Jakubovic [10].

Theorem 2.1 (Existence of J-Spectral Factorization): Suppose that the multiplicity of the zeros on the imaginary axis of each of the invariant polynomials of the para-Hermitian polynomial matrix \( Z \) is even. Then \( Z \) has a spectral factorization \( Z = P^* JP \).
For the definition of the invariant polynomials of a polynomial matrix see for instance Kailath [12]. The condition of the theorem is violated if and only if any of the invariant factors is not factorizable by itself. An example of a nonfactorizable polynomial is \( 1 + s^2 \). A sufficient condition which implies that of the theorem is that \( \det Z \) has no roots on the imaginary axis. This assumption is often invoked in what follows.

The signature matrix \( J \) of (1) is nonsingular if and only if \( \det Z \) is not identical to zero. \( J \) is a unit matrix (that is, \( J = I \)) if and only if \( Z(j\omega) \) is nonsingular and, hence, \( Z(j\omega) > 0 \) for \( \omega \in \mathbb{R} \). More generally, if \( \det Z \) is not identical to zero, the numbers of positive and negative eigenvalues of \( Z \) on the imaginary axis (that is, the eigenvalues of the Hermitian matrix \( Z(j\omega), \omega \in \mathbb{R} \)) are constant and equal the dimensions of \( I_1 \) and \( I_2 \), respectively.

In the sequel it is often assumed that \( Z \) is diagonally reduced [3], [2].

Definition 2.2 (Diagonal Reducedness): Suppose that the \( n \times n \) para-Hermitian polynomial matrix \( Z(j\omega) \) is diagonally reduced (that is, \( Z(j\omega) = 0 \) for \( \omega \in \mathbb{R} \)) and define the diagonal leading coefficient matrix \( Z_L \) of \( Z \), if it exists, as

\[
Z_L = \lim_{|s| \to \infty} E(s)^{-1} Z(s) E(s)^{-1}
\]

where \( E \) is the polynomial matrix defined by

\[
E(s) = \begin{bmatrix} s^\delta_1 & 0 & \cdots & 0 \\
0 & s^\delta_2 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & s^\delta_n \end{bmatrix}
\]

\( Z \) is diagonally reduced if \( Z_L \) is nonsingular.

Denote the degrees of the \( (i,j) \) entry of \( Z \) as \( \delta_{ij} \). It is easy to see that the diagonal leading coefficient matrix \( Z_L \) of \( Z \) exists if and only if \( 2\delta_{ij} \leq \delta_i + \delta_j \) for \( i,j = 1, 2, \cdots, n \).

Example 2.3 (Diagonal Reducedness): The polynomial matrix

\[
Z(s) = \begin{bmatrix} 0 & s \\
-s & s^2 \end{bmatrix}
\]

has half diagonal degrees \( \delta_1 = 0 \) and \( \delta_2 = 1 \), and leading coefficient matrix

\[
Z_L = \begin{bmatrix} 0 & 1 \\
1 & -1 \end{bmatrix}
\]

\( Z_L \) is nonsingular and, hence, \( Z \) is diagonally reduced. The polynomial matrix

\[
Z(s) = \begin{bmatrix} 0 & s \\
-s & 0 \end{bmatrix}
\]

has half diagonal degrees 0 and 0 but its diagonal leading coefficient matrix is undefined.

If a para-Hermitian polynomial matrix \( Z \) with finite diagonal leading coefficient matrix \( Z_L \) is not diagonally reduced, there always exists a unimodular polynomial matrix \( U \) such that \( U^* Z U \) is diagonally reduced. For the construction of \( U \) see Section III.

Factorizations of diagonally reduced polynomial matrices may or may not be canonical.

Definition 2.4 (Canonical Factorization): The spectral factorization of a diagonally reduced para-Hermitian polynomial matrix \( Z \) is canonical if there exists a spectral factor \( P \) that is column reduced with column degrees equal to the half diagonal degrees of \( Z \).

For the definition of the canonical factorization of monic polynomial matrices see Gohberg et al [7]. Our definition of polynomial canonical factorizations is closely related to the definition of the canonical factorization of rational matrices (see, for instance, [6]).

Factorizations of polynomial matrices that are positive-definite on the imaginary axis are always canonical [2], but this is not true in the indefinite case. Noncanonical factorizations arise in \( H_\infty \) optimization when optimal solutions (as opposed to suboptimal) are computed [15].

Example 2.5 (Canonical and Noncanonical Factorizations): Consider the polynomial matrix

\[
Z(s) = \begin{bmatrix} \varepsilon & 1-s \\ 1+s & 1-s^2 \end{bmatrix}
\]

with \( \varepsilon \) real such that \( |\varepsilon| < 1 \). For \( \varepsilon \neq 0 \) the matrix \( Z \) has the spectral factor and signature matrix

\[
P(s) = \begin{bmatrix} -1 & -\frac{2}{\varepsilon} + 1+s \\ \frac{2}{\varepsilon} \sqrt{1-\varepsilon} & 2 \varepsilon \sqrt{1-\varepsilon} \end{bmatrix}, \quad J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
\]

(compare Example 5.2) \( P \) is column reduced with column degrees equal to the half diagonal degrees of \( Z \). Hence, for \( \varepsilon \neq 0 \) the factorization of \( Z \) is canonical. Note that as \( \varepsilon \) approaches 0, some of the coefficients of \( P \) grow without bound.

For \( \varepsilon = 0 \) the matrix \( Z \) reduces to

\[
Z(s) = \begin{bmatrix} 0 & 1-s \\ 1+s & 1-s^2 \end{bmatrix}
\]

which has the spectral factor and signature matrix

\[
P(s) = \frac{1}{2} \begin{bmatrix} 1+s & \frac{3-s^2}{2} \\ 1+s & \frac{-3-s^2}{2} \end{bmatrix}, \quad J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
\]

(again compare Example 5.2.) The spectral factor \( P \) is not column reduced, and its column degrees 1 and 2 do not equal the half diagonal degrees \( \delta_1 = 0 \) and \( \delta_2 = 1 \) of \( Z \). By direct computation it may be verified that there exists no spectral factor of \( Z \) with column degrees 0 and 1. Hence, for \( \varepsilon = 0 \) the factorization of \( Z \) is noncanonical. Apparently, the spectral factorization of \( Z \) has an essential discontinuity at \( \varepsilon = 0 \).

On the other hand, there does exist a canonical cofactorization of (10) with (left) spectral factor and signature matrix given by

\[
P(s) = \begin{bmatrix} 1 & 1 \\ 1+s & 0 \end{bmatrix}, \quad J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
\]

1 A polynomial matrix is monic if its highest degree coefficient matrix is the unit matrix.

2 A cofactorization is canonical if there exists a row reduced left spectral factor whose row degrees equal the half diagonal degrees.
Of the various algorithms discussed in this paper, only that based on diagonalization (Section IV) can handle factorizations that are noncanonical and factorizations of polynomial matrices \( Z \) such that \( \det Z \) has roots on the imaginary axis. The algorithm based on symmetric factor extraction (Section V) can deal with noncanonical factorizations. The two other algorithms (interpolation, Section VII and the one based on solution of a Riccati equation, Section VIII) can handle nearly noncanonical factorizations. Such factorizations are the subject of Section IX.

It remains to discuss the uniqueness of spectral factorizations.

**Theorem 2.6 (Nonuniqueness of J-Spectral Factorizations):** Let the polynomial matrix \( P \) be a spectral factor of the para-Hermitian polynomial matrix \( Z \) with corresponding signature matrix \( J \).

1. All other spectral factors of \( Z \) are of the form \( UP \), with \( U \) unimodular such that

   \[
   U^* J U = J. \tag{13}
   \]

   \( U \) is said to be a \( J \)-unitary unimodular matrix.

2. If the factorization is canonical (that is, \( P \) is column reduced), any other canonical spectral factor is of the form \( UP \) with \( U \) constant \( J \)-unitary [18].

Given \( J \) there are many \( J \)-unitary unimodular matrices. In the \( 2 \times 2 \) case, with \( J = \text{diag}(1,-1) \), all constant \( J \)-unitary matrices are given by

\[
U = \begin{bmatrix} c_1 \cosh \alpha & c_2 \sinh \alpha \\ c_1 \sinh \alpha & c_2 \cosh \alpha \end{bmatrix} \tag{14}
\]

with \( c_1 = \pm 1, c_2 = \pm 1, \) and \( \alpha \in \mathbb{R} \). An example of a nonconstant \( J \)-unitary unimodular matrix with this signature matrix is

\[
U(s) = \begin{bmatrix} 1 + s & s \\ s & -1 + s \end{bmatrix}. \tag{15}
\]

We conclude this section with a few comments about the factorization of constant square symmetric matrices. Such a factorization is needed in most of the algorithms that we discuss for polynomial matrices. Positive-definite symmetric constant matrices \( S \) may conveniently be factored by Choleski decomposition (see for instance [8]) in the form \( S = C^T C \), with \( C \) lower- or upper-triangular. The Choleski algorithm, based on successively clearing the nondiagonal entries of each row and corresponding column by using the diagonal entry, may be generalized straightforwardly to nondefinite matrices, but fails whenever all diagonal entries are zero, such as in

\[
S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{16}
\]

Although this difficulty may be remedied, the most reliable method to factor \( S \) is to find its Schur decomposition [8, p. 192]

\[
S = QTQ^H \tag{17}
\]

where the superscript \( H \) indicates the complex conjugate transpose. \( Q \) is unitary, that is, \( Q^HQ = QQ^H = I \), and, in general, \( T \) is upper-triangular, with the eigenvalues of \( S \) on its diagonal. The eigenvalues may be arranged in any order. If \( S \) is symmetric, \( T \) is diagonal and real, and \( Q \) may be arranged to be orthogonal, that is, real such that \( QTQ = QQ^T = I \) [8, p. 268]. There exist reliable algorithms for computing the Schur decomposition [8, ch. 7]. The MATLAB Robust-Control Toolbox [4] provides a routine.

Once the Schur decomposition (17) of \( S \) is available, with \( T \) diagonal, it is simple to bring \( S \) into the form \( S = P^T JP \).

### III. Factorization of a Unimodular Matrix

In this section we consider the factorization of unimodular para-Hermitian matrices. Such factorizations arise in the more general algorithms based on diagonalization (Section IV) and on successive extraction (Section V).

The algorithm we present consists of the successive application of three types of steps.

#### Algorithm 3.1 (Factorization of a Unimodular Polynomial Matrix): For a given nonsingular para-Hermitian \( n \times n \) polynomial matrix \( Z \) with finite diagonal leading coefficient matrix, we calculate the spectral factor \( P \) and the signature matrix \( J \) of the factorization \( Z = P^T JP \) by the following steps.

**Step 1. Deflation** [2]. Find the half diagonal degrees \( \delta_1, \delta_2, \ldots, \delta_n \) of \( Z \), and let \( P := I \).

a) Determine the diagonal leading coefficient matrix \( Z_L \) of \( Z \) corresponding to the half diagonal degrees \( \delta_1, \delta_2, \ldots, \delta_n \). If \( Z_L \) is nonsingular (which only happens if \( Z \) is a constant matrix), go to Step 2. Else, compute a real null vector \( e = \text{col}(e_1, e_2, \ldots, e_n) \) such that \( Z_L e = 0 \).

b) Determine the active index set \( A = \{ i : e_i \neq 0 \} \) and the highest degree active index set \( M \subset A \) as \( M = \{ i \in A : \delta_i \geq \delta_j \text{ for all } j \in A \} \). Choose \( k \in M \). If \( M \) has several elements, for numerical reasons it is best to select \( k \) such that \( |e_k| \) is maximal on \( M \).

c) If \( \delta_k = 0 \) for every null vector \( e \), go to Step 2. Else, let \( a = e/e_k \), and construct the unimodular matrix \( U \) and its inverse \( V = U^{-1} \) as follows. Form \( V \) and \( U \) by replacing the \( k \)th column of the \( n \times n \) unit matrix with the polynomial column vectors

\[
\begin{bmatrix}
  a_1 s^{\delta_k - \delta_1} \\
  a_2 s^{\delta_k - \delta_2} \\
  \vdots \\
  a_k s^{\delta_k - \delta_{k-1}} \\
  1 \\
  a_{k+1} s^{\delta_k - \delta_{k+1}} \\
  \vdots \\
  a_n s^{\delta_k - \delta_n}
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
  -a_1 s^{\delta_k - \delta_1} \\
  -a_2 s^{\delta_k - \delta_2} \\
  \vdots \\
  -a_k s^{\delta_k - \delta_{k-1}} \\
  1 \\
  -a_{k+1} s^{\delta_k - \delta_{k+1}} \\
  \vdots \\
  -a_n s^{\delta_k - \delta_n}
\end{bmatrix}
\]

respectively.

d) We now have the factorization \( Z = U^*(V^*ZV)U \), where the \( k \)th half diagonal degree of \( V^*ZV \) is one less than the corresponding half diagonal degree of \( Z \). Let \( Z := V^*ZV, P := UP, \delta_k := \delta_k - 1 \) and return to a).
Step 2. Clearing. Use all nonzero nonpolynomial diagonal entries of $Z$ to clear the corresponding row and column. That is, suppose that the $k$th diagonal entry $Z_{kk}$ of $Z$ is a nonzero constant. Construct the unimodular polynomial matrix $V$ and its inverse $U = V^{-1}$ by replacing the $k$th row of the $n \times n$ unit matrix with

$$
\frac{1}{Z_{kk}} (Z_{k1} Z_{k2} \cdots Z_{k,k-1} Z_{kk} Z_{k,k+1} \cdots Z_{kn})
$$

and

$$
\frac{1}{Z_{kk}} (-Z_{k1} - Z_{k2} \cdots -Z_{k,k-1} Z_{kk} -Z_{k,k+1} \cdots -Z_{kn})
$$

respectively. Then $Z = U^{-1}(V^{-1}ZV)U$, where the $k$th row and column of $V^{-1}ZV$ except the diagonal entry consist of zeros. Let $Z := V^{-1}ZV$, $P := PU$, and repeat this step for all nonzero nonpolynomial diagonal entries.

Step 3. Finalization. If at this point $Z$ has been reduced to a constant diagonal matrix, perform a $J$-factorization as described at the end of Section II and terminate the algorithm. Else, bring the polynomial matrix $Z$ by suitable symmetric row and column permutations (possibly nonunique) into the block matrix form

$$
Z = \begin{bmatrix}
0 & Z_{12} \\
Z_{12} & Z_{22}
\end{bmatrix}
$$

Algorithm 3.2 may be invoked.

Step 1 of Algorithm 3.1 may also be used to transform a nonunimodular para-Hermitian polynomial matrix with finite leading diagonal coefficient matrix that is not diagonally reduced unimodularly into a diagonally reduced polynomial matrix [2].

Algorithm 3.2 (Factorization of a Special Unimodular Polynomial Matrix): We next study the factorization $Z = P^{-1}P$ of the nonsingular unimodular para-Hermitian polynomial matrix

$$
Z = \begin{bmatrix}
0 & Z_{12} \\
Z_{12} & Z_{22}
\end{bmatrix}
$$

Note that $Z_{12}$ is wide (that is, has at least as many columns as rows), because otherwise $Z$ would be singular.

Step 1. Use the standard greatest left divisor procedure [13] to find a unimodular polynomial matrix $U$ such that $Z_{12}U = [z_{12} \ 0]$, with $z_{12}$ square nonsingular. In fact, we show in what follows that $z_{12}$ is unimodular and, hence, can be taken to be the unit matrix. Let

$$
Z = \begin{bmatrix}
I & 0 \\
U^{-1}Z_{21} & U^{-1}Z_{22}U
\end{bmatrix}
$$

and

$$
P := \begin{bmatrix}
I & 0 & 0 \\
U^{-1} & 0 & 0
\end{bmatrix}
$$

It may easily be found that $\det Z = \det z_{12} \cdot \det z_{23} \cdot \det z_{21}$. Since by assumption $Z$ is unimodular, so are $z_{12}$, $z_{23}$, and $z_{21}$. This proves that $z_{12}$ can be taken to be the unit matrix, and we now have reduced $Z$ to the form

$$
Z = \begin{bmatrix}
0 & I & 0 \\
I & z_{22} & z_{23} \\
0 & z_{23} & z_{33}
\end{bmatrix}
$$

Step 2. Use the two unit matrices in $Z$ to remove the blocks $z_{23}$ and $z_{32}$ symmetrically and unimodularly and correspondingly let

$$
Z = \begin{bmatrix}
0 & I & 0 \\
I & z_{22} & 0 \\
0 & z_{23} & 0
\end{bmatrix}
$$

where $Z_{00}$ is constant diagonal, the diagonal entries of $Z_{22}$ are either zero or strictly polynomial (that is, polynomial with nonzero degree), and $U$ a suitable permutation matrix. Let $Z := U^{-1}ZV$, $P := PU$, and repeat this step for all nonzero nonpolynomial diagonal entries.

Algorithm 3.1, Step 1. The half diagonal degrees of $Z$ are $\delta_1 = \delta_2 = 1$, and, accordingly, its leading diagonal coefficient matrix is

$$
Z_L = \begin{bmatrix}
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
$$

The leading coefficient matrix has the null vector $e = \text{col}(1, 1)$. The highest degree active index set is $\mathcal{M} = \{1, 2\}$, and we may select the pivot as $k = 1$. Correspondingly, we write $Z = U^{-1}Z_{11}U_1$, where

$$
Z_1(s) = \begin{bmatrix}
0 & \frac{i}{2} & 0 \\
\frac{i}{2} & \frac{1 + s^2}{4} & \frac{1 + s^2}{4}
\end{bmatrix}, \quad U_1(s) = \begin{bmatrix}
1 & 0 \\
0 & -1 & 1
\end{bmatrix}
$$

Example 3.3 (Factorization of a Unimodular Matrix): By way of example we consider the factorization of the unimodular para-Hermitian polynomial matrix

$$
Z(s) = \begin{bmatrix}
\frac{-3}{4} & s^2 \\
\frac{-1}{4} & \frac{1}{4}
\end{bmatrix}
$$

The factorization of the unimodular para-Hermitian polynomial matrix $Z_{23}$ follows (recursively) by Algorithm 3.1.
Since the degree of the diagonal entry is zero on the highest
degree active set \{1\} of the null vector of the leading diagonal
coefficient matrix of $Z_1$, we terminate Step 1.

**Step 2.** $Z_1$ has no nonzero constant diagonal entries so
Step 2 is skipped.

**Step 3.** We invoke Algorithm 3.2 to factor $Z_1$.

**Algorithm 3.2.** **Step 1.** Transforming the $(1, 2)$ and $(2, 1)$
entries of $Z_1$ to 1 we obtain $Z_1 = U_2^2 Z_2 U_2$, with

$$Z_2(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 + s^2 \end{bmatrix}, \quad U_2(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}. \quad (32)$$

**Step 2.** $Z_2$ may be factored as $Z_2 = U_3^2 J U_3$, with

$$J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad U_3(s) = \frac{1}{2} \sqrt{2} \begin{bmatrix} 1 & \frac{3 + s^2}{2} \\ 1 & -\frac{1 + s^2}{2} \end{bmatrix}. \quad (33)$$

Thus, we have the factorization $Z = P^- JP$, with

$$P(s) = U_3(s) U_2(s) U_1(s) = \frac{1}{8} \sqrt{2} \begin{bmatrix} 1 - s^2 & 3 + s^2 \\ 5 - s^2 & -1 + s^2 \end{bmatrix}. \quad (34)$$

**IV. Diagonalization**

In this section we discuss an algorithm for the factorization of $Z$
that is based on diagonalization. The algorithm avoids
calculation of the zeros of $Z$ needed in the extraction and interpolation
algorithms but involves more elaborate polynomial operations than the other procedures. On the other hand, it works whenever the conditions of Theorem 2.1 are satisfied, whether or not the factorization is canonical, whether or not $Z$ is singular, and whether or not $Z$ is diagonally reduced. The algorithm follows Jakubovič's proof of Theorem 2.1 [10], except for the first step, where instead of the Smith diagonal form used by Jakubovič we allow more general diagonal forms.

**Algorithm 4.1 (Diagonalization):** For a given $n \times n$ para-Hermitian polynomial matrix $Z$ that satisfies the conditions of Theorem 2 we calculate the desired spectral factor $P$ and the signature matrix $J$ by the following steps:

**Step 1.** Find a diagonal polynomial matrix $D = \text{diag}(d_1, d_2, \ldots, d_n)$ and corresponding unimodular polynomial matrices $V$ and $W$ such that

$$Z = V D W \quad (35)$$

with $D$ para-Hermitian and nonnegative definite on the imaginary axis. If some diagonal entry of $D$ is a zero polynomial (because $Z$ is singular), replace it with one while at the same time setting the corresponding column of $V$ equal to zero. By this device, $D$ is always nonsingular while $V$ is a generalized unimodular matrix.\(^3\) $V$ is singular if $Z$ is.

This diagonalization is standard (see for instance [13, Sec. 7.7]). It is not unique, and, accordingly, the unimodular matrices $V$ and $W$ may vary.

**Step 2.** Perform scalar spectral factorizations of the diagonal entries of $D$ in the form $d_i = f_i^* f_i, i = 1, 2, \ldots, n$, and form a (nonsingular) Hurwitz polynomial matrix $F = \text{diag}(f_1, f_2, \ldots, f_n)$. Clearly,

$$D = F^* F. \quad (36)$$

There are several ways to do the scalar spectral factorizations. The iterative method based on Newton-Raphson approximation is recommended [11].

**Step 3.** Compute the matrices

$$X = (W^{-1})^* V, \quad Y = (F^{-1})^* X F^{-1}. \quad (37)$$

which both are polynomial (rather than rational) and generalized unimodular.\(^4\) Moreover, $Y$ is para-Hermitian.

**Step 4.** Find a nonsingular (unimodular) polynomial matrix $U$ along with a (possibly singular) constant matrix $J$ so that

$$Y = U^* J U. \quad (38)$$

The factorization of unimodular para-Hermitian polynomial matrices is discussed in Section III.

**Step 5.** $P = U F W$ is a spectral factor of $Z$.

The factorization produced by this algorithm is not necessarily canonical, even if one exists.

**Example 4.2 (Factorization by Diagonalization):** By way of example we consider the (noncanonical) factorization of the polynomial matrix

$$Z(s) = \begin{bmatrix} 0 & 1 - s \\ 1 + s & 1 - s^2 \end{bmatrix} \quad (39)$$

which was also discussed in Example 2.5. This matrix may be diagonalized as

$$Z = \begin{bmatrix} 0 & 1 - s \\ 1 + s & 1 - s^2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1 + s}{2} \\ \frac{1 + s}{2} & \frac{1 + s^2}{2} \end{bmatrix} \quad (40)$$

Given $V, D,$ and $W$, the polynomial matrices $F, X,$ and $Y$
easily follow as

$$F(s) = \begin{bmatrix} 1 + s & 0 \\ 0 & 1 \end{bmatrix}, \quad X(s) = \begin{bmatrix} -\frac{3 + s^2}{4} & (1 - s)^2 + (1 + s) \\ 1/4 + s^2 & 1/4 + s^2 \end{bmatrix}. \quad (41)$$

$Y(s) = \begin{bmatrix} -\frac{3 + s^2}{4} & \frac{1 - s^2}{4} \\ \frac{1 - s^2}{4} & \frac{1 - s^2}{4} \end{bmatrix}.$

$Y$ is para-Hermitian as expected. According to Example 3.3, the factorization of $Y$ is $Y = U^* J U$, with

$$U(s) = \frac{1}{8} \sqrt{2} \begin{bmatrix} 1 - s^2 & 3 + s^2 \\ 5 - s^2 & -1 + s^2 \end{bmatrix}, \quad J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (42)$$

It follows that the desired spectral factor is given by

$$P(s) = U(s) F(s) W(s) = \frac{1}{2} \sqrt{2} \begin{bmatrix} 1 + s & (1 - s)(3 + s) \\ 1 + s & 1 + s^2 \end{bmatrix}. \quad (43)$$

\(^3\)A (possibly singular) polynomial matrix is called generalized unimodular if all its invariant polynomials are real constants or zero.

\(^4\)For the proof see Jakubovič [10] in case $D$ has the Smith form and Šebek [19] for the more general case.
This spectral factor is not the same as that given in Example 2.5. The two spectral factors are related by

$$P_{\text{Example 2.5}} = U_0 P_{\text{this example}}$$

with $U_0$ the J-unitary polynomial matrix

$$U_0(s) = \frac{1}{2} \begin{bmatrix} 2 + s & -s \\ s & 2 - s \end{bmatrix}. \quad (44)$$

V. SUCCESSIVE FACTOR EXTRACTION

The algorithm presented in this section is an adaption of a procedure developed by Callier [2] for standard spectral factorization.

Algorithm 5.1 (Successive Factor Extraction): The method requires that $\det Z$ have no roots on the imaginary axis (and, a fortiori, be nonsingular) and that $Z$ be diagonally reduced. Since any para-Hermitian matrix with finite leading diagonal coefficient matrix can be made diagonally reduced by a symmetric unimodular transformation (see Section III) the second assumption causes little loss of generality.

Step 0. The algorithm is initialized by determining the half diagonal degrees $\delta_1, \delta_2, \cdots, \delta_n$ of $Z$ as defined in Section II. Furthermore, the zeros $\zeta_1, \zeta_2, \cdots, \zeta_m$ of $Z$, that is, the roots of $\det Z$, need to be calculated. The zeros also play an important role in the interpolation method for factorization, and their computation is discussed in Section VI. Note that because $Z$ is para-Hermitian, if $\zeta$ is a zero, so is $-\zeta$. Because $Z$ has real coefficients, if $\zeta$ is a complex zero, so is its complex conjugate $\overline{\zeta}$.

Steps 1--m The extraction algorithm consists of extracting symmetrically and successively elementary factors that correspond to the each of the zeros. The elementary factor is real only if the root is real. If the zero is complex, complex conjugate pairs may be combined to form a real factor of degree two [2], [15]. In this paper, we limit ourselves to first-order, possibly complex elementary factors.

If $T_j$ is the elementary factor that is extracted at step $j$ of the algorithm we have

$$Z_{m-j+1} = T_j^* Z_{m-j} T_j, \quad j = 1, 2, \cdots, m \quad (45)$$

with $m$ the number of zeros and $Z_m = Z$. The subscript on $Z$ indicates the number of remaining extractions, and if $A$ is any polynomial matrix with real or complex coefficients, $A^*$ is the polynomial matrix defined by $e^2 A^*(s) = (A(s))^H$. Each factor $T_j$ corresponds to a zero $\zeta_j$ of $Z$. To simplify the presentation we consider the extraction

$$Z = T^* Z^* T \quad (46)$$

without identifying the sequential number of the extractions. The elementary factor $T$ corresponding to the real or complex zero $\zeta$ has the form

$$T(s) = \begin{bmatrix} I_1 & -a_1 & 0 \\ 0 & s - \zeta & 0 \\ 0 & -a_2 & I_2 \end{bmatrix}. \quad (47)$$

$I_1$ and $I_2$ are unit matrices of generally different dimensions. The second column block is the $k$th column of $T$, with $k$ to be determined. The entries $a_1$ and $a_2$ in this column are constant (generally complex valued) vectors that are obtained as follows. Because $\zeta$ is a root of $\det Z$, clearly $Z(\zeta)$ is singular, and there exists a nontrivial vector $e$ such that $Z(\zeta)e = 0$. The vector $e$ is called the null vector of $Z$ corresponding to the zero $\zeta$.

Because $Z(\zeta)e = T^*(-\zeta)Z^*(-\zeta)T(\zeta)e = 0$, we may determine $a_1$ and $a_2$ by letting $T(\zeta)e = 0$. When writing out this identity component-by-component, it is easily found that $a_1$ and $a_2$ follow from

$$a = e / e_k = \begin{bmatrix} a_1 \\ 1 \\ a_2 \end{bmatrix} \quad (48)$$

where $e_k$ is the $k$th component of $e$ and the 1 is in the $k$th position of $a$. The elementary factor $T$ as given by (47) may be viewed as a variant of the Hermite standard form [12] of a polynomial matrix of degree one.

The following rule determines which column $k$ is selected. Before doing an extraction define the active index set $A$ as $A = \{ i : e_i \neq 0 \}$, where $e_i$ is the $i$th entry of the null vector $e$. The active index set contains the indices of all nonzero entries of the null vector $e$. Next, introduce the highest degree active index set $M \subset A$ as $M = \{ i \in A : \delta_i \geq \delta_j \text{ for all } j \in A \}$. The highest degree active index set contains the indices of the diagonal elements of $Z$ of highest degree within the active set $A$. The column index $k$ may be now chosen as any element of $M$. If there are several such elements, for numerical reasons it is recommended to choose the element such that the magnitude $|e_k|$ of the $k$th entry of the null vector $e$ is maximal.

For spectral factorization, naturally each zero $\zeta$ that is extracted on the right is chosen to have negative real part.

We next discuss how to determine the "remaining factor" $Z^*$ in (46). To this end, we first extract the factor $T$ "on the right" and write $Z = Z'T$, with the square polynomial matrix $Z'$ to be determined. Multiplying the equality $Z = Z'T$ out element-by-element it is easy to see that all entries of $Z$ and $Z'$ are equal except those in their $k$th columns. Denoting the $k$th column of $Z'$ as $z_k$, it follows that

$$Z(s)a = z_k(s)(s - \zeta). \quad (49)$$

From this, $z_k$ may easily be computed by dividing the left-hand side by $s - \zeta$.

The polynomial matrix $Z^*$ may now be obtained by the left extraction $Z' = T^*Z^*$, which we rewrite as the right extraction $(Z')^* = (Z^*)^*T$. This extraction follows by the same procedure as before. Because $Z'$ is para-Hermitian, it is sufficient to compute the $k$th diagonal entry $z_{kk}$ of $Z'$. This entry may be obtained by solving the equation

$$z_{kk}^*(s)a = z_{kk}(s)(s - \zeta) \quad (50)$$

for $z_{kk}$ by dividing the left-hand side by $s - \zeta$. The nondiagonal elements of the $k$th column of $Z^*$ equal the corresponding entries of $z_k$, the nondiagonal elements of the $k$th row of $Z^*$
follow by adjugation, while the remaining elements of $Z''$ equal the corresponding elements of $Z$. This defines $Z''$. As the last step in the extraction we modify the half diagonal degree of the 4th diagonal element to $b_4 := b_4 - 1$.

The extraction procedure is repeated until the supply of left-half plane zeros is exhausted. The order in which the factors are extracted is not important.

Step $m+1$. Eventually, $Z$ is reduced to the form

$$Z = T_1^* T_2^* \cdots T_m^* Z_0 T_m \cdots T_2 T_1 = T^* Z T$$

with $T = T_m T_{m-1} \cdots T_1$. Generally, $T$ and $Z_0$ are complex (that is, polynomial matrices with complex coefficients).

If the factorization is canonical the half diagonal degrees remain nonnegative during the process of extracting elementary factors. In this case the half diagonal degrees eventually are reduced to zero, so that $Z_0$ is a constant Hermitian matrix. Moreover, $T$ is column reduced with column degrees equal to the half diagonal degrees of $Z$. $T$ and $Z_0$ may both be made real by replacing $T$ with $U^{-1} T$ and $Z_0$ with $U^H Z_0 U$, with $U$ a suitable constant matrix. A convenient choice is to take $U$ equal to the leading column coefficient matrix $\mathbf{I}$ of $T$ so that $Z_0$ becomes the leading diagonal coefficient matrix of $Z$. Alternatively, $U$ may be chosen as the constant coefficient matrix of $T$, so that $Z_0$ becomes the constant coefficient matrix of $Z$. The final step of the factorization is to factor the constant matrix $Z_0$.

If the factorization is non-canonical, during the extraction process one or several of the half diagonal degrees become negative. The result is that the "remaining factor" $Z_0$ is no longer a constant matrix, but polynomial unimodular. This unimodular para-Hermitian matrix may be J-factored as polynomial matrices with complex coefficients.

We consider the example studied in Example 2.5, where

$$Z(s) = \begin{bmatrix} \varepsilon & 1 - s \\ 1 + s & 1 - s^2 \end{bmatrix}$$

with $\varepsilon$ a real number such that $|\varepsilon| < 1$. Since det $Z(s) = (\varepsilon - 1)(1 - s^2)$, the polynomial matrix $Z$ has the roots $1 \pm 1$. The null vector corresponding to the left-half plane root $-1$ is $\mathbf{e} = \mathbf{col}(2, -\varepsilon)$.

Case 1: $\varepsilon \neq 0$. If $\varepsilon \neq 0$, the active index set is $\mathcal{A} = \{1, 2\}$, and the highest degree active index set is $\mathcal{M} = \{2\}$. Since as a result we need to take $k = 2$, it follows that $a = \mathbf{col}(-2/\varepsilon, 1)$, and we extract the factor

$$T(s) = \begin{bmatrix} 1 & 2 \\ 0 & 1 + s \end{bmatrix}.$$  

(53)

By straightforward computation it follows from (49) that $z_2(s) = \mathbf{col}(-1, -2/\varepsilon + 1 - s)$, while from (50) we obtain that $z_{22}(s) = 1$. Thus, after extracting the factor $T$ we are left with

$$Z_0(s) = \begin{bmatrix} \varepsilon & -1 \\ -1 & 1 \end{bmatrix}.$$  

(54)

This is a constant matrix and, hence, the factorization of $Z$ is canonical. Since

$$Z_0(s) = \begin{bmatrix} \varepsilon & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & \sqrt{1 - \varepsilon} \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ \sqrt{1 - \varepsilon} & 0 \end{bmatrix}$$

(55)

the desired spectral factor $P$ is

$$P(s) = \begin{bmatrix} -1 & 1 \\ \sqrt{1 - \varepsilon} & 0 \end{bmatrix}.$$  

(56)

This is the result shown in Example 2.5.

Case 2: If $\varepsilon = 0$, the active and highest degree index sets reduce to $\mathcal{A} = \mathcal{M} = \{1\}$, so that $k = 1$ and $a = \mathbf{col}(1, 0)$.

As a result, we extract the factor

$$T(s) = \begin{bmatrix} 1 + s \\ 0 \\ 1 \end{bmatrix}.$$  

(57)

By inspection, the remaining factor is seen to be given by

$$Z_0(s) = \begin{bmatrix} 1 \\ 1 - 1 - s^2 \end{bmatrix}.$$  

(58)

$Z_0$ is unimodular polynomial, so that the factorization of $Z$ is noncanonical. By Algorithm 3.2, Step 2, we have

$$Z_0(s) = \frac{1}{2} \sqrt{\begin{bmatrix} 1 & 3 - x^2 \\ 1 & 1 - \frac{x^2}{2} \end{bmatrix}} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$  

(59)

Hence, the desired spectral factor $P$ is

$$P(s) = \frac{1}{2} \sqrt{\begin{bmatrix} 1 & 3 - x^2 \\ 1 & 1 - \frac{x^2}{2} \end{bmatrix}} \cdot \begin{bmatrix} 1 + s \\ 0 \\ 1 \end{bmatrix}.$$  

(60)

which is the result shown in Example 2.5.

VI. ZEROS OF POLYNOMIAL MATRICES

We consider the determination of the zeros of the matrix $A \in \mathbb{C}^{n \times n}$ square polynomial matrix $P$ (compare [23]). The zeros of the matrix $P$ are needed in the successive extraction algorithm of Section V and also play a role in the interpolation algorithm of Section VII. Given a zero $\zeta$ of $P$, the corresponding null vector $e$ satisfies $P(e) = 0$. Writing $P$ in terms of its coefficient matrices as $P(s) = P_0 + P_1 s + \cdots + P_N s^N$ is equivalent to

$$(P_0 + P_1 \zeta + \cdots + P_N \zeta^N) e = 0.$$  

(61)

Defining the vectors $e_j = \zeta^j e$, $j = 0, 1, \cdots, N - 1$, we equivalently have $e_j = \zeta e_j - 1$, $j = 1, 2, \cdots, N - 1$, and

$$P_0 e_0 + P_1 e_1 + \cdots + P_{N-1} e_{N-1} + \zeta P_N e_{N-1} = 0.$$  

(62)
These relations may be arranged as
\[
\begin{bmatrix}
0 & I & 0 & \cdots & 0 \\
0 & 0 & I & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & 0 \\
-P_0 & -P_1 & \cdots & \cdots & -P_{N-1}
\end{bmatrix}
\begin{bmatrix}
e_0 \\
e_1 \\
e_2 \\
\vdots \\
e_{N-2} \\
e_{N-1}
\end{bmatrix}
= \zeta
\begin{bmatrix}
I & 0 & 0 & 0 & 0 \\
0 & I & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & 0 & I & 0 \\
0 & 0 & 0 & \cdots & 0 & P_N
\end{bmatrix}
\begin{bmatrix}
e_0 \\
e_1 \\
e_2 \\
\vdots \\
e_{N-2} \\
e_{N-1}
\end{bmatrix}
. \quad (63)
\]

This shows that the zeros of \( P \) can be found as the eigenvalues of the generalized eigenvalue problem defined by \( Cw = \lambda Dw \).

Given the eigenvector \( v \) corresponding to an eigenvalue \( \zeta \) (which is a zero of \( P \)) the corresponding null vector \( e \) of \( P \) consists of the first \( n \) components of \( v \).

The computation reduces to a usual eigenvalue computation if the highest coefficient matrix \( P_N \) is a unit matrix, because then \( D = I \). If \( P_N \) is not the unit matrix we can make it equal to the unit matrix. Assume that \( P \) is column reduced (if not, make it column reduced by multiplication on the right by a suitable unimodular polynomial matrix) and has a unit leading column coefficient matrix (if not, multiply the matrix on the left by the inverse of the leading column coefficient matrix). Let the column degrees of the polynomial matrix \( P \) be given by \( \delta_1, \delta_2, \ldots, \delta_n \). Then multiplication of \( P \) on the right by the diagonal matrix \( \text{diag}(s^{N-\delta_1}, s^{N-\delta_2}, \ldots, s^{N-\delta_n}) \) results in a polynomial matrix whose highest coefficient matrix is the \( n \times n \) unit matrix.

This multiplication on the right causes certain columns in the lower row of blocks in the block companion matrix \( C \) to be zero. In particular, the \( i \)-th columns of the blocks with subscripts 0 through \( N - \delta_i - 1 \) are zero, with \( i = 1, 2, \ldots, n \). These zero columns correspond to the \( N - \delta_i \) spurious zeros at the origin introduced by multiplying the \( i \)-th column of \( P \) by \( s^{N-\delta_i} \). The spurious zeros can be removed by deleting the corresponding column, together with the row with the corresponding number, from the companion matrix \( C \). The eigenvalues of the remaining "decimated" companion matrix are the roots of the polynomial matrix \( P \).

**Example 6.1 (Zeros):** Consider the polynomial matrix
\[
P(s) = \begin{bmatrix}
1 & 1 & 0 \\
0 & 0 & 1 \\
0 & 1 & s^2
\end{bmatrix}.
\]

By inspection, we see that the zeros of \( P \) are 0 and \( \pm j \). \( P \) is column reduced with leading column coefficient matrix the 3 \( \times \) 3 unit matrix. Since the column degrees of \( P \) are 0, 1, and 2, respectively, we multiply on the right by \( \text{diag}(s^2, s, 1) \) so that we obtain the polynomial matrix
\[
\begin{bmatrix}
s^2 & s & 0 \\
0 & s^2 & 1 \\
0 & 0 & 1 + s^2
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{bmatrix}

P_0
\]
whose highest coefficient matrix is the 3 \( \times \) 3 unit matrix. The corresponding companion matrix and its resolvent are
\[
C = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1
\end{bmatrix},
\]
\[
\lambda I - C = \begin{bmatrix}
\lambda & 0 & -1 & 0 & 0 \\
1 & \lambda & 0 & 0 & 0 \\
0 & 1 & 0 & \lambda & 0 \\
0 & 0 & 1 & 0 & \lambda
\end{bmatrix}.
\]

The roots of the determinant \( \lambda(\lambda^2 + 1) \) of this matrix are the zeros 0 and \( \pm j \) of \( P \).

We summarize the algorithm for the computation of the zeros of a polynomial matrix as follows.

**Algorithm 6.2 (Computation of the Zeros of a Polynomial Matrix):** Let \( P \) be an \( n \times n \) column reduced square polynomial matrix with column degrees \( \delta_1, \delta_2, \ldots, \delta_n \) and coefficient matrices \( P_0, P_1, \ldots, P_N \).

1. **Step 1.** Form the block row matrix \( [P_0, P_1, \ldots, P_N] \).
2. **Step 2.** Multiply \( P \) on the right by \( \text{diag}(s^{N-\delta_1}, s^{N-\delta_2}, \ldots, s^{N-\delta_n}) \):
   a) For \( i = 1, 2, \ldots, n \) and \( k = \delta_i, \delta_i - 1, \ldots, 0 \) replace column \( (k + N - \delta_i)n + i \) of the block row matrix with column \( kn + i \).
   b) Next, for \( i = 1, 2, \ldots, n \) and \( k = 0, 1, \ldots, N - \delta_i - 1 \) set column \( kn + i \) equal to zero.
3. **Step 3.** Make \( P \) monic: Denoting the result as \( [P_0 P_1, \ldots, P_N] \), premultiply by \( P_N^{-1} \) and delete the final \( n \times n \) row block (which is the unit matrix).
4. **Step 4.** Use the resulting row block matrix to form the block companion matrix \( C \).
5. **Step 5.** Remove the spurious zeros at 0: Delete the rows and columns numbered \( kn + i \) with \( i = 1, 2, \ldots, n \) and \( k = 0, 1, \ldots, N - \delta_i - 1 \), from \( C \).
6. **Step 6.** Compute the zeros: The eigenvalues of the remaining matrix are the zeros of the polynomial matrix \( P \).
VII. INTERPOLATION

Suppose that the para-Hermitian polynomial matrix \( Z \) is diagonally reduced and has no zeros on the imaginary axis (and, hence, necessarily is nonsingular). As explained in Section V, by successively extracting elementary factors corresponding to each of the zeros of \( Z \) with negative real part we obtain a factorization of \( Z \) of the form \( Z = T^\tau Z_0 T \). If the factorization is canonical, the matrix \( Z_0 \) is constant, and \( T \) is column reduced with column degrees equal to the half diagonal degrees \( \delta_1, \delta_2, \ldots, \delta_n \) of \( Z \).

The factor \( T \) may be obtained in one calculation, without the successive procedure of the symmetric extraction algorithm, by exploiting the property that if \( \zeta \) is any of the left-half plane zeros of \( Z \), the corresponding null vector \( e \) of \( Z \) is also a null vector of \( T \). Let \( \zeta_1, \zeta_2, \ldots, \zeta_N \) be the left-half plane zeros of \( Z \), and \( e_1, e_2, \ldots, e_N \) the corresponding null vectors. Their computation is discussed in Section VI. Then

\[
T(\zeta_i)e_i = 0, \quad i = 1, 2, \ldots, N. \tag{68}
\]

Along with the fact that the column degrees of \( T \) are known, this permits the calculation of \( T \) as follows. The degree of \( T \) equals \( M = \max_i \delta_i \). Define \( T \) in terms of its coefficient matrices as \( T(s) = T_0 + T_1 s + T_2 s^2 + \cdots + T_M s^M \). Then from (68) we have

\[
\begin{bmatrix}
T_0 & T_1 & \cdots & T_M
\end{bmatrix} E = 0 \tag{69}
\]

where the constant matrix \( E \) is given by

\[
E = \begin{bmatrix}
 e_1 & e_2 & \cdots & e_N \\
e_1 \zeta_1 & e_2 \zeta_2 & \cdots & e_N \zeta_N \\
e_1 \zeta_1^2 & e_2 \zeta_2^2 & \cdots & e_N \zeta_N^2 \\
\vdots & \vdots & \ddots & \vdots \\
e_1 \zeta_1^M & e_2 \zeta_2^M & \cdots & e_N \zeta_N^M
\end{bmatrix}. \tag{70}
\]

It follows from the fact that the degree of the \( k \)th column of \( T \) is \( \delta_k \) that the columns with numbers \( k + in \), \( i = 0, 1, \ldots, M \) of the matrix \( \begin{bmatrix} T_0 & T_1 & \cdots & T_M \end{bmatrix} \) consist of zeros only. Hence, the constant matrix \( \tilde{T} \) by removing from the matrix \( \begin{bmatrix} T_0 & T_1 & \cdots & T_M \end{bmatrix} \) all the columns numbered \( k + in \), \( k = 1, 2, \ldots, n \), where \( i = \delta_k + 1, \ldots, M \). Similarly, define the matrix \( \tilde{E} \) by removing from the matrix \( E \) all the rows numbered \( k + in \), \( k = 1, 2, \ldots, n \) with \( i = \delta_k + 1, \ldots, M \). Then

\[
\tilde{T}\tilde{E} = 0 \tag{71}
\]

and the constant matrix \( \tilde{T} \) may be computed such that its rows form a real basis for the left null space of \( \tilde{E} \). Once \( \tilde{T} \) has been obtained, the coefficient matrix \( \begin{bmatrix} T_0 & T_1 & \cdots & T_M \end{bmatrix} \), and, hence, the polynomial matrix \( T \), follows by reinserting the appropriate zero columns.

Complex arithmetic may be avoided as follows. If \( \zeta \) is a complex left-half plane zero with null vector \( e \), also its complex conjugate \( \bar{\zeta} \) is a left-half plane zero with null vector \( \bar{e} \). We may then replace the two corresponding complex conjugate columns of \( E \) with two real columns, the first of which is the real part and the second the imaginary part of the complex valued column.

The algorithm yields a unique solution \( T \) (within multiplication on the left by a nonsingular constant matrix), provided \( \tilde{E} \) has full column rank. Under the assumption that the factorization is canonical, this condition is always satisfied if the zeros of \( Z \) are all distinct. If the zeros are nondistinct the condition may be violated.

It remains to complete the factorization \( Z = T^\tau Z_0 T \) by finding the matrix \( Z_0 \). A simple way of doing this is to left multiply the factor \( T \) by the inverse of its leading column coefficient matrix, so that the leading coefficient matrix of \( T \) is normalized to the unit matrix. Then \( Z_0 \) is the leading diagonal coefficient matrix of the diagonally reduced para-Hermitian matrix \( Z \). The factorization may be put into standard form by transforming \( Z_0 \) to its signature matrix.

It is not difficult to extend the algorithm so that it also applies when there are repeated zeros.

Algorithm 7.1 (Interpolation): Suppose that the \( n \times n \) para-Hermitian polynomial matrix \( Z \) has no zeros on the imaginary axis and is diagonally reduced with half diagonal degrees \( \delta_1, \delta_2, \ldots, \delta_n \).

Step 1. Let \( M = \max_i \delta_i \) and define the diagonal matrix \( D(s) = \text{diag}(s^{M-\delta_1}, s^{M-\delta_2}, \ldots, s^{M-\delta_n}) \). Make the highest coefficient matrix of \( Z \) nonsingular by considering \( S' = D^\tau ZD \). Next, make \( S \) monic by letting \( S = S_2M s, \) with \( S_2M \) the highest coefficient matrix of \( S \).

Step 2. Form the block companion matrix

\[
C = \begin{bmatrix}
0 & I & 0 & \cdots & 0 \\
0 & 0 & I & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & I & 0 \\
0 & \cdots & \cdots & \cdots & 0 \\
\end{bmatrix}
\tag{72}
\]

with \( S_0, S_1, \ldots, S_{2M} \) the coefficient matrices of \( S \).

Step 3. Calculate the Schur decomposition of \( C \) in the form

\[
C = RUR^H \tag{73}
\]

with \( R \) unitary (that is, \( R^H R = RR^H = I \)) and \( U \) upper triangular with the eigenvalues of \( C \) along the diagonal, arranged in order of increasing real part. The nonzero eigenvalues of \( C \) are the zeros of the polynomial matrix \( Z \).

Step 4. Partition \( R = [R_1 \ R_2] \), where the number of columns of \( R_1 \) equals the number of eigenvalues of \( S \) with negative real part, which, in turn, equals the number of zeros of \( Z \) with negative real part. The columns of \( R_1 \) span the maximal invariant subspace of \( C \) corresponding to the eigenvalues with negative real part.

Retain the first \( (M+1)n \) rows of \( R_1 \) and denote the result as \( \tilde{E} \). Next, remove those rows from \( E \) that correspond to the zeros at the origin that were introduced in Step 1. These are the rows numbered \( i + kn, \) \( k = 0, 1, \ldots, \delta_i - 1 \), with \( i = 1, 2, \ldots, n \). The result is denoted \( \tilde{E} \). Next compute a real matrix \( \tilde{T} \) of maximal full row rank whose rows span the left null space of \( \tilde{E} \), that is,

\[
\tilde{T}\tilde{E} = 0. \tag{74}
\]

Step 5. Form the matrix \( T \) by inserting into \( \tilde{T} \) columns consisting of zeros at the column locations corresponding to the rows that have been deleted from \( E \). The resulting
polynomial matrix with matrix coefficients defined by \( T = [T_0 \ T_1 \cdots T_M] \) has a number of spurious zeros at the origin, which were introduced in Step 1. These zeros need to be removed by replacing column \( i + kn \) with column \( i + (M - \delta_i + k)n \), for \( k = 0, 1, \cdots, \delta_i \), with \( i = 1, 2, \cdots, n \), and setting column \( i + kn, \ k = \delta_i + 1, \delta_i + 2, \cdots, M \), with \( i = 1, 2, \cdots, n \), equal to 0.

Step 6. It remains to complete the factorization \( Z = TZ_0T^{-1} \) by finding the constant matrix \( Z_0 \). This may be done by multiplying \( T \) on the left by the inverse of its leading column coefficient matrix, which is nonsingular if and only if the factorization of \( Z \) is canonical. \( Z_0 \) then is the leading diagonal coefficient matrix \( Z_L \) of \( Z \). The factorization is finalized by factoring \( Z_L \).

If the factorization is noncanonical, the factor \( T \) as computed in Step 5 is well defined, but its leading column coefficient matrix is singular. As a result, Step 6 fails. The nearly noncanonical case, that is, when the factorization is close to noncanonical, is discussed in Section IX.

**Example 7.2 (Factorization by Interpolation):**
We consider the factorization of the matrix
\[
\begin{bmatrix}
\epsilon & 1 - s \\
1 + s & 1 - s^2
\end{bmatrix}
\]
with \( |\epsilon| < 1 \), which was also considered in Example 5.2. Since \( \delta_1 = 1 \) and \( \delta_2 = 0 \), we let \( D(s) = \text{diag}(s, 1) \), so that, before and after making \( S \) monic as in Step 1, we have
\[
S(s) = \begin{bmatrix}
-\epsilon s^2 & -s(1-s) \\
\epsilon(1+s) & 1 - s^2
\end{bmatrix},
\]
with \( \epsilon = 0.2 \), which was also considered in Example 5.2.

As a result, the block companion matrix \( C \) is given by
\[
C = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & -\frac{1}{\epsilon} - \frac{1}{\epsilon^2} & -\frac{1}{\epsilon} & 1 \\
0 & -\frac{1}{\epsilon} & -\frac{1}{\epsilon^2} & 1 - \frac{1}{\epsilon^2}
\end{bmatrix}
\]

Retaining the first \( (M + 1)n = 4 \) rows of \( R_1 \) yields \( E = R_1 \).

\[
R_1 = \begin{bmatrix}
1 \\
\frac{1}{\epsilon} \\
\frac{1}{\epsilon^2} \\
\frac{1}{\epsilon}
\end{bmatrix}
\]

Retaining the first \( (M + 1)n = 4 \) rows of \( R_1 \) yields \( E = R_1 \).

The only row of \( E \) that needs to be deleted is the first, which results in \( E \) and the corresponding \( T \) given by
\[
E = \begin{bmatrix}
\frac{1}{\epsilon} \\
-\frac{1}{\epsilon^2} \\
\frac{1}{\epsilon}
\end{bmatrix}, \quad T = \begin{bmatrix}
1 & \frac{1}{\epsilon} & 0 \\
1 & 0 & 1
\end{bmatrix}
\]

Reinserting the first column of zeros into \( T \) and interchanging the first and third columns of the result we obtain
\[
[T_0 \ T_1] = \begin{bmatrix}
\frac{1}{\epsilon} & 1 & 0 \\
0 & 1 & 0
\end{bmatrix}
\]

This defines the polynomial matrix factor
\[
T(s) = \begin{bmatrix}
\frac{1}{\epsilon} & 1 \\
0 & 1 + s
\end{bmatrix}
\]

Premultiplication by the inverse of the leading column coefficient matrix results in
\[
T(s) = \begin{bmatrix}
1 & \frac{1}{\epsilon} \\
0 & 1 + s
\end{bmatrix}
\]

Factoring the leading diagonal coefficient matrix of \( Z \) as
\[
Z_L = \begin{bmatrix}
\epsilon & -1 \\
-1 & 1
\end{bmatrix}
\begin{bmatrix}
\sqrt{\epsilon} & 0 \\
0 & \sqrt{\frac{1}{\epsilon}}
\end{bmatrix}
\begin{bmatrix}
\frac{1}{\epsilon} & 0 \\
0 & 1 + s
\end{bmatrix}
\begin{bmatrix}
\sqrt{\epsilon} & -\sqrt{\frac{1}{\epsilon}} \\
0 & \sqrt{\frac{1}{\epsilon}} 
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & 1 - \frac{1}{\epsilon}
\end{bmatrix}
\]

we find the spectral factor
\[
P(s) = \begin{bmatrix}
\sqrt{\epsilon} & -\frac{1}{\epsilon} \\
0 & \sqrt{\frac{1}{\epsilon}}
\end{bmatrix}
\begin{bmatrix}
1 & 2 \\
0 & 1 + s
\end{bmatrix}
\begin{bmatrix}
\sqrt{\epsilon} & \frac{1}{\epsilon} \\
0 & \sqrt{\frac{1}{\epsilon}} 
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & 1 - \frac{1}{\epsilon}
\end{bmatrix}
\]

This result is not the same as that found in Case 1 of Example 5.2. The two (canonical) spectral factors are related by \( P_{\text{Example 5.2}} = UP_{\text{this example}} \), where \( U \) is the constant \( J \)-unitary matrix
\[
U = \begin{bmatrix}
-\frac{1}{\epsilon} & -\sqrt{\frac{1}{\epsilon}} \\
\sqrt{\frac{1}{\epsilon}} & \frac{1}{\epsilon}
\end{bmatrix}
\]

**VIII. RICCATI EQUATION**

Factorization by solution of an algebraic Riccati equation is most convenient if the para-Hermitian polynomial matrix \( Z \) is given in the "pre-factored" form
\[
Z = Q^-WQ
\]

where \( W \) is a constant symmetric matrix and \( Q \) a tall polynomial matrix that is column reduced, with column degrees \( \delta_1, \delta_2, \cdots, \delta_n \). This situation is typical for the factorizations that arise in \( H_\infty \) optimization problems [15].
Algorithm 8.1 (Prefactorization): If $Z$ is diagonally reduced nonsingular without zeros on the imaginary axis but not prefactored, a prefactorization may be obtained by introducing an arbitrary square column reduced strictly Hurwitz polynomial matrix $N$ of the same dimensions as $Z$ whose column degrees equal the half diagonal degrees of $Z$ and solving the symmetric bilateral polynomial matrix equation

$$Z = N^* M + M^* N$$  \hspace{1cm} (86)$$

for the polynomial matrix $M$, with column degrees again equal to the half diagonal degrees of $Z$. Then the desired pre-factorization is

$$Z = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} N^* M \\ M \end{bmatrix}.$$  \hspace{1cm} (87)$$

The algorithm discussed in this section relies on the following well-known connection between algebraic Riccati equations and factorizations (compare [16]).

Theorem 8.2 (Algebraic Riccati Equation and Factorization): Consider the linear time-invariant system $\dot{x} = Ax + Bu, y = Cx + Du$, with transfer matrix $H(s) = C(sI - A)^{-1}B + D$, and let $R$ and $W$ be given symmetric constant matrices. Suppose that the algebraic matrix Riccati equation

$$0 = A^TX + XA + C^TW C - (XB + C^TW D) \cdot (D^TW D + R)^{-1} (B^T X + D^T W C)$$  \hspace{1cm} (88)$$

has a symmetric solution $X$. Then

$$R + H^* WH = V^* LV$$  \hspace{1cm} (89)$$

where the constant symmetric matrix $L$ and the rational matrix function $V$ are given by

$$L = R + D^TW D, \quad V(s) = I + F(sI - A)^{-1}B$$  \hspace{1cm} (90)$$

with $F = L^{-1}(B^TX + D^TWC)$. The zeros of the numerator of $V$ are the eigenvalues of the matrix $A - BF$. The factorization algorithm may now be outlined as follows.

Algorithm 8.3 (Factorization of a Prefactored Polynomial Matrix by Solution of a Riccati Equation): Suppose that the diagonally reduced nonsingular para-Hermitian polynomial matrix $Z$ has no zeros on the imaginary axis and is given in the prefactored form

$$Z = Q^* W Q.$$  \hspace{1cm} (91)$$

Step 1. The first step in the algorithm is to convert the polynomial factorization to a rational factorization. To this end, introduce the diagonal polynomial matrix

$$E(s) = \text{diag}(s^{\delta_1}, s^{\delta_2}, \ldots, s^{\delta_n})$$  \hspace{1cm} (92)$$

with $\delta_1, \delta_2, \ldots, \delta_n$ the column degrees of $Q$, and define the rational para-Hermitian matrix $\Pi$ as

$$\Pi = (E^* -1)Q^* W Q E^{-1} = H^* WH.$$  \hspace{1cm} (93)$$

Because $Q$ is column reduced, the rational matrix $H = QE^{-1}$ is proper with full rank at infinity.

Step 2. Next represent $H = Q E^{-1}$ in the form

$$H(s) = D + C(sI - A)^{-1}B$$  \hspace{1cm} (94)$$

with $A, B, C,$ and $D$ constant matrices such that $\dot{x} = Ax + Bu, y = Cx + Du$ is a minimal realization of the system with transfer matrix $H$ (see [12, Ch. 6]). Also determine the polynomial matrix $K$ such that $(sI - A)^{-1}B = K(s)E^{-1}(s)$ by solving the polynomial equation $(sI - A)K(s) = BE(s)$ for $K$.

Step 3. Setting $R = 0$, application of Theorem 8.2 yields the rational factorization $H^*WH = V^*LV$, which follows by solution of the algebraic Riccati equation

$$0 = A^TX + XA + C^TW C - (XB + C^TW D) (D^TW D)^{-1} (B^TX + D^TW C).$$  \hspace{1cm} (95)$$

This rational factorization reduces to the polynomial factorization $Z = T^*LT$, with

$$T(s) = E(s) + FK(s)$$  \hspace{1cm} (96)$$

where $L = D^TW D$ and $F = L^{-1}(B^TX + D^TW C)$. If the solution of the Riccati equation (89) is chosen such that $A - BF$ has all its eigenvalues in the open left-half complex plane, $T$ is strictly Hurwitz.

Step 4. The factorization is completed by the factorization of the constant symmetric matrix $L$. Practically, the algebraic Riccati equation (96) is solved by finding the Schur decomposition (see for instance [17]) of the associated Hamiltonian matrix

$$\mathcal{H} = \begin{bmatrix} A - BL^{-1}S^T & -BL^{-1}B^T \\ -C^TW C + SL^{-1}S^T & (A - BL^{-1}S^T)^* \end{bmatrix}$$  \hspace{1cm} (97)$$

with $S = C^TW D$. The Robust-Control Toolbox [4] provides a routine. The solution of the Riccati equation then is $X = X_2X_1^{-1}$, where the square matrices $X_1$ and $X_2$ follow from the matrix

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$  \hspace{1cm} (98)$$

whose columns form a basis for the maximal stable invariant subspace of the Hamiltonian.

If the Riccati equation has a solution, the factor $P$ is column reduced with the correct column degrees, and the factorization is canonical. The Riccati equation fails to have a solution, however, if the "top" coefficient $X_1$ is singular. In this case, the factorization is noncanonical.
Example 8.4 (Factorization by Solution of a Riccati Equation): Again we consider the factorization of

\[ Z(s) = \begin{bmatrix} \varepsilon & 1 - s \\ 1 + s & 1 - s^2 \end{bmatrix} \]  

(99)

with \(|\varepsilon| < 1\). Choosing

\[ N(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 + s \end{bmatrix} \]  

(100)

it easily follows by solution of (87) that

\[ M(s) = \begin{bmatrix} \frac{\varepsilon}{2} & 1 - s \\ 0 & \frac{1 + s}{2} \end{bmatrix} \]  

(101)

As a result, \( Z \) may be prefactored as

\[ Z_2 = QWQ \]

with

\[ Y_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 + s \end{bmatrix}, \quad Q_1(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 + s \end{bmatrix}. \]  

(102)

Next, choosing the polynomial matrix \( E \) as

\[ E(s) = \text{diag}(1, s) \]

we obtain

\[ H(s) = Q(s)E^{-1}(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 - s^2 \\ \frac{\varepsilon}{2} & \frac{1 + s}{2} \end{bmatrix}. \]  

(103)

It may easily be established that \( H(s) = D + C(sI - A)^{-1}B \), with

\[ D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad A = 0, \quad B = \begin{bmatrix} 1 & 0 \end{bmatrix}. \]  

(104)

After making the appropriate substitutions, the algebraic Riccati equation turns out to have the form \( \varepsilon X^2 + 2(1 + \varepsilon)X + 4 = 0 \), which for \( \varepsilon \neq 0 \) has the two solutions \( X = -2/\varepsilon \) and \( X = -2/\varepsilon \). The corresponding "gains" are

\[ F = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad F = \begin{bmatrix} e_x \\ 1 \end{bmatrix}. \]  

(105)

We need the second, because it makes \( A - BF = -1 \) stable. We find the same result when considering the Hamiltonian, which is given by

\[ \mathcal{H} = \begin{bmatrix} \frac{1 + \varepsilon}{2} & 1 - \frac{1 + \varepsilon}{2} \\ \frac{1 + \varepsilon}{2} & -1 \end{bmatrix}. \]  

(106)

\( \mathcal{H} \) has the eigenvalues \( \pm 1 \). The maximal stable invariant subspace of \( \mathcal{H} \) is spanned by its eigenvector corresponding to the eigenvalue \(-1\), which may be taken as

\[ X_1 = \begin{bmatrix} -\varepsilon \\ 2 \end{bmatrix}. \]  

(107)

For \( \varepsilon \neq 0 \) we have \( X = X_2X_1^{-1} = -2/\varepsilon \), as found before.

For \( \varepsilon = 0 \) the top coefficient \( X_1 \) is zero, and no canonical factorization exists.

For \( \varepsilon \neq 0 \) the polynomial matrix \( T \) as given by (97) equals

\[ T(s) = E(s) + FK(s) = \begin{bmatrix} 1 & 0 \\ 0 & s \end{bmatrix} + \begin{bmatrix} \frac{2}{\varepsilon} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{2}{\varepsilon} \\ 0 & 1 + s \end{bmatrix}. \]  

(108)

The remaining computations are identical to those in Case 1 of Example 5.2, and the result is the same.

IX. NEARLY NONCANONICAL FACTORIZATIONS

In certain applications, in particular in \( H_{\infty} \)-optimization, given a para-Hermitian matrix \( Z \), the purpose of \( J \)-factorization is to find stable rational matrices \( V \) of maximal and full column rank such that

\[ V^*ZV \geq 0 \]

on the imaginary axis. (109)

If \( Z \) has the spectral factorization \( Z = P^*JP \) with \( J = \text{diag}(I_1, -I_2) \), all such \( V \) are given by

\[ V = P^{-1} \begin{bmatrix} A \\ B \end{bmatrix} \]  

(110)

where \( A \) and \( B \) are stable rational matrices, with \( A \) square and nonsingular of the same dimensions as \( I_1 \), such that \( A^*A \geq B^*B \) on the imaginary axis.

A slightly more general form of the factorizations \( Z = P^*JP \) we consider is

\[ Z = P^*K^{-1}P \]  

(111)

with \( K \) a constant diagonal matrix. In the canonical case, \( K = J \). If the factorization is "close to" noncanonical, for numerical reasons it is advisable to use the alternate form (112) of the spectral factorization, with some of the diagonal entries of \( K \) small.

\[ K = \begin{bmatrix} K_1 & 0 \\ 0 & -K_2 \end{bmatrix} \]  

(112)

with \( K_1 \) and \( K_2 \) again diagonal. All stable rational \( V \) such that \( V^*ZV \geq 0 \) on the imaginary axis may now be expressed as

\[ V = P^{-1} \begin{bmatrix} K_1A \\ K_2B \end{bmatrix} \]  

(113)

with \( A \) and \( B \) stable rational such that \( A^*A \geq B^*B \) on the imaginary axis. This expression, in turn, is equivalent to

\[ PV = \begin{bmatrix} K_1A \\ K_2B \end{bmatrix} \]  

(114)

In the case of a noncanonical factorization, the algorithms based on interpolation (Section IV) and the solution of a Riccati equation (Section VIII) are capable of producing a (singular) polynomial matrix \( P \) whose column degrees equal the half diagonal degrees of \( Z \) and a (singular) constant diagonal matrix \( K \), such that all stable \( V \) of maximal column rank that satisfy (115) have the property that \( V^*ZV \geq 0 \) on the imaginary axis. These are not all \( V \) with this property, but all that are needed for the solution of the \( H_{\infty} \) problem.
Example 9.1 (Nearly Noncanonical Factorization): In Example 7.2, we found that for \( \varepsilon \neq 0 \) the polynomial matrix
\[
Z(s) = \begin{bmatrix} \varepsilon & 1 - s \\ 1 + s & 1 - s^2 \end{bmatrix}
\]  
with \( \varepsilon \) real such that \(|\varepsilon| < 1\), has a canonical factorization with spectral factor and signature matrix
\[
P(s) = \begin{bmatrix} \sqrt{\varepsilon} & \frac{1}{\varepsilon} - 1 \\ 0 & \frac{1}{\varepsilon} \end{bmatrix} \quad J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
\]  
Inspection shows that as \( \varepsilon \) approaches 0, some of the coefficients of the spectral factor approach \( \infty \). It is also easily seen that the factorization can be rewritten as \( Z = P^\infty K^{-1} P \), with
\[
P(s) = \begin{bmatrix} \varepsilon & 1 - s \\ 0 & 1 + s \end{bmatrix}, \quad K = \begin{bmatrix} \varepsilon & 0 \\ 0 & \frac{1}{1-s^2} \end{bmatrix}.
\]  
As \( \varepsilon \to 0 \), the matrix \( K \) approaches the zero matrix, and (115) reduces to
\[
\begin{bmatrix} 0 & 1 - s \\ 0 & 1 + s \end{bmatrix} V(s) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]  
This equation has the general stable rational solution
\[
V(s) = \begin{bmatrix} v(s) \\ 0 \end{bmatrix}
\]  
with \( v \) any stable scalar rational function. If \( \varepsilon = 0 \) then \( V^\infty Z V = 0 \).

We now discuss how the interpolation algorithm of Section VII may be used to obtain the alternate form
\[
Z = P^\infty K^{-1} P.
\]  

Algorithm 9.1 (Modification of the Interpolation Algorithm 7.1): Steps 1–5. Steps 1–5 of Algorithm 7.1 may be followed without modification to construct the polynomial matrix \( T \).

Step 6. If the factorization is close to noncanonical, Step VI may become unreliable because the leading column coefficient matrix \( Z_L \) of \( T \) is close to singular. As long as \( Z_L \) is nonsingular we have \( Z = T^{-\infty}(T_L^{-1})^T Z_L T_L^{-1} T, \) where \( Z_L \) is the leading diagonal coefficient matrix of \( Z \). Rewriting this as \( Z = T^{-\infty}(T_L Z_L^{-1} T_L^T)^{-1} T \) and introducing the Schur decomposition
\[
T_L Z_L^{-1} T_L^T = Q K Q^T
\]  
with \( Q \) orthogonal and \( K \) diagonal, we obtain the desired factorization
\[
Z = P^\infty K^{-1} P
\]  
with \( P = Q^T T \). \( K \) is well defined even if the factorization is noncanonical.

Algorithm 8.2 based on the solution of a Riccati equation may also be modified to handle nearly noncanonical factorizations.

Algorithm 9.2 (Modification of the Riccati Equation Algorithm 8.2): Steps 1–2. Steps 1–2 of Algorithm 8.2 may be followed without modification.

Step 3. Step 3 is initiated by computing a real matrix
\[
\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}
\]  
with \( X_1 \) and \( X_2 \) square, whose columns form a basis for the maximal invariant stable subspace of the Hamiltonian matrix (98). When the factorization is nearly noncanonical, the "top coefficient" \( X_1 \) is close to singular. In this case, rather than computing \( F = G X_1^{-1} \), with \( G = B^T X_2 + S^T X_1 \), we let \( G X_1^{-1} = \hat{X}_1^{-1} \hat{G} \), where \( \hat{G} \) and \( \hat{X}_1 \) are computed from \( \hat{X}_1 G = \hat{G} X_1 \), or, equivalently
\[
[\hat{X}_1 - G] \begin{bmatrix} G \\ X_1 \end{bmatrix} = 0.
\]  
This amounts to computing a (real) basis for the left null space of
\[
\begin{bmatrix} B^T X_2 + S^T X_1 \\ X_1 \end{bmatrix}
\]  
which is well defined even if \( X_1 \) is singular. Accordingly, we replace \( T \) as given by (97) with
\[
T(s) = \hat{X}_1 E(s) + \hat{G} K(s).
\]  
Step 4. Step 4 is identical to Step 6 of Algorithm 9.1.

X. CONCLUSIONS

Of the four algorithms for the \( J \)-spectral factorization of para-Hermitian polynomial matrices discussed in this paper, the one based on diagonalization (Section IV) is the most general. It can be applied to matrices which are singular and have zeros on the imaginary axis, does not require diagonal reducedness, and can be used both for canonical and for noncanonical factorizations. Its disadvantages are that it involves rather elaborate polynomial operations and, in the case of canonical factorizations, no control seems to be available over the column degrees of the spectral factor.

The algorithm based on successive factor extraction (Section V) requires the matrix to be diagonally reduced and to have no zeros on the imaginary axis (and, hence, to be nonsingular). The method can handle noncanonical factorizations and produces a column reduced spectral factor for canonical factorizations. On the negative side, it requires explicit computation of the zeros of the polynomial matrix.

The interpolation method (Section VII) also requires the matrix to be diagonally reduced and to have no zeros on the imaginary axis. It appears to be quite efficient and may possibly be further streamlined by eliminating before the Schur decomposition in some way the eigenvalues at the origin that result from the zeros at infinity. Although the algorithm cannot handle noncanonical factorizations, it can deal with nearly noncanonical factorizations.

The factorization algorithm based on the solution of an algebraic Riccati equation (Section VIII) likewise requires diagonal reducedness and does not allow zeros on the imaginary
It can deal with nearly noncanonical factorizations but does not apply to noncanonical factorizations. The algorithm appears quite efficient but requires prefactorization and conversion to a state space representation as preparatory steps. Several of the algorithms have been implemented in MATLAB [14, 19]; further work is in progress. Much work remains to be done on analyzing and improving the numerical efficiency and robustness of the algorithms.

REFERENCES