Stability of a disturbance decoupled rigid body

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Received 11 May 1991
Revised 30 December 1991 and 13 April 1992

Abstract: In this note, we investigate the stability behaviour of torque controlled rotating rigid bodies. The investigation is not only restricted to the angular velocity equations, but also includes the system describing the motion of the body-fixed frame. For the composite system we show that only orbital stability can occur.

Keywords: Disturbance decoupling; feedback; rigid body; stability; Lyapunov functions; orbital stability.

Introduction

As an application of controlled invariance for nonlinear systems, Nijmeijer and Van der Schaft [2,3] solved a disturbance decoupling problem in rigid body dynamics. The object under consideration was a rotating rigid body influenced by three torques – two controls and one disturbance, each acting in one of the principal axes of inertia. The result was a feedback control which decouples the disturbance from the principal axis which it is acting in.

A lot of research has been done in disturbance decoupling with stability, i.e., to find a feedback control which decouples a disturbance from the output while ensuring asymptotic stability of an isolated fixed point of the (feedback modified) drift field [4]. These theories, however, do not apply to rigid body problems as described above due to the fact that in the case of torque control, no isolated fixed points for the full dynamics will appear.

For this reason we reconsider Nijmeijer–Van der Schaft’s feedback transformed rigid body dynamics and look for its stability properties. This will be done not only for the subsystem of the angular velocity equations, but also for the full system which also governs the motion of the body-fixed frame, thus exhibiting properties of (orbital) stability of the rigid body’s attitude.

We describe the rigid body dynamics in the usual way. Let \( \Sigma = (E_1, E_2, E_3) \) be an orthonormal inertial frame, let \( \sigma = (e_1, e_2, e_3) \) be orthonormal and body-fixed such that \( e_i \) spans the \( i \)-th principal axis of inertia, and let \( a_i > 0, i = 1, 2, 3 \), be the corresponding moments of inertia. Then the motions \( t \rightarrow \sigma(t) \) are governed by the ODEs

\[
\begin{align*}
\dot{e}_1 &= \omega_3 e_2 - \omega_2 e_3, \quad \dot{e}_2 = -\omega_3 e_1 + \omega_1 e_3, \quad \dot{e}_3 = \omega_2 e_1 - \omega_1 e_2, \\
a_1 \dot{\omega}_1 &= (a_2 - a_3) \omega_2 \omega_3 + u_1, \quad a_2 \dot{\omega}_2 = (a_3 - a_1) \omega_3 \omega_1 + u_2, \quad a_3 \dot{\omega}_3 = (a_1 - a_2) \omega_1 \omega_2 + d,
\end{align*}
\]

where \( w = \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3 \) is the angular velocity, \( u_1 \) and \( u_2 \) are the controls, and \( d \) is the disturbance.

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Following [2,3], the feedback control

\[ u_1 = (a_1 - a_2 + a_3) \omega_2 \omega_3, \quad u_2 = (a_1 - a_2 - a_3) \omega_3 \omega_1 \]

solves the indicated DDP, transforming (1) into

\[ \begin{align*}
    \dot{e}_1 &= \omega_3 e_2 - \omega_2 e_3, \\
    \dot{e}_2 &= -\omega_3 e_1 + \omega_1 e_3, \\
    \dot{e}_3 &= \omega_2 e_1 - \omega_1 e_2,
\end{align*} \]

(3a)

\[ \begin{align*}
    \dot{\omega}_1 &= \omega_2 \omega_3, \\
    \dot{\omega}_2 &= -\omega_3 \omega_1, \\
    \dot{\omega}_3 &= a \omega_1 \omega_2 + \frac{d}{a_3}, \quad a := \frac{a_1 - a_2}{a_3}.
\end{align*} \]

(3b)

The decoupling of \( d \) from \( e_3 \) will be seen below.

Now passing to the stability analysis we have to consider the feedback modified equations (3), ignoring the disturbance \( d \). We shall first investigate the stability of the relative equilibria of the subsystem (3b) by use of appropriate Lyapunov functions, and then, secondly, the (orbital) stability of the respective uniform rotations utilizing the (almost) explicit solution of the full system (3).

1. Stability of the relative equilibria of the subsystem (3b)

The stability analysis of the relative equilibria

\[ w_1 = (\alpha, 0, 0), \quad w_2 = (0, \alpha, 0), \quad w_3 = (0, 0, \alpha), \quad \alpha \in \mathbb{R}, \]

will be based on the following well known facts about Lyapunov stability.

**Lemma 1.** Let \( x_0 \) be an equilibrium point of the ODE \( \dot{x} = f(x), x \in \mathbb{R}^n \).

(a) If \( f \) has a first integral \( V \) which is (positive or negative) definite in some neighbourhood of \( x_0 \), then \( x_0 \) is stable (\( V \) is a Lyapunov function with vanishing Lie derivative).

(b) If the Jacobian \( Df(x_0) \), describing the linearization of \( f \) about \( x_0 \), has at least one eigenvalue with positive real part, then \( x_0 \) is unstable.

The construction of a Lyapunov function \( V \) can be done by exploiting the first integrals of (3b),

\[ F_1(w) = \omega_1^2 + \omega_2^2, \quad F_2(w) = a_1 \omega_1^2 + a_2 \omega_2^2 - a_3 \omega_3^2. \]

Investigating \( w_1 \), a suitable ansatz is

\[ V(w) = (F_1 + \Phi \circ F_1)(w). \]

Here \( \Phi : \mathbb{R} \to \mathbb{R} \) is some smooth function to be found such that \( V \) (a first integral) vanishes at the equilibrium point \( w_1 \) and is strictly positive or negative in a punctured neighbourhood of \( w_1 \). Now the first and second derivative of \( V \),

\[ 
    D V(w) = \begin{pmatrix}
        2a_1 \omega_1 + 2 \Phi'(F_1(w)) \omega_1, & 2a_1 \omega_2 + 2 \Phi'(F_1(w)) \omega_2, & -2a_3 \omega_3
    \end{pmatrix},
\]

\[ 
    D^2 V(w) = 
    \begin{pmatrix}
        2a_1 \omega_1 + 2 \Phi'(F_1(w)) + 4 \Phi''(F_1(w)) \omega_1^2 & * & *
    \\
        4 \Phi''(F_1(w)) \omega_1 \omega_2 & 2a_1 + 2 \Phi'(F_1(w)) + 4 \Phi''(F_1(w)) \omega_2^2 & *
    \\
        0 & 0 & -2a_3
    \end{pmatrix}
\]

yield

\[ 
    DV(w_1) = \begin{pmatrix}
        2a_1 \alpha + 2 \Phi'(\alpha^2) \alpha, & 0, & 0
    \end{pmatrix} = 0 \quad \text{iff} \quad a_1 + \Phi'(\alpha^2) = 0,
\]

(4a)

and then

\[ 
    D^2 V(w_1) = \begin{pmatrix}
        4a_1 \alpha^2 \Phi''(\alpha^2), & 2(a_2 - a_1), & -2a_3
    \end{pmatrix}
\]

(negative) definite iff \( a_2 < a_1 \) and \( \Phi''(\alpha^2) < 0 \).

Let \( \Phi : \Phi(s) := -a_1[s - \frac{1}{2}(s - \alpha^2)^2] \), then \( V(w_1) = 0 \) is a proper maximum, thus \( w_1 \) is stable if \( a_2 < a_1 \).
If $a_1 < a_2$, then the linearization of (3b) around $\mathbf{w}_1$ has a positive eigenvalue, thus $\mathbf{w}_1$ is unstable in this case. The same instability occurs for $a_1 = a_2$ (i.e. $a = 0$) which is easily checked by considering the elementary solution of (3b). The investigation of $\mathbf{w}_2$ is done in the same way.

The stability of $\mathbf{w}_3$ can be shown by means of $V(\mathbf{w}) := (F_1 + \Phi \circ F_2)(\mathbf{w})$. Now

$$DV(\mathbf{w}) = 2(\mathbf{w}_1 + \Phi'(F_2(\mathbf{w}))a_1 \mathbf{w}_1, \mathbf{w}_2 + \Phi'(F_2(\mathbf{w}))a_2 \mathbf{w}_2, -\Phi'(F_2(\mathbf{w}))a_3 \mathbf{w}_3)$$

vanishes at $\mathbf{w}_3$ iff $\Phi'(-a_3 \mathbf{a}_2^2) = 0$, and in that case $D^2V(\mathbf{w}) = \text{diag}(2, 2, 4a_3^2\Phi'(-a_3 \mathbf{a}_2^2))$ is (positive) definite iff $\Phi''(-a_3 \mathbf{a}_2^2) > 0$.

Let $\Phi(s) := (s + a_3 \mathbf{a}_2^2)^2$. Then $V$ has the proper minimum $V(\mathbf{w}_3) = 0$ and this implies stability of $\mathbf{w}_3$ whatever the values of $a_1$, $a_2$, $a_3$ are. Altogether, this leads to:

**Proposition 1.** If $a_2 < a_1$ then $\mathbf{w}_1$ is stable, $\mathbf{w}_2$ is unstable; if $a_2 > a_1$ then $\mathbf{w}_1$ is unstable, $\mathbf{w}_2$ is stable; for $a_1 = a_2$ both $\mathbf{w}_1$ and $\mathbf{w}_2$ are unstable; $\mathbf{w}_3$ is stable in any case.

**Remark.** The method used to construct a Lyapunov function is also used in [1], where it is called the ‘Energy–Casimir method’. This is due to the (hidden) differential geometric feature of (3b): it is in fact a hamiltonian vector field (with Hamiltonian $F_2$: energy) on a Poisson manifold (granting the existence of a family of ‘Casimir functions’ as first integrals, $\Phi \circ F_1$ in the present context). But obviously the essence of this method is just the knowledge of (at least) two first integrals comprising a family of integrals which fortunately contains a Lyapunov function.

2. The solutions of the full system (3)

The feedback modified system (3) has the agreeable feature that, up to the integration of a pendulum-type ODE, its solutions can be given in explicit form.

Let $(\mathbf{\bar{w}}, \dot{\mathbf{\bar{w}}}) = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \dot{\mathbf{w}}_1, \dot{\mathbf{w}}_2, \dot{\mathbf{w}}_3)$ denote the initial values at $t = 0$. Since $F_1: F_1(\mathbf{w}) = \omega_1^2 + \omega_2^2$ used above is also a first integral for the full system (3) with disturbance $d \neq 0$, there holds, for any solution,

$$\omega_1^2(t) + \omega_2^2(t) = \dot{\omega}_1^2 + \dot{\omega}_2^2 =: A^2, \quad (5)$$

whence a representation

$$\omega_1 = A \cos \theta, \quad \omega_2 = A \sin \theta, \quad A = (\dot{\omega}_1^2 + \dot{\omega}_2^2)^{1/2} \quad (6)$$

with some smooth $\theta: t \to \theta(t) \in \mathbb{R}$ follows. Then, together with (6), the first two equations of (3b) yield

$$\dot{\theta} = -\omega_3 \quad (7)$$

while the third equation leads to

$$\ddot{\theta} = -\frac{1}{2}aA^2 \sin 2\theta - d/a_3. \quad (8)$$

This ODE of pendulum type has to be seen with initial conditions

$$\theta(0) = \hat{\theta}, \quad \dot{\theta}(0) = -\hat{\omega}_3, \quad (9a)$$

where $\hat{\theta}$ is, modulo $2\pi$, given by

$$\hat{\omega}_1 = A \cos \hat{\theta}, \quad \hat{\omega}_2 = A \sin \hat{\theta}. \quad (9b)$$

Thus each solution $\mathbf{w}(t)$ of the subsystem (3b) is uniquely determined by (6) through to (9).

A straightforward calculation using (3) and (5) yields

$$\dot{e}_3 = -A^2 e_3. \quad (10)$$
Remark. Since (10) also holds for a disturbance \( d \neq 0 \), it clearly shows that \( d \) does not influence \( e_3 \).

The integration of (3a) is now straightforward. (10) yields

\[
e_3(t) = \dot{e}_3 \cos A t + \dot{e}_3(0) \frac{1}{A} \sin A t
\]

(covering the case \( A = 0 \) by \( A \to 0 \)). Introduce the auxiliary vectors

\[
e_0 := \cos \dot{\theta} \dot{e}_1 + \sin \dot{\theta} \dot{e}_2, \tag{11a}
\]

\[
\dot{e}_0 := -\sin \dot{\theta} \dot{e}_1 + \cos \dot{\theta} \dot{e}_2. \tag{11b}
\]

Then, using the third equation of (3a) at \( t = 0 \), \( e_3(t) \) can be written as

\[
e_3(t) = \dot{e}_3 \cos At - \dot{e}_0 \sin At. \tag{11c}
\]

Due to (6), (7) and (12c) the first two equations of (3a) turn into an inhomogeneous linear ODE,

\[
\begin{align*}
\dot{e}_1 &= -\theta e_2 - A \sin \theta(t) (\dot{e}_3 \cos At - \dot{e}_0 \sin At), \\
\dot{e}_2 &= \theta e_1 + A \cos \theta(t) (\dot{e}_3 \cos At - \dot{e}_0 \sin At),
\end{align*}
\]

which has the solution

\[
\begin{align*}
e_1(t) &= e_0 \cos \theta(t) - (\dot{e}_3 \sin At + \dot{e}_0 \cos At) \sin \theta(t), \\
e_2(t) &= e_0 \sin \theta(t) + (\dot{e}_3 \sin At + \dot{e}_0 \cos At) \cos \theta(t). \tag{12a, 12b}
\end{align*}
\]

(12) describes the motions \( t \to \sigma(t) = (e_1(t), e_2(t), e_3(t)) \), where \( \theta \) is governed by (8) and (9). The orthonormality of \( \sigma(t) \) is granted by orthonormality of \((e_0, \dot{e}_0, \dot{e}_3)\).

It is evident that the investigation of the stability of motions can now be done by discussion of (12) taking into consideration the qualitative behaviour for \( t \to \infty \) of the solutions of (8) with disturbance \( d = 0 \).

3. Stability of uniform rotations

Note that, under feedback control (2) and zero disturbance, the rigid body performs the same uniform rotations about the principal axes as the uncontrolled rigid body does. Clearly, however, the stability properties of these rotations may change under feedback: if, for example, \( a_1 < a_2 < a_3 \), then the rotation \( \omega_2 \) about the intermediate axis of inertia becomes stable while the rotation \( \omega_1 \) about the minor axis has lost its stability.

In what follows the emphasis is on the stability of uniform rotations, including the behaviour in time of the body-fixed frame \( \sigma = (e_1, e_2, e_3) \). Let \( (\sigma, \omega) \) denote the state and consider first the uniform rotation \( (\sigma_1, \omega_1) \) about the 1-axis,

\[
\begin{align*}
e_1(t) &= \dot{e}_1, \quad \omega_1(t) = \alpha, \quad \alpha \neq 0, \\
e_2(t) &= \dot{e}_2 \cos \alpha t + \dot{e}_3 \sin \alpha t, \quad \omega_2(t) = 0, \tag{13a, 13b} \\
e_3(t) &= -\dot{e}_2 \sin \alpha t + \dot{e}_3 \cos \alpha t, \quad \omega_3(t) = 0. \tag{13c}
\end{align*}
\]

In the framework of Section 2 this solution of (3) is determined by the initial condition \( (\sigma, \omega)(0) = (\dot{e}_1, \dot{e}_2, \dot{e}_3, (\alpha, 0, 0)) \). Then (9b) yields \( A = \alpha, \theta = 0 \) whence, by (8), \( \theta(t) = 0 \) follows.

The corresponding orbit could be visualized by the resting unit vector \( \dot{e}_1 \) and a great circle of \( S^2 \subset \mathbb{R}^3 \) in a plane perpendicular to \( \dot{e}_1 \), together with the resting point \( (\alpha, 0, 0) \) in the \( \omega \) 3-space. An initial
perturbation \((\dot{e}_1, \dot{e}_2, \dot{e}_3, (\alpha, 0, 0)) \rightarrow (\dot{e}_1, \dot{e}_2, \dot{e}_3, (\alpha + \delta, 0, 0))\) essentially preserves this orbit while the circle is now run through with deviation arising from the \(\delta t\) argument. This shows that at most orbital stability and not Lyapunov stability can be expected.

Orbital stability means that, under sufficiently small initial perturbations, the orbit of the perturbed motion is in an arbitrary \(\varepsilon\)-neighbourhood of the unperturbed orbit. Since \(w\) is governed by the subsystem (3b), it becomes clear by the above description that \((\sigma_1, w_1)\) is orbitally stable iff, under slight initial perturbation, \(e_i(t)\) remains in an \(\varepsilon\)-cone about \(\dot{e}_i\) (implying, by virtue of orthonormality, that the curves \(t \rightarrow e_i(t), i = 2, 3\), are in an \(\varepsilon\)-neighbourhood in \(S^2\) of the great circle) and \((\omega_1(t), \omega_2(t), \omega_3(t))\) remains in an \(\varepsilon\)-neighbourhood of \((\alpha, 0, 0)\) in the \(\omega\)-space. But this clearly means the following.

**Lemma 2.** \((\sigma_1, w_1)\) is orbitally stable iff, for every \(\varepsilon > 0\), there is a \(\delta > 0\) such that (with appropriate norm)

\[
\| (e_1(0), e_2(0), e_3(0), \omega_1, \omega_2, \omega_3) \) \(\rightarrow\) \( (\dot{e}_1, \dot{e}_2, \dot{e}_3, \alpha, 0, 0) \| < \delta
\]

implies \(\| e_1(t) - \dot{e}_1 \| < \varepsilon\) and \(\| w(t) - (\alpha, 0, 0) \| < \varepsilon\) for all \(t > 0\).

**Proof.** We omit the proof (which can easily be given analytically) for reasons of space and refer to the description above.

The \(\omega\)-part of the latter implication yields the following:

**Corollary.** Lyapunov stability of \(w_1\) with respect to the ODE (3b) is necessary for orbital stability of \((\sigma_1, w_1)\).

**Remark.** Mutatis mutandis the same statements hold for the rotations about the 2- and 3-axis.

It is easy to see that an initial perturbation \(\delta \rightarrow \sigma(0)\) of the frame is of no interest. So let \((\alpha, 0, 0) \rightarrow \dot{\sigma} = (\alpha + \beta_1 \delta, \beta_2 \delta, \beta_3 \delta)\) with some \(\beta_i \in \mathbb{R}, |\beta_2| + |\beta_3| > 0\) and small \(\delta > 0\), be a perturbation of the initial velocity. Then (9) implies \(A = \alpha + O(\delta), \dot{\theta} = O(\delta), \) where the big O symbols are with \(\delta \rightarrow 0\). Now the perturbed motion is governed by the initial value problem

\[
\dot{\theta} = -\frac{1}{2} a A^2 \sin 2 \theta, \quad \theta(0) = O(\delta), \quad \dot{\theta}(0) = -\beta_3 \delta.
\]

If \(a > 0\), the differential equation is of normal pendulum type, the solution \(\theta(\cdot)\) of (14) is periodic with an amplitude tending to zero for \(\delta \rightarrow 0\). In this case, using Euclidean norm, (12a) yields for all \(t > 0,\)

\[
\| e_1(t) - \dot{e}_1 \|^2 = 2(1 - \cos \theta(t) \cos \dot{\theta}) - 2 \sin \theta(t) \cos At \sin \dot{\theta} = O(\delta^2).
\]

Therefore, by Lemma 2, orbital stability of \((\sigma_1, w_1)\) follows. The investigation of the uniform rotation \((\sigma_2, w_2)\) is done in the same way.

Finally, since \(w_3\) is Lyapunov stable, orbital stability of \((\sigma_3, w_3)\) may be expected. Initial perturbation \((0, 0, \alpha) \rightarrow (\beta_1 \delta, \beta_2 \delta, \alpha + \beta_3 \delta), |\beta_1| + |\beta_2| > 0\), yields, by (9b), \(A = \sqrt{\beta_1^2 + \beta_2^2} \delta\) and \(\dot{\theta}\) independent of \(\delta\) (\(\tan \dot{\theta} = \beta_2/\beta_1\)). Then, with (12c) and (11b),

\[
e_3(t) = \dot{e}_3 \cos At + \left(\sin \dot{\theta} \dot{e}_1 - \cos \dot{\theta} \dot{e}_2\right) \sin At
\]

shows that \(e_3(t)\) drifts away from \(\dot{e}_3\) with small angular velocity \(A > 0\), thus it does not remain in an \(\varepsilon\)-cone about \(\dot{e}_3\): \((\sigma_3, w_3)\) is orbitally unstable.

Summing up, we find the following:

**Proposition 2.** \((\sigma_1, w_1)\) is orbitally stable iff \(a_2 < a_1; (\sigma_2, w_2)\) is orbitally stable iff \(a_1 < a_2; (\sigma_3, w_3)\) is orbitally unstable in any case.
Remarks. (a) The last statement shows that Lyapunov stability of $w_i$, $i = 1, 2, 3$, with respect to the ODE (3b) is not sufficient for orbital stability of $(\sigma_i, w_i)$ with respect to (3).

(b) The investigation of Lyapunov stability of $w_i$, $i = 1, 2, 3$, could of course also be done by closer inspection of the solutions of (8).

Acknowledgement

The authors wish to thank Henk Nijmeijer for stimulating discussions.

References


