Second-order asymptotics in level crossing for differences of renewal processes

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We consider level crossing for the difference of independent renewal processes. Second-order expansions for the distribution function of the crossing time of level \( n \) are found, as \( n \to \infty \). As a by-product several other results on the difference process are found. The expected minimum of the difference process appears to play an important role in the analysis. This makes this problem essentially harder than the level crossing for the sum process which was studied earlier.

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difference of renewal processes * boundary crossing * second-order approximations

1. Introduction

Although renewal processes are among the most important and basic processes in applied probability, various areas in renewal theory are still rather unexploited. One of these more or less "blank spots on the map" seems to be the difference process of two independent renewal processes, which occurs frequently in queueing and reliability theory. Let \( N_1 = (N_1(t)) \) and \( N_2 = (N_2(t)) \) be two independent renewal processes and let the difference process \( M \) be defined by \( M(t) = N_1(t) - N_2(t) \), \( t \geq 0 \). Assume that \( M \) has an upward drift. We are in particular interested in the crossing time \( \tau_n \) of level \( n \in \mathbb{N} \). The situation resembles in a way the model of Kroese and Kallenberg (1989) [KK] where level crossing was considered for the sum process of independent renewal processes. For that model second-order approximations to the distribution function (d.f.) of the crossing time of level \( n \) were found, as \( n \to \infty \). These approximations depend only on the first three moments of the d.f.'s defining the renewal sequences.

Both for the sum process and for the difference process a first-order approximation, based on asymptotic normality, can be easily established using Cox (1962, p. 73). However, as seen in [KK], first order approximations are not very accurate and a big improvement is obtained by applying second-order approximations, leading to quite satisfactory approximations even for very small values of \( n \).

The main result of the paper is a second-order expansion for the d.f. of \( \tau_n \), when suitably standardized, as \( n \to \infty \). In the derivation of this expansion we come across...
several results, which are of independent interest. First of all, in Theorem 4.2 the d.f. of \( M(t) \) for large values of \( t \) is approximated in a similar way as the d.f. of the sum process, using expansions for the individual processes. In the case of the sum process a classical inversion step makes it possible to give a second-order expansion for the d.f. of the crossing time of level \( n \). However, for the difference process, such a relatively simple inversion step is not available, due to the fact that in this situation the process is not increasing. To tackle these complications we rely on an idea of Anscombe (1953) to “look backwards in time”. This idea is most easily used when both \( N_1 \) and \( N_2 \) are stationary renewal processes. In the recent past the idea of Anscombe has been applied frequently in boundary crossing problems for random walks (cf. Woodroofe (1982), Lalley (1984), Siegmund (1985) and Woodroofe and Keener (1986)).

Apart from a term which may be seen as the equivalent of the inversion step in the expansion for the sum process (cf. (3.2) and p. 486 of \([KK]\)) now an extra term comes in, where the expectation of the minimum of \( M \) emerges. It is shown that this expectation exists finitely, due to the positive drift of \( M \) and the finiteness of the second moments of the corresponding renewal sequences. Finally a delicate investigation of the local behavior of \( M \) at time \( t \) for large \( t \) is given for the finishing touch.

From a theoretical point of view the second-order expansion of the d.f. of \( \tau_n \) gives insight in the structure of the problem. The approximation does not only depend on the first three moments of the d.f.'s defining the renewal sequences. The somewhat unexpected appearance of the expected minimum of \( M \) in the expansion shows that the d.f. of the crossing time for the difference process is essentially more complicated than for the sum process.

From a computational point of view, (3.1) shows that for a second-order approximation to the d.f. of \( \tau_n \) it is not sufficient to know the first three moments of the d.f.'s involved. Fortunately, in many practical cases the expected minimum of \( M \) can be derived, whereas the true d.f. of \( \tau_n \) is practically always intractable. Numerical results and special cases are given in Kroese (1992). If the positive drift is not too small, the approximations are close to the (estimated) true d.f. even for rather small \( n \).

We conclude the Introduction with an outline of the rest of the paper. In the next section the basic definitions and model assumptions are given. Section 3 lists the main result and sketches its proof. The formal proof of the main result is given in Sections 4–5.

2. Model, definitions

In this section we formalize the model. Throughout this paper we conform to the definitions and assumptions that are given here, unless otherwise specified. There exist several definitions of a renewal process. Here we use the following:
Let $X = (X_i)_{i \geq 1}$ be a sequence of independent nonnegative random variables, such that $X_2, X_3, \ldots$ are identically distributed with some d.f. $G$. Let $\Gamma$ denote the d.f. of $X_1$. We call $X$ the renewal sequence corresponding to the delayed renewal process $N = (N(t))_{t \geq 0}$, which is defined for $t \geq 0$ by
\[
N(t) = \begin{cases} 
0, & \text{if } X_1 > t, \\
\sup\{n \geq 1: X_1 + \cdots + X_n \leq t\}, & \text{else}.
\end{cases}
\]

$N$ is called an ordinary renewal process if $F = G$, and a stationary renewal process if
\[
F(x) = F(x) = 1 - \int_0^x (1 - G(y)) \, dy, \quad \text{where } \mu = EX_2 < \infty.
\]

Let $N_1 = (N_1(t))$ and $N_2 = (N_2(t))$ denote two independent renewal processes with renewal sequences $(X_k^{(1)})$ and $(X_k^{(2)})$, respectively. If we omit the index set in the definition of a stochastic process, it is either $\mathbb{N}_+$ (renewal sequences) or $\mathbb{R}_+$ (otherwise). The difference process $M = (M(t))$ of $N_1$ and $N_2$ is defined by
\[
M(t) = N_1(t) - N_2(t), \quad t \geq 0.
\]
Let $\tau_n$ be the time at which $M$ crosses level $n \in \mathbb{N}$, i.e.
\[
\tau_n = \inf\{t \geq 0: M(t) \geq n\}.
\]

Similarly, let $T_k$ be the first time at which $M$ crosses level $k \in \mathbb{Z}_-$,
\[
T_k = \inf\{t \geq 0: M(t) \leq k\}.
\]

We do not a priori assume $N_1$ and $N_2$ to be stationary renewal processes. Although the main result is stated for such processes, most of the results hold true for delayed renewal processes. Let $F$ be the d.f. of $X_1^{(i)}$ and let $G_i$ be the d.f. of $X_2^{(i)}$, $i = 1, 2$. We assume that the expectation $\mu$, the variance $\sigma_i^2$ and the third central moment $\mu_3$ of $X_1^{(i)}$ exist finitely and that $G_i$ is nonlattice, $i = 1, 2$. This also ensures the finiteness of the expectation and variance of $X_1^{(i)}$ when $N_i$ is stationary, $i = 1, 2$. Let
\[
\zeta_i = \sup\{x \geq 0: F_i(x) < 1\}, \quad i = 1, 2.
\]

The probability space in the background is denoted by $(\Omega, \mathcal{F}, P)$. We introduce also a family of probability measures $(P^{r,s}; 0 \leq r < \zeta_1, 0 \leq s < \zeta_2)$ on $(\Omega, \mathcal{F})$. Under measure $P^{r,s}$, $N_1$ and $N_2$ are independent renewal processes, such that $X_2^{(i)}$ has d.f. $G_i$, $i = 1, 2$, and, for $x, y \geq 0$,
\[
P^{r,s}(X_1^{(1)} \leq x, X_1^{(2)} \leq y) = \frac{F_1(x + r) - F_1(r) F_2(y + s) - F_2(s)}{1 - F_1(r)} \frac{1 - F_2(s)}{1 - F_2(s)}.
\]

The expectation operator corresponding to $P^{r,s}$ is denoted by $E^{r,s}$.

**Remark 2.1.** When $F_i(0) = 0$, for $i = 1, 2$ (as is the case for stationary $N_1$ and $N_2$), then probability measures $P$ and $P^{0,0}$ coincide.

The following notation for conditional probabilities is used. Let $(\Omega, \mathcal{F}, Q)$ be the probability space under use. Let $H$ be an event and let $\mathcal{B}$ be a sub $\sigma$-algebra of $\mathcal{F}$.
The conditional probability of the event $H$ given $\mathcal{G}$ is denoted by $Q_\mathcal{G}(H)$. If $\mathcal{G}$ is generated by a random variable $V$, we write $Q_V(H)$ for $Q_\mathcal{G}(H)$.

We assume from now on that $\mu_1 < \mu_2$. In that case $M$ has a positive drift $\alpha^{-1}$,

$$\alpha^{-1} = \mu_1^{-1} - \mu_2^{-1}.$$ 

The following parameters are used frequently in the expansions below. We therefore list them in this section. We define

$$\gamma = \sqrt{\sigma_1^2 \mu_1^{-3} + \sigma_2^2 \mu_2^{-3}},$$

and, for $x \in \mathbb{R}$,

$$t_n(x) = n\alpha + x\sqrt{n} \alpha^{3/2},$$

$$c_i = \frac{1}{2} \sigma_i \mu_i^{-1/2}, \quad d_i = \frac{1}{2} \sigma_i \mu_i^{-1/2}, \quad \alpha_i = \mu_i^{-1} \sigma_i^{-1}, \quad i = 1, 2. \quad (2.1)$$

The standard normal d.f. is denoted by $\Phi$, its density by $\varphi$. To conclude this section, Figure 1 gives an illustration of the various definitions (note that $k \leq 0$).

![Figure 1. Realization of $(M(t), t \geq 0)$.](image)

### 3. Main result

**Theorem 3.1.** Let $N_1$ and $N_2$ be independent stationary renewal processes that satisfy the conditions of Section 2. Then the following second-order expansion for $\tau_n$ holds true.

$$\sup_{x \in \mathbb{R}} \left| P\left( \frac{\tau_n - n\alpha}{\sqrt{n}} \leq x \right) - \Phi(x) \right|$$

$$= \frac{\varphi(x)}{\sqrt{n}} \left\{ p(1 - x^2) + \frac{1}{2} \gamma \sqrt{\alpha} - \frac{1}{2} \gamma \sqrt{\alpha} - \frac{1}{2} \gamma \sqrt{\alpha} E \inf_{t>0} M(t) \right\}$$

$$= o(1/\sqrt{n}) \quad \text{as } n \to \infty,$$

where

$$p = \frac{(c_1 - d_1)(\gamma a_1)^{-3} - (c_2 - d_2)(\gamma a_2)^{-3}}{\sqrt{\alpha}} + \frac{1}{2} \gamma \sqrt{\alpha}.$$
Remark 3.1. First we show that the terms in (3.1) are well-defined (i.e. that \( \inf_{t \geq 0} M(t) \) has finite expectation). This can be seen as follows. Let \( S_n^{(i)} = X_1^{(i)} + \cdots + X_n^{(i)}, \ i = 1, 2 \). For \( k \in \mathbb{N}_+ \),

\[
P\left( -\inf_{t \geq 0} M(t) \geq k \right)
= P(S_n^{(1)} > S_n^{(2)} / k - 1 \text{ for some } n)
\leq P\left( \sup_{j \geq k} j^{-1} |S_j^{(2)} - j \mu_2| > \varepsilon \right) + P(S_n^{(1)} > (n + k - 1)(\mu_2 - \varepsilon) \text{ for some } n)
\leq P\left( \sup_{j \geq k} j^{-1} |S_j^{(2)} - j \mu_2| > \varepsilon \right)
+ P\left( \sup_{n > 0} \{S_n^{(1)} - n(\mu_2 - \varepsilon)\} > (\mu_2 - \varepsilon)(k - 1) \right),
\]

where \( \varepsilon > 0 \) is so small that \( \mu_1 < \mu_2 - \varepsilon \). Now

\[
- E \inf_{t \geq 0} M(t) = \sum_{k=1}^{\infty} P\left( -\inf_{t \geq 0} M(t) \geq k \right)
\leq \sum_{k=1}^{\infty} P\left( \sup_{j \geq k} j^{-1} |S_j^{(2)} - j \mu_2| > \varepsilon \right)
+ (\mu_2 - \varepsilon)^{-1} E \sup_{n > 0} \{S_n^{(1)} - n(\mu_2 - \varepsilon)\} + 1.
\]

The sum in the previous line is finite by Theorem 3 of Baum and Katz (1965), while the second expression is finite e.g. by Theorem 5 of Kiefer and Wolfowitz (1956).

Next we give a broad sketch of the proof of Theorem 3.1, of which the details can be found in the remaining sections.

Sketch of the proof of Theorem 3.1. By definition of \( t_n \) and \( \tau_n \),

\[
P\left( \frac{\tau_n - n \alpha}{\gamma \alpha^{3/2} \sqrt{n}} \leq x \right) = P(\tau_n \leq t_n)
= P(M(t_n) \geq n) + P(M(t_n) < n, \tau_n \leq t_n).
\]

We tackle the first term in (3.2) by using techniques that were developed in [KK]. That is, first we derive an expression for \( P(N_i(t) \leq x) \), as \( t \to \infty \), \( i = 1, 2 \). Then, in order to obtain an expansion for \( P(M(t) \leq x) \) it suffices to convolve the expansions of \( P(N_1(t) \leq x) \) and \( P(N_2(t) \leq x) \). This first term may be seen as the equivalent of
the inversion step in the expansion for the sum process. For the second term in (3.2) we have,

\[ P(M(t_n) < n, \tau_n \leq t_n) \]
\[ = \sum_{k=-\infty}^{-1} P(M(t_n) = n + k, \tau_n \leq t_n) \]
\[ = \sum_{k=-\infty}^{-1} P\left( M(t_n) - M(t_n - t_n) = n + k, \min_{0 \leq u \leq t_n} \{ M(t_n) - M(t_n - u) \} \leq k \right). \]

Since \( N_1 \) and \( N_2 \) are independent stationary renewal processes, \( (M(t_n) - M(t_n - u))_{0 \leq u \leq t_n} \) has the same probability law as \( (M(u))_{0 \leq u \leq t_n} \). This is a direct consequence of the reversibility property of stationary renewal sequences and Anscombe's (1953) idea to "look backwards in time". This idea is frequently used in level crossing problems for random walks (cf. Section 1), and is most easily illustrated by turning the realization of \( M \) in Figure 1 upside down and viewing the process "backwards in time", starting at time \( t_n \). Hence by definition of \( T_k \) in Section 2, we have

\[ P(M(t_n) < n, \tau_n \leq t_n) = \sum_{k=-\infty}^{-1} P(M(t_n) = n + k, T_k \leq t_n). \] (3.3)

We will show that, up to \( o(n^{-1/2}) \), we may restrict attention to a finite sum in (3.3) and that the summand may be replaced by

\[ P(M(t_n) = n + k) \cdot P(T_k \leq t_n). \]

Moreover,

\[ P(M(t_n) = n + k) \approx \varphi(x) \gamma^{-1}(n\alpha)^{-1/2} \]

for fixed \( k \) and large enough \( n \), and finally

\[ \sum_{k=-\infty}^{-1} P(T_k \leq t_n) \approx \sum_{k=-\infty}^{-1} P(T_k < \infty) = -E \inf_{t > 0} M(t). \]

The approximation sign \( \approx \) will be made more precise in the next sections, where we will give the formal proof of Theorem 3.1. The first term in (3.2) is considered in Section 4, whereas the second part is treated in Section 5. The actual proof of the theorem is given at the end of Section 5.

4. Expansions, first part

In order to expand the first term in (3.2), \( P(M(t_n) \geq n) \), for large \( n \), we start with an expansion for \( P(M(t) \leq x) \) for large \( t \).

Throughout this section \( N = (N(t)) \) denotes a delayed renewal process with renewal sequence \( (X_i) \). Let \( F \) be the d.f. of \( X_1 \) and \( G \) the d.f. of \( X_i, i = 2, 3, \ldots \).
We assume that $F$ and $G$ have finite expectation, variance and third central moment and that $G$ is nonlattice. Let $\xi = \sup\{x \geq 0: F(x) < 1\}$. The probability measure $P^r, 0 \leq r < \xi,$ is defined in the same manner as $P^\cdot$ in Section 2.

Let $\mu = EX_2, \sigma^2 = \text{Var} X_2, \mu_3 = E(X_2 - \mu)^3$ and $\nu = \nu(r) = E^r X_1, 0 \leq r < \xi$. Define parameters $a, c$ and $d$ similarly to (2.1). Note that $\nu(0)$ is not necessarily equal to $EX_1$, cf. Remark 2.1. We have the following lemma.

**Lemma 4.1.** Let $S_n = X_1 + \cdots + X_n$. For every fixed $0 \leq y < \xi$,

$$\sup_{0 \leq r < y} \sup_{x \in \mathbb{R}} \left| P^r \left( \frac{S_n - (n-1)\mu - \nu}{\sigma \sqrt{n}} \leq x \right) - \Phi(x) - \frac{\varphi(x)\mu_3}{\sqrt{n} \sigma^3} (1-x^2) \right|$$

$$= o\left( n^{-1/2} \right) \text{ as } n \to \infty.$$  

**Proof.** Let $\tilde{S}_{n-1} = X_2 + \cdots + X_n$ and $Y_n = \left( \sigma \sqrt{n} X_1 + \nu \right) \sigma^{-1} (n-1)^{-1/2}$, then

$$P^r \left( \frac{S_n - (n-1)\mu - \nu}{\sigma \sqrt{n}} \leq x \right) = P^r \left( \frac{\tilde{S}_{n-1} - (n-1)\mu}{\sigma \sqrt{n-1}} \leq Y_n \right)$$

$$= E^r_\mu \left( \frac{\tilde{S}_{n-1} - (n-1)\mu}{\sigma \sqrt{n-1}} \leq Y_n \right).$$

Consequently,

$$\left| P^r \left( \frac{S_n - (n-1)\mu - \nu}{\sigma \sqrt{n}} \leq x \right) - \Phi(x) - \frac{\varphi(x)\mu_3}{\sqrt{n} \sigma^3} (1-x^2) \right|$$

$$\leq E^r \left| P^r_\mu \left( \frac{\tilde{S}_{n-1} - (n-1)\mu}{\sigma \sqrt{n-1}} \leq Y_n \right) - \Phi(Y_n) - \frac{\varphi(Y_n)\mu_3}{\sqrt{n-1} \sigma^3} (1-Y_n^2) \right|$$

(4.1)

$$+ \left| E^r_\mu \varphi(Y_n) - \varphi(x) \right|$$

(4.2)

$$+ E^r \left| \frac{\varphi(Y_n)\mu_3}{\sqrt{n-1} \sigma^3} (1-Y_n^2) - \frac{\varphi(x)\mu_3}{\sqrt{n} \sigma^3} (1-x^2) \right|. \quad (4.3)$$

It is clear that (4.1) is of the order $o\left( n^{-1/2} \right)$ uniformly in $x$ and $r$ (cf. Feller, 1971, Theorem XVI 4.1, p. 539). Moreover, for $|x| = |x_n| \leq \log n$,

$$Y_n - x = \frac{\nu - X_1}{\sigma \sqrt{n-1}} + O((n - 1)^{-1} \log n).$$

It follows from the mean value theorem that both (4.2) and (4.3) are $o\left( n^{-1/2} \right)$ uniformly for $0 \leq r \leq y$ and $|x| \leq \log n$. In particular,

$$\sup_{0 \leq r \leq y} P^r \left( \frac{S_n - (n-1)\mu - \nu}{\sigma \sqrt{n}} \leq -\log n \right) = o\left( n^{-1/2} \right)$$

and

$$\sup_{0 \leq r \leq y} P^r \left( \frac{S_n - (n-1)\mu - \nu}{\sigma \sqrt{n}} \leq \log n \right) = 1 - o\left( n^{-1/2} \right),$$

so that Lemma 4.1 follows by the monotonicity of d.f.'s. \qed
We now have the means to derive an expansion for the d.f. of the delayed renewal process \( N(t) \) starting with 'age' \( r \).

**Theorem 4.1.** For \( 0 \leq y < \zeta \) fixed,

\[
\sup_{0 \leq r \leq y} \sup_{x \in \mathbb{R}} \left| P' \left( \frac{N(t) - t/\mu}{\sqrt{t}/a} \leq x \right) - \Phi(x) \right|
\]

\[
- \frac{\varphi(x)}{\sqrt{t}} \left\{ (c - d)(x^2 - 1) + e + a \left\{ \theta \left( \frac{t}{\mu} + xa^{-1}\sqrt{t} \right) - \frac{1}{2} \right\} \right\}
\]

\[
= o(t^{-1/2}) \quad \text{as } t \to \infty,
\]

where \( e = -d - \frac{1}{2} \alpha + \mu^{1/2} \sigma^{-1} \nu \) and \( \theta(y) = \lfloor y \rfloor + 1 - y \) with \( \lfloor y \rfloor \) the integer part of \( y \).

**Proof.** Let \( |x| = |x| \leq \log t \).

\[
P' \left( \frac{N(t) - t/\mu}{\sqrt{t}/a} \leq x \right) = P' \left( N(t) \leq t/\mu + xa^{-1}\sqrt{t} \right)
\]

\[
= P' \left( N(t) \leq m_i - \theta(t/\mu + xa^{-1}\sqrt{t}) \right) = P'(S_{m_i} > t),
\]

where

\[
m_i = \left\lfloor \frac{t}{\mu} + xa^{-1}\sqrt{t} \right\rfloor + 1.
\]

Let \( y_i = (t - (m_i - 1)\mu - \nu)\sigma^{-1}m_i^{-1/2} \), then by Lemma 4.1,

\[
P'(S_{m_i} > t) = 1 - \Phi(y_i) - \frac{\varphi(y_i)\mu_s}{\sqrt{m_i}6\sigma} (1 - y_i^2) + o(m_i^{-1/2}),
\]

uniformly for \( 0 \leq r \leq y \). Moreover,

\[
y_i = -x + \frac{x^2\sigma\mu^{-1/2}}{2\sqrt{t}} + \frac{\mu^{1/2}(\mu - \nu)}{\sigma\sqrt{t}} - \frac{\mu^{3/2}}{\sigma\sqrt{t}} \theta(t/\mu + xa^{-1}\sqrt{t})
\]

\[
+ O(t^{-1}\log^3 t),
\]

uniformly for \( 0 \leq r \leq y, \quad |x| \leq \log t \). Therefore, by the mean value theorem and the finiteness of \( \nu \), we have,

\[
P' \left( \frac{N(t) - t/\mu}{\sqrt{t}/a} \leq x \right) = \Phi(x) - \frac{\varphi(x)}{\sqrt{t}} c(1 - x^2)
\]

\[
- \frac{\varphi(x)}{\sqrt{t}} \left\{ x^2d + a - \frac{\mu^{1/2}\nu}{\sigma} - a\theta(t/\mu + xa^{-1}\sqrt{t}) \right\}
\]

\[
+ o(t^{-1/2}),
\]

uniformly for \( 0 \leq r \leq y, \quad |x| \leq \log t \). The uniformity for \( x \in \mathbb{R} \) now follows in the usual way from the monotonicity of d.f.'s. \( \square \)
Corollary 4.1. Let \( N_1 \) and \( N_2 \) be delayed renewal processes and let \( e_1 \) and \( e_2 \) be defined similarly as \( e \) in Theorem 4.1. For all \( 0 \leq y < \xi_1 \) and \( 0 \leq z < \xi_2 \) we have,

\[
\sup_{0 \leq r < y, 0 \leq s < z} \sup_{x \in \mathbb{R}} \left| P_{r,s} \left( \frac{N_1(t) - t/\mu_1}{\gamma \sqrt{t}} \leq x \right) - \Phi(\gamma a_1 x) \right|
\]

\[
- \frac{\varphi(\gamma a_1 x)}{\sqrt{t}} \left\{ (c_1 - d_1)((\gamma a_1 x)^2 - 1) + e_1 \right. \\
\left. + a_1 \left\{ \theta \left( \frac{t}{\mu_1} + \gamma x \sqrt{t} \right) - \frac{1}{2} \right\} \right\} \\
= o(t^{-1/2})
\]

and

\[
\sup_{0 \leq r < y, 0 \leq s < z} \sup_{x \in \mathbb{R}} \left| P_{r,s} \left( \frac{-N_2(t) + t/\mu_2}{\gamma \sqrt{t}} \leq x \right) - \Phi(\gamma a_2 x) \right|
\]

\[
+ \frac{\varphi(\gamma a_2 x)}{\sqrt{t}} \left\{ (c_2 - d_2)((\gamma a_2 x)^2 - 1) + e_2 \right. \\
\left. - a_2 \left\{ \theta \left( \frac{-t}{\mu_2} + \gamma x \sqrt{t} \right) - \frac{1}{2} \right\} \right\} \\
= o(t^{-1/2}).
\]

Proof. (4.4) is a direct consequence of Theorem 4.1 by substituting \( \gamma a_1 x \) for \( x \). (4.5) is obtained in the same way after observing that \( \lim_{h \downarrow 0} \theta(z - h) = 1 - \theta(-z) \), for all \( z \in \mathbb{R} \).

The following theorem is proved in exactly the same way as Theorem 4.1 of [KK], using Corollary 4.1. Since in Theorem 4.2 \( \sup_{0 \leq r < y, 0 \leq s < z} |e_1| < \infty \) and \( \sup_{0 \leq s < z} |e_2| < \infty \), the uniformity in \( r \) and \( s \) follows easily.

Theorem 4.2. Let \( M(t) = N_1(t) - N_2(t) \), where \( N_1 \) and \( N_2 \) are independent delayed renewal processes. Let \( e_1 \) and \( e_2 \) be defined as in Corollary 4.1, then for fixed \( 0 \leq y < \xi_1 \) and \( 0 \leq z < \xi_2 \),

\[
\sup_{0 \leq r < y, 0 \leq s < z} \sup_{x \in \mathbb{R}} \left| P_{r,s} \left( \frac{M(t) - t/\alpha}{\gamma \sqrt{t}} \leq x \right) - \Phi(x) \right|
\]

\[
- \frac{\varphi(x)}{\sqrt{t}} \left\{ q(x^2 - 1) + \tilde{\varepsilon} + \frac{1}{\gamma} \left\{ \theta \left( \frac{t}{\alpha} + \gamma x \sqrt{t} \right) - \frac{1}{2} \right\} \right\} \\
= o(t^{-1/2}),
\]

as \( t \to \infty \), where

\[
q = (c_1 - d_1)(\gamma a_1)^{-3} - (c_2 - d_2)(\gamma a_2)^{-3}
\]

and

\[
\tilde{\varepsilon} = e_1(\gamma a_1)^{-1} - e_2(\gamma a_2)^{-1}.
\]
Corollary 4.2. If \( M \) is the difference of two independent stationary renewal processes, then

\[
\sup_{x \in \mathbb{R}} \left| P(M(t_n) \geq n) - \Phi(x) \right|
\]

\[
- \frac{\varphi(x)}{\sqrt{n}} \left( \left( \frac{q}{\sqrt{\alpha}} + \frac{1}{2} \alpha^{1/2} \gamma \right) \left( 1 - x^2 \right) + \frac{1}{2 \gamma \sqrt{\alpha}} - \frac{1}{2} \alpha^{1/2} \gamma \right) = o(n^{-1/2})
\]

as \( n \to \infty \), where \( q \) is defined in Theorem 4.2 and the other parameters in Section 2.

Proof. First let \( |x| \leq \log n \). We have

\[
P(M(t_n) \geq n) = 1 - P \left( \frac{M(t_n) - t_n / \alpha}{\gamma \sqrt{t_n}} \leq y_n \right),
\]

where, by the choice of \( x \),

\[
y_n = \frac{n - 1 - t_n / \alpha}{\gamma \sqrt{t_n}} = -x + \frac{x^2 \alpha^{1/2} \gamma}{2 \sqrt{n}} - \frac{1}{\gamma \sqrt{\alpha}} + O(n^{-1} \log^2 n).
\] (4.6)

Observe that in this stationary case, \( \nu = \frac{1}{2} (\mu + \sigma^2 / \mu) \), so that \( c_1 = c_2 = \delta = 0 \) in Theorem 4.2. Moreover, the argument of \( \theta \) in Theorem 4.2 now equals \( n - 1 \), and thus the \( \theta \)-part simply becomes 1. Consequently,

\[
P(M(t_n) \geq n) = 1 - \Phi(y_n) - \frac{\varphi(y_n)}{\sqrt{t_n}} \left( q(y_n^2 - 1) + \frac{1}{2 \gamma} \right) + o(t_n^{-1/2}),
\]

so that by (4.6),

\[
P(M(t_n) \geq n) = \Phi(x) - \frac{\varphi(x)}{\sqrt{n \alpha}} \left( q(x^2 - 1) + \frac{1}{2 \gamma} \right)
\]

\[- \frac{\varphi(x)}{\sqrt{n}} \left( \frac{1}{2} x^2 \gamma \sqrt{\alpha} - \frac{1}{\gamma \sqrt{\alpha}} \right) + o(n^{-1/2}),
\]

as \( n \to \infty \), uniformly for \( |x| \leq \log n \). The corollary follows by the monotonicity of d.f.'s. \( \square \)

A local expansion for the distribution of \( M \) is given in the next lemma. Although the result below can be extended in several ways, we only list it in the form that is sufficient for our purposes. Note that we again do not specify the distribution of the first component of \( N_1 \) and \( N_2 \).

Lemma 4.2. Let \( y \geq 0 \) be fixed and let \( r_0 < \min \{ \gamma, \zeta \} \), then

\[
\sup_{0 \leq r \leq r_0} \sup_{0 \leq u \leq y} \sup_{|x| \leq \log n} \left| P^{\leq 0}(M(t_n - u) = n) - \frac{\varphi(x)}{\gamma \sqrt{n \alpha}} \right| = o(n^{-1/2}).
\]
Proof. Let \( y_n = (n - (t_n - u)/\alpha) \gamma^{-1}(t_n - u)^{-1/2} \) and \( z_n = y_n - \gamma^{-1}(t_n - u)^{-1/2} \), then

\[
P^{r,0}(M(t_n - u) = n)
\]

\[
= P^{r,0}(M(t_n - u) \leq n) - P^{r,0}(M(t_n - u) \leq n - 1)
\]

\[
= P^{r,0}\left( \frac{M(t_n - u) - (t_n - u)/\alpha}{\gamma \sqrt{t_n - u}} \leq y_n \right)
\]

\[
- P^{r,0}\left( \frac{M(t_n - u) - (t_n - u)/\alpha}{\gamma \sqrt{t_n - u}} \leq z_n \right)
\]

Since \( t_n = n\alpha + xy\sqrt{n} \alpha^{3/2} \),

\[
|y_n - z_n| = \gamma^{-1}(t_n - u)^{-1/2} = \gamma^{-1}(n\alpha)^{-1/2} + O(n^{-1} \log n), \quad (4.7)
\]

uniformly for \( 0 \leq r \leq r_0, 0 \leq u \leq y \) and \( |x| \leq \log n \). Also,

\[
y_n = -x + O(n^{-1/2} \log^2 n), \quad (4.8)
\]

uniformly for \( 0 \leq r \leq r_0, 0 \leq u \leq y \) and \( |x| \leq \log n \). Now

\[
|P^{r,0}(M(t_n - u) = n) - \varphi(x) \gamma^{-1}(n\alpha)^{-1/2}| \leq |y_n - z_n| \varphi(y_n) \gamma^{-1}(n\alpha)^{-1/2} \leq (4.9)
\]

\[
+ \Phi(y_n) - \Phi(z_n) - \varphi(x) \gamma^{-1}(n\alpha)^{-1/2} \leq (4.10)
\]

\[
+ \frac{\varphi(y_n)}{\sqrt{t_n - u}} p(y_n) - \frac{\varphi(z_n)}{\sqrt{t_n - u}} p(z_n) \leq (4.11)
\]

\[
+ \frac{\varphi(y_n)}{\sqrt{t_n - u}} p(y_n) - \frac{\varphi(z_n)}{\sqrt{t_n - u}} p(z_n) \leq (4.12)
\]

where \( p(x) = q(x^2 - 1) + \tilde{c} + \frac{1}{2} \gamma^{-1} \) (\( q \) and \( \tilde{c} \) are defined in Theorem 4.2). By Theorem 4.2 it follows that (4.9) and (4.10) are of order \( O((t_n - u)^{-1/2}) = O(n^{-1/2}) \) uniformly for \( 0 \leq r \leq r_0, 0 \leq u \leq y \) and \( |x| \leq \log n \). The same result holds for (4.11) and (4.12), by the mean value theorem and (4.7) and (4.8), which completes the proof. \( \square \)

The following lemma provides additional information about the local behavior of a delayed renewal process.

Lemma 4.3. Let \((N(t))\) be a delayed renewal process as defined at the beginning of this section, then

\[
\sup_{t \geq 0} \sup_{0 \leq r \leq \zeta} P^r(N(t) = n) = O(n^{-1/2}).
\]
Proof. Throughout the proof $C$ denotes a generic constant. Let $(X_n)$ be the renewal sequence of $N$, and define

$$\tilde{S}_{n-1} = (X_2 + \cdots + X_n - (n-1)\mu)(n-1)^{-1/2} \sigma^{-1}, \quad n \geq 2.$$ 

Then

$$P'(N(t) = n) = P'(\tilde{S}_{n-1} \leq Y) - P'(\tilde{S} \leq Z),$$

where $Y = (t - X_1 - (n-1)\mu)\sigma^{-1}(n-1)^{-1/2}$ and $Z = (t - X_1 - n\mu)\sigma^{-1}n^{-1/2}$. Therefore,

$$P'(N(t) = n) \leq E'[|\Phi(Y) - \Phi(Z)| + E'[P'_{X_1}(\tilde{S}_{n-1} \leq Y) - \Phi(Y)]$$

$$+ E'[P'_{X_1}(\tilde{S} \leq Z) - \Phi(Z)].$$

By the Berry–Esseen inequality and the independence of $X_1, X_2, \ldots$ we have

$$\sup_{y \in \mathbb{R}} |P'_{X_1}(\tilde{S}_{n-1} \leq y) - \Phi(y)| \leq \frac{C}{\sqrt{n}},$$

independently of $r$, so that the last two terms in (4.13) are $O(n^{-1/2})$ uniformly in $t$ and $0 \leq r < \zeta$. The same holds for the first term in the right-hand side of (4.13), since it can be established by

$$\sup_{x \in \mathbb{R}} |\Phi\left(\frac{x}{\sqrt{n-1}}\right) - \Phi\left(\frac{x}{\sqrt{n}} - \frac{\mu}{\sigma\sqrt{n}}\right)| = O(n^{-1/2}).$$

This completes the proof. \qed

5. Expansions, second part; proof of main result

Throughout this section, $M$ denotes the difference of two independent stationary renewal processes. In (3.3) we found

$$P(M(t_n) < n, \tau_n \leq t_n) = \sum_{k=-\infty}^{-1} P(M(t_n) = n + k, T_k \leq t_n).$$

We first show that it suffices to consider $P(M(t_n) = n + k, T_k \leq t_n)$ for fixed $k$.

Lemma 5.1. Let $M$ and $T_k$ be defined as in Section 2, then

$$\lim_{A \to \infty} \limsup_{n \to \infty} \sup_{|x| \leq \log n} \sqrt{n} \sum_{k=-\infty}^{-A} P(M(t_n) = n + k, T_k \leq t_n) = 0.$$ 

Proof. By using conditional expectation with respect to the $\sigma$-algebra $\mathcal{F}_{T_k}$ generated by the stopping time $T_k$ we obtain

$$P(M(t_n) = n + k, T_k \leq t_n) = E[I_{[T_k \leq t_n]} P_{\mathcal{F}_{T_k}}(M(t_n) = n + k).$$

(5.1)
Let $Q^{r_s}$ denote the probability measure $P^{r_s}$ under which both $N_1$ and $N_2$ are ordinary renewal processes and $X_2^{(i)}$ has d.f. $G_i$, $i = 1, 2$. Denote $\xi_1$ in this case by $\xi_1^*$. On the other hand let the probability measure $\tilde{Q}^{r_s}$ correspond to $P^{r_s}$ such that $N_1$ is a stationary and $N_2$ is an ordinary renewal process with again $X_2^{(i)}$ having d.f. $G_i$, $i = 1, 2$. Now $\xi_1$ is denoted by $\tilde{\xi}_1$. If we denote the 'lifetime' of $N_1$ (i.e. the time elapsed since the last renewal of $N_1$, or, if there was no renewal, the time elapsed since $t = 0$) at time $T_k$ by $R_k$ then we have

$$P_{\tilde{T}_k}(M(t_k) = n + k) = \mathbb{1}_{\{T_k = R_k\}} \tilde{Q}^{R_0}(M(t_n - T_k) = n) + \mathbb{1}_{\{T_k = R_k\}} Q^{R_0}(M(t_n - T_k) = n). \quad (5.2)$$

Let $(P^{r_0}, \xi)$ be $(\tilde{Q}^{r_0}, \tilde{\xi}_1^*)$ or $(Q^{r_0}, \xi_1^*)$. Then, applying Lemma 4.3, we have in either case

$$P^{r_0}(M(t) = n + k) = \sum_{n_1 = n}^{\infty} P^{r_0}(N_1(t_n) = n_1) P^{r_0}(N_2(t) = n_1 - n)$$

$$= O(n^{-1/2}) \sum_{n_1 = n}^{\infty} P^{r_0}(N_2(t) = n_1 - n) = O(n^{-1/2}), \quad (5.3)$$

uniformly in $0 < r < \xi_1^*$ or $0 < r < \tilde{\xi}_1$, respectively and $t$. Hence, combining (5.1) (5.3),

$$P(M(t_n) = n + k, T_k \leq t_n) = P(T_k \leq t_n) O(n^{-1/2}), \quad (5.4)$$

where $O(n^{-1/2})$ is independent of $x$ and $k$. In particular,

$$\lim_{n \to \infty} \sqrt{n} \sum_{k = -\infty}^{t_n - 1} P(M(t_n) = n + k, T_k \leq t_n) \leq \lim_{n \to \infty} \sum_{k = -\infty}^{t_n - 1} P(T_k \leq t_n) \leq -CE \inf_{t > 0} M(t) \quad (5.5)$$

by the monotone convergence theorem and the definition of $T_k$. Since $E \inf_{t > 0} M(t)$ is finite by Remark 3.1, Lemma 5.1 follows from (5.5).

Lemma 5.2. For fixed $k \in \mathbb{Z}_-$,

$$\sup_{|x| \leq \log n} \left| P(M(t_n) = n + k, T_k \leq t_n) - \frac{\varphi(x)}{\gamma \sqrt{n \alpha}} P(T_k < \infty) \right| = o(n^{-1/2}).$$

Proof. Let $\mathcal{T}_T$, $\tilde{Q}^{r_0}, Q^{r_0}, \tilde{\xi}_1^*$ and $\xi_1^*$ be defined as in the proof of Lemma 5.1. Fix $y > 0$ and $\delta > 0$. For $n$ large enough

$$P(M(t_n) = n + k, T_k \leq y)$$

$$\leq P(M(t_n) = n + k, T_k \leq t_n)$$

$$= P(M(t_n) = n + k, T_k \leq y) + P(M(t_n) = n + k, y < T_k \leq t_n). \quad (5.6)$$
Using Lemmas 4.3 and 4.2 we have for \( T_k \leq y \), similarly to (5.2) and (5.3),

\[
P_{X_k}(M(t_n) = n + k) = (I_{(T_k = R, R \leq R^* - \delta)} + I_{(T_k = R, R \leq R^* + \delta)}) \frac{\varphi(x)}{\gamma \sqrt{na}} + o_\delta(n^{-1/2})
\]

\[+ (I_{(T_k = R, R^* - \delta < R < R^* + \delta)} + I_{(T_k = R, R^* + \delta < R < R^* + 1)}) \alpha(n^{-1/2}) = f_R(\delta) \frac{\varphi(x)}{\gamma \sqrt{na}} + o_\delta(n^{-1/2}) + g_R(\delta) O(n^{-1/2}),
\]

where \( f_R(\delta) = 1 \) and \( g_R(\delta) = 0 \) if \( \delta \) is small enough (depending on \( R \)) and \( f_R(\delta), g_R(\delta) \in \{0, 1\} \). Hence,

\[
P(M(t_n) = n + k, T_k \leq y) = E I_{(T_k < y)} P_{X_k}(M(t_n) = n + k)
\]

\[
= E I_{(T_k < y)} \left\{ \frac{\varphi(x)}{\gamma \sqrt{na}} f_R(\delta) + o_\delta(n^{-1/2}) + g_R(\delta) O(n^{-1/2}) \right\}
\]

uniformly for \(|x| \leq \log n\). Therefore

\[
\lim_{y \to \infty} \lim_{\delta \downarrow 0} \lim_{n \to \infty} \left[ \sqrt{n} P(M(t_n) = n + k, T_k \leq y) \right] = 0.
\]

(Note that in taking the limit w.r.t. \( \delta \) we apply the dominated convergence theorem.) Moreover, since, similarly to (5.4),

\[
P(M(t_n) = n + k, y < T_k \leq t_n) \leq P(y < T_k \leq t_n) O(n^{-1/2}),
\]

where \( O(n^{-1/2}) \) does not depend on \( y \), we have

\[
\lim_{y \to \infty} \sqrt{n} P(M(t_n) = n + k, y < T_k \leq t_n) = 0.
\]

Combination of (5.6)-(5.8) and \( \lim_{y \to \infty} P(T_k \leq y) = P(T_k < \infty) \) now yields the result. \( \square \)

**Corollary 5.1.** Let \( M \) be the difference process of two independent stationary renewal processes and let \( T_k \) be defined as in Section 2, then

\[
\sup_{|x| = \log n} \left| \sum_{k = -\infty}^{t_n} P(M(t_n) = n + k, T_k \leq t_n) + \frac{\varphi(x)}{\gamma \sqrt{na}} E \inf_{t \geq 0} M(t) \right| = o(n^{-1/2}) \quad \text{as} \quad n \to \infty.
\]
Proof. For every fixed $A \in \mathbb{N}$,

$$
\sum_{k=-\infty}^{-1} P(M(t_n) = n+k, T_k \leq t_n) = \sum_{k=-\infty}^{-A-1} P(M(t_n) = n+k, T_k \leq t_n) + \sum_{k=-A}^{-1} P(M(t_n) = n+k, T_k \leq t_n).
$$

The result now follows from Lemmas 5.1 and 5.2. Note also that

$$
\sum_{k=-\infty}^{-1} P(T_k < \infty) = -E \inf_{t>0} M(t). \quad \square
$$

We now prove the main result, Theorem 3.1.

Proof of Theorem 3.1. Since $E \inf_{t>0} M(t)$ is well defined by Remark 3.1, we have

$$
sup_{|x| = \log n} \left| P\left(\frac{\tau_n - na}{\gamma \sqrt{n}} \leq x\right) - \Phi(x)\right| = o(n^{-1/2}) \quad \text{as } n \to \infty,
$$

by (3.2), Corollary 4.2, (3.3) and Corollary 5.1. Uniformity for all $x \in \mathbb{R}$ follows by the monotonicity of d.f.'s. $\square$

References