A SIMPLE MORPHODYNAMIC MODEL FOR SAND BANKS AND LARGE-SCALE SAND PITS SUBJECT TO ASYMMETRICAL TIDES

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ABSTRACT: We extend existing knowledge on theoretical growth characteristics of tidal sand banks by including asymmetrical tides with an $M_0$, $M_2$ and $M_4$-constituent, thus allowing for migration. Furthermore, in the context of the continuously increasing demand on the Dutch sand market, we show that creating a large-scale offshore sand pit has long-term morphological implications, both for the pit itself and the surrounding area. The pit deepens, while around it a sand bank pattern emerges, spreading at a constant rate of the order of tens to hundred metres per year.

1 INTRODUCTION

Nowadays offshore sand extraction is a common activity in the North Sea, be it at a relatively small scale. However, in order to meet the continuously increasing demand for purposes like beach nourishments and (large) infrastructural projects, future offshore sand extraction is likely to occur at a much larger scale. More details on the Dutch sand market can be found in Section 2. As a result, typical sand pit dimensions will rise to the order of $10^8$ m³. Hence, as the current legislation limits pit depth to about two meters, horizontal pit dimensions must be of the order of (tens of) kilometres. Moreover, in view of coastal stability, mining within the 20 m water depth contour is prohibited. These large-scale sand pits will constitute a significant disturbance of the offshore sea bed, which may have morphodynamic implications.

The offshore sea bed exhibits a wide variety of rhythmic bottom features of different length scales, such as tidal sand banks, sandwaves, megaripples and sand ripples. In the present paper we restrict our attention to tidal sand banks, which have a typical wavelength of some kilometres, i.e. of the order of the large-scale sand pits introduced above (see Fig. 1). The crest orientation of sand banks is slightly counterclockwise with respect to the main tidal motion. Their formation can be explained as a morphodynamic instability of a sandy bed subject to tidal motion, see Huthnance [7][8], Hulscher et al. [5] and Hulscher [6]. The linear stability analysis by Hulscher et al. [5] serves as the starting point for this study. It considers a sandy sea bed subject to an $M_2$-tide and studies its response to wavy bed perturbations. The thus obtained fastest growing mode (FGM) was shown to predict wavelength and orientation of the sand banks in a satisfactory way. However, the model is unable to predict the (slow) bank migration observed in nature (see e.g. Dyer & Huntley [3]), as the model does not allow for tidal asymmetry.

The research question addressed in this study is twofold: (1) What new information on tidal sand banks is obtained by extending the model of Hulscher et al. [5] to allow for tidal asymmetry? And (2) what does this model tell us about the evolution of a large-scale sand pit?

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Therefore, in the present paper, tidal conditions are incorporated in a more realistic manner by allowing for asymmetry; besides the $M_2$, here $M_0$ and $M_4$-components are included. See Sections 3 and 4, where the model and linear analysis are presented, respectively. The dependence of the FGM on tidal asymmetry is described in Section 5. In Section 6, the model is applied to the case of an offshore sand pit, which can be seen as a local disturbance of the sea bed, small with respect to the water depth. A rather novel feature is that the results of this linear analysis, obtained in Fourier space, are numerically transformed back into the physical space. This enables us to study both the hydrodynamical and morphodynamical aspects of the pit evolution in a direct way. Moreover, the dependence on pit characteristics and tidal conditions is studied. The results of the linear approach are presented in Section 7, along with a physical interpretation of the hydrodynamic behaviour. Section 8 contains the discussion and conclusions.

2 THE DUTCH SAND MARKET

In the Netherlands a future deficit of sand is expected for (i) filling sand and (ii) concrete and construction industry sand (Peters [10]). This deficit is caused by the increasing need for sand, as well as the depletion of inland resources. The pressure on the Dutch sand market originates from a growing economy and welfare, combined with the scarcity of space.

To answer the needs of society and to stimulate a sustainable development of the economy, more and more sand will be needed to build roads, railways, industries, houses, etcetera (Rijkswater-
staat [13]). Therefore, the regular need for sand increases \((125 \cdot 10^6 \text{ m}^3\) for activities on land and \(10 - 15 \cdot 10^6 \text{ m}^3\) for beach nourishment). An expanding economy requires space, which can be created by the replanning of existing land or by land reclamation (e.g. creating an island for a future airport). These large infrastructural projects result in an annual extra demand of filling sand of \(100 - 300 \cdot 10^6 \text{ m}^3\) (Hoogewoning [4]).

Due to the scarcity of land, sand extraction has to compete with other types of land use, such as housing, industry, nature and recreation. Sand extraction usually does not get the highest priority and as a consequence the supply decreases. Moreover, during the last decade the south-east regions of the Netherlands, which are the main suppliers of sand, have suffered from river floods. Consequently, they are unwilling to point out new sand sources just outside the river (main channel and flood plain) as it might increase the risks of future flooding. This public awareness especially harms the concrete and construction industry, forcing it to search for other sources.

Because it is hard to decrease the demand for sand without harming the economy, solutions to establish equilibrium between supply and demand should be sought on the supply side of the sand market. Importing sand could be a solution, but one prefers to solve the problems within the Netherlands. As the sources on the Dutch land decrease, one is forced to explore the possibilities of extracting large amounts of sand offshore. This requires an investigation of its effects, such as possible damage to objects in the coastal zone, ecological damage and threats to coastal protection.

The two types of sand introduced above require different strategies of extraction. The quality demands for concrete and construction industry sand are rather high: it should consist of different grain sizes, with a sufficient amount of coarse sand (Rijkswaterstaat [14]). As a result, this type of sand can only be found at a limited number of locations in the North Sea (Rijkswaterstaat [12]). Moreover, it should be extracted from rather deep sand layers, which clearly complicates the extraction procedure. Filling sand has less strict quality demands and can therefore be found more easily throughout the North Sea, furthermore without resorting to deep sand layers. In the present paper, we focus on modelling the extraction of filling sand.

3 THE MODEL

In this section, we present the morphodynamic model which will be used to study both tidal sand banks and large-scale sand pits. We consider a shallow sea of depth \(H^*\), in which a tidal wave of frequency \(\sigma^*\) and maximum velocity \(U^*_0\) is active. We define an orthogonal coordinate system with horizontal coordinates \(x^* = (x^*, y^*)\), and the \(z^*-\)axis pointing upward. The free surface is denoted by \(z^* = \zeta^*\) and the bed level by \(z^* = -H^* + h^*\). We take a depth-averaged approach, introducing horizontal flow velocities \(u^* = (u^*, v^*)\). The bed is assumed to consist of noncohesive sediment of uniform grain size, representative for filling sand, which is mainly transported as bed load. Upon defining the nondimensional variables

\[
\begin{align*}
  u &= \frac{u^*}{U^*}, & t &= \sigma^*t^*, & x &= \frac{\sigma^*x^*}{U^*}, & h &= \frac{h^*}{H^*}, & \zeta &= \frac{g^*}{U^*2} \zeta^*,
\end{align*}
\]
we find that the scaled morphodynamic model can be written as

\[
\frac{\partial \zeta}{\partial x} + \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - f v + \frac{r u}{1 + \delta^2 \zeta - h} = 0, \tag{2}
\]

\[
\frac{\partial \zeta}{\partial y} + \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + f u + \frac{r v}{1 + \delta^2 \zeta - h} = 0, \tag{3}
\]

\[
\frac{\partial^2 \zeta}{\partial t^2} - \frac{\partial h}{\partial t} + \frac{\partial}{\partial x} \left[ (1 + \delta^2 \zeta - h) u \right] + \frac{\partial}{\partial y} \left[ (1 + \delta^2 \zeta - h) v \right] = 0, \tag{4}
\]

\[
\frac{\partial h}{\partial t} + \nabla \cdot \left[ \alpha |u|^b \left( \frac{u}{|u|} - \lambda \nabla h \right) \right] = 0. \tag{5}
\]

Eqs. (2)-(5) show a number of nondimensional parameters: a Froude number \( \delta = U^* / (g^* H^*)^{1/2} \), a Coriolis parameter \( f \), a friction parameter \( r \) and sediment transport parameters \( \alpha, b \) and \( \lambda \). Values of these nondimensional parameters, typical for the North Sea, are given by

\[
\delta = 0.01, \quad f = 0.8, \quad r = 0.6, \quad \alpha = 0.5 \times 10^{-6} - 0.5 \times 10^{-5}, \quad b = 3, \quad \lambda = 0.01, \tag{6}
\]

and further discussed in Hulscher et al. [5]. The uncertainty in predicting the magnitude of the sediment transport rate has forced us to consider a range of \( \alpha \)-values. The fact that \( \alpha \) is small shows us that bed evolution is slow, occurring on a time scale much longer than that of the hydrodynamics. Hence the term \( \frac{\partial h}{\partial t} \) can be omitted from (4), so we effectively decouple the hydrodynamic part from the morphodynamic part of the model. Since we are interested in the long-term bed development, we introduce a slow time \( \tau = \alpha t \) for the bed evolution equation (5).

To get an idea of the morphological time scale, within our range of \( \alpha \), \( \tau = 1 \) corresponds to a time of 45-450 years. In terms of \( \tau \), and after averaging over a tidal cycle to eliminate intratidal fluctuations, we obtain

\[
\frac{\partial h}{\partial \tau} + \left\langle |u|^b \left( \frac{u}{|u|} - \lambda \nabla h \right) \right\rangle = 0. \tag{7}
\]

where \( \langle \cdot \rangle \) represents averaging over a tidal cycle. Boundary conditions are not imposed, since the boundaries of the offshore system are assumed to be infinitely far away.

The solution to the problem is symbolically written as \( \phi = (u, v, \zeta, h) \). It can be readily shown that

\[
\phi_0 = (u_0(t), v_0(t), \zeta_0(x, y, t), 0) \tag{8}
\]

is a solution of the system (2)-(5) if terms of order \( \mathcal{O}(\delta) \) are dropped. We consider a tidal flow that is a generalization of the \( M_2 \)-tide considered by Hulscher et al. (1993). Adding an \( M_0 \)-component (a residual current \( u_r = (u_r, v_r) \)) along with an \( M_4 \)-component (periodic with double frequency \( 2 \sigma^* \)), thus allowing for tidal asymmetry, we obtain

\[
\begin{align*}
u_0(t) &= u_r + \epsilon_u \cos t + \mu_u \cos(2t - \varphi_u), \\
v_0(t) &= v_r + \epsilon_v \sin t + \mu_v \sin(2t - \varphi_v).
\end{align*} \tag{9}
\]

Here, the velocity amplitudes of the \( M_2 \)-ellipse are given by \( \epsilon_u \) and \( \epsilon_v \), with \( \epsilon_v / \epsilon_u \) denoting its tidal eccentricity. Finally \( \mu_u, \mu_v, \varphi_u \) and \( \varphi_v \) describe an \( M_4 \)-ellips with arbitrary amplitude, ellipticity, orientation and phase relative to the other constituents. Note that the parameters can be varied freely, provided that the maximum value of \( |u_0| \) during the tidal cycle equals unity. The state \( \phi_0 \) describes a tidal flow over a flat bed and is called the basic state, the stability of which will be investigated in the next section.
4 THE PERTURBATIONS

We consider a small wavy perturbation of the flat sea bed of the form

$$h(x, y, \tau) = \gamma A(\tau)e^{-i k \cdot x} + c.c..$$

(10)

Here $\gamma$ is a small parameter (strictly speaking, infinitesimally small), $A(\tau)$ a function of order one, $k = (k, \ell)$ the wave vector with wave numbers $k$ and $\ell$ and $c.c.$ means the complex conjugate of a complex number. The bottom profile (10) can be seen as a perturbation of the basic state (8). Since $\gamma$ is small and $A$ of order one, we may expand the solution around the basic state $\phi_0$ according to

$$\phi = \phi_0 + \gamma \phi_1 + O(\gamma^2).$$

(11)

At order $\gamma$ a set of equations, linear in $\phi_1$, is found (see Appendix). This system can easily be formulated in terms of the vorticity $\eta_1 \equiv \frac{\partial v_1}{\partial x} - \frac{\partial u_1}{\partial y}$ (see Appendix). We can expect that the hydrodynamic response has a spatial structure similar to that of the bed perturbation (10) itself:

$$\eta_1 = \tilde{\eta}_1 A(\tau)e^{-i k \cdot x} + c.c..$$

Next, we use the technique of harmonic truncation to find the vorticity

$$\hat{\eta}_{1, \text{trunc}} = \tilde{h}_1 \left[ d_0 + \sum_{p=1}^{N} (d_{sp} \sin pt + d_{cp} \cos pt) \right].$$

(12)

De Swart & Zimmerman [2] have shown that the cut-off must take place after an odd mode; Hulscher et al. [5] take $N = 3$. We take $N = 5$ as we include an $M_4$-component in the basic state. The constants $d = (d_0, d_{s1}, d_{c1}, \ldots)^T$ follow from the linear system that appears after inserting (12) into (25). More information on the solution procedure can be found in Roos & Hulscher [11]. The bed evolution equation has the structure

$$\frac{dA(\tau)}{d\tau} = \omega A(\tau),$$

(13)

in which the growth rate $\omega = \omega_r + i \omega_i$ is a complex number depending on the perturbation wavenumbers $k = (k, \ell)$ as well as on the other problem parameters. Solving (13) with initial condition $A(0) = 1$ and inserting the result into (10) provides insight in the role of both the real and imaginary part of the growth rate:

$$h_1 = e^{\omega_r \tau} e^{-i |k| (e_k \cdot x - c \tau)} + c.c., \quad e_k = \frac{k}{|k|}, \quad c = \frac{\omega_i}{|k|}.$$

(14)

Hence, the real part $\omega_r$ is related to the amplitude growth of the perturbation, while a nonzero imaginary part $\omega_i$ causes a nonzero celerity $c$, thus migration of the wavy bed features. When the tide is fully symmetric, as is the case in Hulscher et al. (1993), symmetry arguments show that bed forms do not migrate: $\omega_i = 0$. However, as emphasized in Section 1, now also asymmetrical tides are considered, so migrating bed features can indeed be expected. An example of an asymmetric tide causing a nonzero migration celerity is given in Fig. 2. In a linear stability analysis, the mode with the largest growth rate $\omega_r$ is usually considered to be the most interesting one, see Dodd et al. [1]. This mode is called the fastest growing mode (FGM), which is also depicted in Fig. 2.

5 FASTEST GROWING MODE AND SAND BANK MIGRATION

As pointed out in the previous section, tidal asymmetry causes a nonzero celerity and hence migration of the sand bank features. Here we present two examples in which asymmetry is
caused by (i) an \( M_0 \) plus \( M_2 \) or (ii) an \( M_2 \) plus \( M_4 \):

\[
\begin{align*}
(i) \quad (u_0, v_0) &= \left( u_r + \sqrt{2(1 - u_r^2)} \cos t, 0 \right), \\
(ii) \quad (u_0, v_0) &= \left( \sqrt{2 - \mu_u^2} \cos t + \mu_u \cos 2t, 0 \right).
\end{align*}
\]

These special cases of (9) are such that, when \( u_r \) and \( \mu_u \) are varied, the average tidal kinetic energy \( \langle |u_0|^2 \rangle \) equals unity. Note that in this way the characteristic tidal velocity \( U^* \) is redefined: rather than the maximum value of the tidal velocity it is now the velocity corresponding to the average kinetic energy. We feel that this is a more appropriate scaling in view of comparing tides with different characteristics, like in (15) and (16).

We restrict our attention to the migration of the fastest growing mode, so let \( \hat{c} \) denote the celerity of the FGM. Fig. 3 shows how the migration celerity depends on the parameters \( u_r \) and \( \mu_u \) in (15) and (16), respectively. The two tides (15) and (16) seem to have similar behaviour, as far as the celerity of the FGM is concerned. In dimensional terms, a celerity \( \hat{c} = 0.1 \) (typical for a weak residual current of about 5% of the \( M_2 \)-amplitude) corresponds to a migration rate of about 1-7 m per year. Other properties of the FGM, such as its wavelength, orientation angle and growth rate are found to vary hardly with the parameters in (15) and (16).

Increasing the nonlinear character of the sediment transport enhances the effects of tidal asymmetry. See also Fig. 3, where for two special cases of the examples (i) and (ii), the celerity of the FGM is shown to increase weakly with the sediment transport power \( b \). For an \( M_2 \) plus an \( M_4 \) the tidally averaged water flux vanishes, so migration clearly originates from nonlinear transport only. Indeed, \( \hat{c} \) tends to vanish for \( b \downarrow 1 \).

6 APPLICATION TO SAND PITS: THEORY

In Section 4 we studied the evolution of wavy bed perturbations of small amplitude. A sand pit can be seen as a superposition of these wavy features, which enters the problem as an initial bed profile at order \( \gamma \):

\[
h |_{\tau=0} = \gamma h_{\text{pit}}.
\]

With a pit depth of approximately 2 m and a water depth of roughly 30 m we obtain \( \gamma = \frac{2}{30} \ll 1 \) (\( h_{\text{pit}} \) is of order one). Hence, even though the parameter \( \gamma \) now has a finite value, it can still be...
considered small. Obviously one can think of many details of the pit geometry, but we restrict our attention to the dimension of the pit. The depth is given by the expansion parameter $\gamma$, so varying the depth merely affects the validity of the theory and does not provide qualitative physical insight. We propose a circular Gaussian pit shape, given by

$$h_{\text{pit}}(x) = -\exp \left[ -\frac{\pi}{L^2} (x^2 + y^2) \right].$$

(18)

Here, the nondimensional diameter $L$ is defined such that the nondimensional volume of the pit is simply given by $V = \gamma L^2$, its dimensional counterpart being given by $V^* = \gamma L^2 H^* U^* \sigma^{*-2}$. The problem will be solved in Fourier space, so we transform the problem according to

$$(u, v, h, \zeta) = \int \int (\tilde{u}, \tilde{v}, \tilde{h}, \tilde{\zeta}) e^{-i k \cdot x} d k + c.c..$$

(19)

The solution procedure is exactly parallel to Section 4, keeping in mind that (13) now depends directly on the wave numbers $k$. Hence, each point in the Fourier spectrum grows or decays exponentially with an individual growth rate, i.e.

$$\tilde{h}_{\text{pit}}(k, \tau) = \tilde{h}_{\text{pit}}(k) e^{\omega(k) \tau},$$

(20)

with $\tilde{h}_{\text{pit}}$ denoting the Fourier transform of the initial pit shape (18). Note that we study the evolution of a local disturbance in an otherwise flat bed. Even though we know from Huthnance [7] and Hulscher et al. [5] that this flat bed is unstable, causing the growth of rhythmic patterns, we wish to isolate the pit effect here. This means that we may safely assume the globally present perturbations to be initially a higher order effect as compared to the pit of 2 m depth.

Equation (20) indicates that the whole spectrum comes into play and not only the fastest growing mode (FGM). A straightforward numerical procedure is applied to transform the results in Fourier space back into the physical space, i.e. to map a window in the $(k, \ell)$-plane onto an area of finite extent in the physical space.

7 APPLICATION TO SAND PITS: RESULTS

Here we present the evolution of a sand pit subject to an $M_2$ plus $M_4$-tide, both in the $x$-direction. See Fig. 4, where the bed evolution as well as the streamlines of the residual current ($u_1$) are shown. Clearly, the presence of a pit triggers the formation of three circulation cells. As a result, the bed develops gradually forming a pattern of sand banks. As time evolves, the

Figure 3: Celerity $\hat{c}$ of the FGM after introducing tidal asymmetry. Left two panels: (i) adding an $M_0$ to an $M_2$ according to (15), (ii) adding an $M_4$ to an $M_2$ according to (16). The two cases denoted with a cross and a circle are further investigated in the right two panels. They show the sensitivity of $\hat{c}$ to the sediment transport power $b$ around $b = 3$. Values of the physical parameters taken from (6).
patch of sand banks spreads and migrates, alternatingly adding troughs and crests, while the existing sand banks elongate further. In response, also the pattern of residual cells is modified. The pit itself deepens and the pattern spreads at a rate of about 12-120 m per year. (Here, the pattern spreading is defined as the elongation rate of the central trough.)

An explanation of the hydrodynamic response to the pit can be found in terms of vorticity dynamics. As shown by the source terms in (25), vorticity can be generated either by Coriolis and streamwise bed slopes or by bottom friction and transverse bed slopes. See also Zimmerman [16] and Pattiaratchi & Collins [9]. For the case of a sand pit, the generation of the residual cells due to bottom friction is outlined in Fig. 5.

8 DISCUSSION AND CONCLUSIONS

The continuously rising demand on the Dutch sand market increases the pressure on offshore sand-mining. The main conclusion of the present study is that creating a large-scale offshore sand pit has a significant morphodynamic impact, both on the pit itself and on the surrounding area. The pit itself deepens and around the pit a pattern of tidal sand bank appears, with a spreading rate up to 12-120 m per year. The underlying theoretical framework, i.e. the stability analysis by Hulscher et al. [5], has been extended allowing for tides with an $M_0$, an $M_2$ and an $M_4$-component. We showed that this tidal asymmetry causes bank migration, at a predicted rate of about 1-7 m per year. This agrees with observations presented in the review paper by Dyer & Huntley [3]. If present equally, $M_0$ and $M_4$ contribute also equally to this migration. The migration rate of the pit, i.e. the centre of the pattern, is of the same order of magnitude, showing that the horizontal impact of the pit is mainly due to the (not necessarily circular) pattern spreading. Unfortunately, no observations are available to test this interesting result.
Figure 5: Top view sketch of the sand pit to illustrate the physical mechanism causing the secondary residual cells. One circular depth contour of the pit is drawn as well as the pit axis, along which arrows denote the direction of the tide. First, we neglect the Coriolis force. (a) Consider the small fluid column on the slope of the pit, as depicted in the figure, where the flow is from left to right. Continuity forces the flow, when entering the pit, to slow down. This causes the bottom friction, which is proportional to some power of the flow velocity, to be smaller on the deeper side of the column. As a result, the fluid column experiences a torque, which tends to bend the flow to the right. Repeating this argument throughout the pit shows that below the pit axis the fluid experiences a clockwise rotation, while above the pit axis it experiences a counterclockwise rotation. (b) The thus generated vorticity induces residual cells, that are carried slightly downstream by advection. (c) Conversely, when the tidal flow is from right to left, the same mechanism generates two other residual cells on the other side of the pit. (d) In tidally averaged sense, a four-cell pattern emerges showing lateral inflow and longitudinal outflow of the pit, with respect to the direction of the tide. (e) Including the Coriolis force affects the pattern of the four residual cells. In particular, on the Northern Hemisphere the two clockwise cells are amplified, which leads to the merging of the slightly damped counterclockwise cells into one larger elongated cell.

The numbers presented above strongly depend on morphological time scales, which are related to the magnitude of the sediment transport. As a result, the rates of migration and pattern spreading are proportional to the parameter $\alpha$, for which we merely have a range of estimates. Therefore, further investigating the uncertainties in the parameterization of sediment transport is a suggestion for future research. Finally, we note that the present theory is linear, and another suggestion is to include the nonlinear effects of tidal sand bank formation and pit evolution, as well.

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APPENDIX: THE LINEAR PROBLEM

The linear problem, introduced in Section 4, is given by

\[
\begin{align*}
\frac{\partial \zeta_1}{\partial x} + \frac{\partial u_1}{\partial t} + u_0 \frac{\partial u_1}{\partial x} + v_0 \frac{\partial u_1}{\partial y} - f v_1 + r(h_1 u_0 + u_1) &= 0, \\
\frac{\partial \zeta_1}{\partial y} + \frac{\partial v_1}{\partial t} + u_0 \frac{\partial v_1}{\partial x} + v_0 \frac{\partial v_1}{\partial y} + f u_1 + r(h_1 v_0 + u_1) &= 0, \\
\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} - u_0 \frac{\partial h_1}{\partial x} - v_0 \frac{\partial h_1}{\partial y} &= 0, \\
\frac{\partial h_1}{\partial \tau} + \nabla \cdot \left( |u_0|^{b-1} u_1 + (b-1)|u_0|^{b-3}(u_0 \cdot u_1)u_0 - |u_0|^b \lambda \nabla h_1 \right) &= 0.
\end{align*}
\]
The vorticity equation, formulated in terms of $\eta_1$ and $h_1$ only, reads

$$\frac{\partial \eta_1}{\partial t} + u_0 \frac{\partial \eta_1}{\partial x} + v_0 \frac{\partial \eta_1}{\partial y} + r\eta_1 + f \left( u_0 \frac{\partial h_1}{\partial x} + v_0 \frac{\partial h_1}{\partial y} \right) + r \left( -u_0 \frac{\partial h_1}{\partial y} + v_0 \frac{\partial h_1}{\partial x} \right) = 0. \quad (25)$$

REFERENCES


