Dynamics of large-scale bed forms in coastal seas

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Abstract

The formation of offshore sand ridges and sand waves in coastal seas is considered. The growth of these bed forms can be described as an instability mechanism due to tide-topography interactions. First extensions of the depth-integrated model proposed by Huthnance (1982) are studied. The most preferred bed forms resemble sand ridges of which the crests are rotated counterclockwise (on the Northern Hemisphere) with respect to the principal current direction. This orientation can become clockwise, as is observed for shore-face connected ridges, by allowing for different directions of depth-averaged currents and sediment transport. Finally a model is analyzed which includes the effect of secondary circulations in the vertical plane on the sediment transport. As a result both sand ridges and sand waves are found. Conditions are discussed for which a weakly nonlinear analysis can be carried out.

1 Introduction

The bottom of many coastal seas and estuaries exhibits patches of rhythmic patterns with a broad spectrum of spatial scales. An example of small-scale undulations are the ripples which can often be observed at sandy beaches during low-tide conditions. They have a characteristic wave-length of 10 cm and a typical amplitude of 1 cm. Larger-scale phenomena close to the beach are discussed by e.g. Lippman and Holman (1990).

In this paper we focus on offshore sand waves and sand ridges which both have horizontal length scales much larger than the local water depth. The sand ridges have wave-lengths in the order of 5 km, amplitudes vary between 1 and 15m and their crests are rotated slightly cyclonically (counterclockwise on the Northern Hemisphere) with respect to the principal direction of the tidal currents. Furthermore, historical data indicate that the ridges hardly propagate (Van der Meene, 1994). In figure 1 the location of various sand ridge systems in the North Sea is shown. Sand waves have smaller wave-lengths (usually a few hundreds of meters), their crests are perpendicular to the tidal current direction and they propagate several meters per year. Figure 2 shows sand waves in the southern part of the North Sea detected with an echosounder.

Sand ridges and sand waves are often observed simultaneously, see Huntley et al. (1993). The profiles of these bed forms can be highly asymmetric, where the steepest slope of the sand waves indicates the direction of the mean sediment transport. Understanding the behaviour of these large-scale morphological patterns is relevant
for the management of coastal zones, the offshore industry (trenching of pipe-lines), navigation, etc..

Figure 2: Observation of sand waves in the North Sea (15 km west of The Hague) using an echosounder.

A simple model for the formation of sand ridges was developed by Huthnance (1982) and modifications were studied by De Vriend (1990) and Hulscher et al. (1993). Here the water motions are described by depth-integrated shallow water equations and forcing is due to tides. The model is supplemented with a parameterization for the sediment transport (assumed to be of bedload type) and an evolution equation for the erodible bottom. Huthnance studied the linear stability properties of a basic state, describing a tidal current over a flat bottom, with respect to arbitrary bed form perturbations. He found that the largest feedback between water motions and bed forms is obtained for perturbations having the characteristic wave-lengths and crest orientations corresponding to sand ridges. The physical mechanism of instability is that,
for bedforms with their crests turned cyclonically with respect to the principal tidal current direction, Coriolis and bottom frictional forces generate anticyclonic residual horizontal circulations around the bank, see Zimmerman (1981) and figure 3a. They cause the actual flow velocities on the upstream side of the bank to be larger than those on the down-stream side. Since larger flow velocities imply larger sediment transport rates there will be a convergence of sediment transport at the crest, both during the ebb and the flood period. In case the bank crests are turned anticyclonically with

Figure 3: a. Horizontal residual circulations (indicated by arrows) near sand ridges on the Northern Hemisphere (top view) b. Vertical residual circulations near sand waves (side view). For further explanation see text.

respect to the tidal current direction the residual circulations induced by Coriolis and bottom frictional forces are of opposite sign and the instability mechanism will be weaker. Thus earth rotation effects explain the preferred orientation of the banks, but they are not necessary to understand their formation. The reason that sand ridges have the largest growth rates is that for larger-scale bed forms residual circulations are weaker, whereas for smaller scale bed forms the instability mechanism is counteracted by a diffusion process related to the preferred down-slope direction of the sediment transport.

A model for sand waves is discussed in Fredsoe and Deigaard (1992) which is a synthesis of earlier work by these authors, see the references in this book. They consider fully developed sand waves, generated by steady currents, which are assumed to be form-preserving and which propagate in the downstream direction. Using a bed load transport parameterization and a given shear stress distribution in terms of the local water depth they are able to compute the shape, length and height of the sand wave. Alternative models are proposed by Hulscher et al. (1993) and Hulscher (1995). Here it is assumed that the sand waves are driven by oscillating (tidal) currents and the feedback between bed forms and water motions is explicitly computed. An important conclusion is that the presence of sand waves is related to secondary circulations in the vertical plane. As can be seen from figure 3b the latter cause an increase of the currents moving up-slope (and thus of sediment transport rates). Similarly, there is a corresponding decrease in the flow velocity and sediment transport on the downstream
side. Consequently, both during ebb and flood conditions there will be a convergence of the sediment flux at the crest of the bed form.

A limitation of these studies is that either only a linear stability analysis is carried out, whence no information on the amplitude behaviour is obtained, or a parameterized description of the water motion is used. The stability studies indicate that for realistic parameter values there is a broad spectrum of unstable modes. This implies that a strongly nonlinear analysis is required to describe the amplitude behaviour. On the other hand the observed sand ridges and sand waves all have a well defined length scale with only moderate spatial and temporal modulations which are characteristics of weakly nonlinear phenomena. Therefore the aim of the present paper is to investigate whether conditions can be found for which a weakly nonlinear analysis of sand waves and sand ridges can be carried out.

In section 2 the equations of motion for the currents, sediment transport and bottom evolution are presented and in section 3 the stability concepts are discussed. Use is made of the averaging method to obtain a relatively simple system of partial differential equations. In section 4 a more extensive analysis of depth-averaged models of the type proposed by Huthnance (1982) will be presented. This includes elliptical tides, threshold of sediment movement and lateral mixing. The basic assumption of this two-dimensional model is that direction of the sediment transport coincides with that of the depth-averaged tidal current. However, this is not correct because Coriolis forces (due to earth rotation) cause the veering of the horizontal velocity field over the vertical. This effect is parameterically included in the extended two-dimensional model of section 5. As a consequence bed forms can be found which have their crest rotated anticyclonically (clockwise on the Northern Hemisphere) with respect to the principal tidal current direction. This might be important to understand the characteristics of the shore-face connected ridges, such as those observed near the Dutch Coast, see figure 1.

Finally in section 6 the effect of secondary circulations, which occur in the vertical plane, on the dynamics of bed forms is investigated. These circulations are the principal driving mechanism for sand waves. It is shown how the effect of secondary currents can be incorporated in a depth-integrated model. The result will be referred to as the parameterized three-dimensional model and it extends the one discussed in Hulscher et al. (1993).

2 Model formulation

2.1 Equations of motion

In order to describe the dynamics of morphological patterns in coastal seas a model is required which describes both the water motions, the sediment transport and the evolution of the bed. As sand ridges and sand waves have horizontal length scales which are large compared to the local water depth a model based on the shallow water equations can be used. The water motions are assumed to be driven by tidal forces,
hence the currents will be oscillatory with a characteristic amplitude $U$ and frequency $\sigma$. The coupling between water motions and bed form changes is described by a bottom evolution equation, which follows from mass conservation of the sediment, together with a parameterization of the sediment transport. The domain is an open coastal sea, hence horizontal boundaries are absent. Only a brief outline of the derivation of the model will be presented, more details can be found in Hulscher et al. (1993).

It is assumed that a characteristic length scale of the bed forms is the tidal excursion length $L_e = U/\sigma$, which is much smaller than the tidal wave-length. The equations of motion are made dimensionless by introducing time scale $1/\sigma$, length scale $L_e$ and velocity scale $U$. Furthermore, the bottom elevations are scaled with the undisturbed water depth $H$ and $U^2/g$ is a scale for the free surface variations. Finally it is assumed that the morphological time scale is much larger than the tidal period. This implies that the averaging method can be applied such that the bottom topography may be considered as time independent on the fast time scale whereas its evolution on the slow time scale is determined by tidally averaged rather than instantaneous sediment fluxes. The resulting nondimensional model is

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} + f \vec{e}_z \times \vec{u} = -\nabla \zeta - \frac{\tau}{1 - h} + A \nabla^2 \vec{u}, \quad (1)$$

$$\nabla [(1 - h) \vec{u}] = 0, \quad (2)$$

$$\frac{\partial h}{\partial T} + \nabla \cdot (\vec{S}) = 0. \quad (3)$$

Here $t$ and $T$ are fast and slow time coordinates, respectively, $\langle \cdot \rangle$ denotes an average over a tidal period and $\nabla$ is a vector differential operator with components $\partial/\partial x$ and $\partial/\partial y$, where $x$ and $y$ are horizontal coordinates. Furthermore $\vec{u}$ is the depth-averaged velocity field (components $u$ and $v$), $\zeta$ the elevation of the free surface and $h$ the elevation of the bed with respect to the rest level. Finally $\vec{e}_z$ is a unity vector in the vertical direction, $\tau$ is the bottom shear stress (scaled by $\rho H U \sigma$, where $\rho$ is the density of the sea water) and $\vec{S}$ is the volumetric sediment flux. The momentum equations (1) contain two parameters: the Coriolis number $f$ (ratio of Coriolis parameter and tidal frequency $\sigma$) and the diffusion coefficient $A$ (ratio of tidal period and horizontal diffusion time scale). Their characteristic values at midlatitudes are $f \sim 0.8$ and $A \sim 0.1$.

22 Bed shear stress and sediment transport

In this paper the following parameterizations for the volumetric sediment flux $\vec{S}$ and the bed shear stress $\tau$ will be studied:

$$\vec{S} = \mu H (|\tau| - \tau_c) (|\tau| - \tau_d)^b \left[ \frac{\tau}{|\tau|} - \lambda \nabla \vec{u} \right], \quad (4)$$

$$\tau = R_1 \vec{u} (t - \phi_1) + R_2 \vec{e}_z \times \vec{u} (t - \phi_2), \quad (5)$$
where $H$ is the Heaviside function. Note that the sediment flux is assumed to depend on local flow variables only. This is a good approximation for material which is mainly transported as bed load such that the particles keep contact with the bed. The transport increases with increasing intensities of the bed shear stress. Furthermore, the flux is mainly in the direction of the bed shear stress, but there is a bed slope correction term involved which measures the effect that the sediment transport has a preferred down-hill component. Below a critical shear stress $\tau_c$ there is no transport. Many parameterizations of this type are discussed in Van Rijn (1993). In the present context realistic values of the morphological parameters are $\mu \sim 1$ (this is a scaling parameter, to be discussed later), $2 < b < 6, \tau_c \sim 0.1, \lambda \sim 0.01 - 0.02$.

The bed shear stress is expressed as a linear combination of the depth-averaged velocity $\bar{u}$ and the horizontal velocity field perpendicular to $\bar{u}$. Moreover phase lags $\phi_1, \phi_2$ are introduced. The motivation for these choices is as follows. First of all turbulence theory and observations indicate that bed shear stresses should be represented by a quadratic friction law. In many idealized models for tidal flow they are replaced by linear terms where the coefficients are chosen such that the average dissipation during a tidal cycle remains unchanged, see e.g. Zimmerman (1982). The reason that equation (5) contains two terms on its right-hand side is that in general the direction of the bed shear stress will not coincide with that of the depth-averaged velocity. In the present setting this is primarily due to earth rotation effects which cause the veering of the horizontal velocity field over the vertical (Ekman spiral effect, cf. Prandle (1982)). Finally the bed shear stress will not have the same phase as the depth-averaged velocity field due to the presence of internal friction in the fluid. The coefficients $R_1, R_2, \phi_1, \phi_2$ (phases in rad), which are all of order 1, can be computed from a three-dimensional shallow water model.

In this paper the following cases will be investigated:

* $R_1 = r = \text{constant, } \mu = r^{-b}, R_2 = \phi_1 = \phi_2 = 0$.
* $R_1 = r_1, \phi_1 = \varphi_1, R_2 = r_2, \phi_2 = \varphi_2$ and all variables are constant.
* $R_1, R_2, \phi_1$ and $\phi_2$ depend on the flow state.

The first case represents the two-dimensional depth-averaged model of Huthannce (1982) a.o., in which the directions of sediment transport and principal tidal currents coincide. The second is the extended two-dimensional model which accounts for Ekman veering effects. The final case results in the parameterized three-dimensional model which also incorporates the effects of secondary currents in the vertical plane on the sediment transport.

3 Stability analysis

We will now investigate the stability properties of a simple basic state of the model (1)-(3) with the parameterizations (4) and (5). As can be seen there exist basic states of the type

$$\bar{u} = \bar{u}_0(t), \quad \bar{\nabla} \zeta = \bar{\nabla} \zeta_0(t), \quad h = h_0 \equiv 0. \quad (6)$$
They describe a spatially uniform tidal flow over a flat bottom. Next small bed form perturbations are introduced and we investigate whether there is a positive feedback between the bed forms and the water motions. Thus we substitute in the equations of motion

\[ \vec{u} = \vec{u}_0 + \vec{u}' , \quad \vec{\nabla} \zeta = \vec{\nabla} \zeta_0 + \vec{\nabla} \zeta' , \quad h = h' , \quad (7) \]

where the perturbations are indicated by accents and are assumed to be small. After linearization we obtain

\[ \frac{\partial \vec{u}'}{\partial t} + (\vec{u}_0 \cdot \vec{\nabla}) \vec{u}' + \int \vec{e}_z \times \vec{u}' = -\vec{\nabla} \zeta' - \vec{r}' - \vec{\nabla} h' + A \nabla^2 \vec{u}' , \quad (8) \]

\[ \vec{\nabla} \cdot \vec{u}' - (\vec{u}_0 \cdot \vec{\nabla}) h' = 0 , \quad (9) \]

\[ \frac{\partial h'}{\partial T} + \mu \vec{\nabla} \cdot (\vec{S}') = 0 , \quad (10) \]

where the perturbed sediment flux for \(|\vec{r}_0| > \tau_c\) reads

\[ \vec{S}' = \left[ \frac{|\vec{r}_0| - \tau_c}{|\vec{r}_0| - \tau_c} \right] \left[ \frac{\vec{r}' - \lambda \vec{\nabla} h'}{|\vec{r}_0|^2} \right] + \left[ \frac{|\vec{r}_0| - \tau_c}{|\vec{r}_0|^2} \right] \left[ b - 1 + \frac{\tau_c}{|\vec{r}_0|} \right] (\vec{r}_0 \cdot \vec{r}') \vec{r}_0 \quad (11) \]

and for \(|\vec{r}_0| < \tau_c\) the transport rate is zero. The perturbed shear stress follows from replacing in equation (5) the variable \(\vec{u}\) by \(\vec{u}'\). Note however that this expression becomes more complicated if the parameterized three-dimensional model is studied since in that case the parameters \(R_1, R_2, \phi_1\) and \(\phi_2\) depend on the flow state.

The pressure terms can be eliminated by cross-differentiation of the momentum equations. This yields the vorticity equation

\[ \frac{\partial \eta'}{\partial t} + (\vec{u}_0 \cdot \vec{\nabla}) \eta' + \int \vec{\nabla} \cdot \vec{u}' = -\vec{e}_z \cdot \left[ \vec{\nabla} \times \vec{r}' \right] + \vec{\nabla} \times \vec{\nabla} h' + A \nabla^2 \eta' , \quad (12) \]

where

\[ \eta' = \vec{e}_z \cdot \left[ \vec{\nabla} \times \vec{u}' \right] \quad (13) \]

is the vertical component of the relative vorticity. The equations (9), (10) and (12) allow for spatially periodic solutions, thus

\[ \phi'(\vec{k}, t) \equiv (\vec{u}', \eta', h') = \int \int \tilde{\phi}(\vec{k}, t) \exp(-i \vec{k} \cdot \vec{r}) d\vec{k} . \quad (14) \]

This reduces the problem to the analysis of linear algebraic equations and ordinary differential equations in time.

We will now consider a specific basic state for which the velocity components read

\[ u_0 = \sin t , \quad v_0 = \beta \cos t . \quad (15) \]

This represents a simple \(M_2\) (period 12:25 h) elliptical tide. It is assumed that \(|\beta| \leq 1\), thus the principal current direction is along the \(z\)-axis. Positive (negative) values of
\( \beta \) indicate velocity vectors rotating in a clockwise (counterclockwise) direction. In this basic state the solutions of the spatially Fourier transformed flow equations can be written as Fourier series in time, e.g., the transformed vorticity becomes

\[
\tilde{\eta} = \bar{h} \sum_{n=0}^{\infty} \{ \eta_{nc} \cos(nt) + \eta_{ns} \sin(nt) \},
\]

and similar expressions for \( \tilde{u} \) and \( \tilde{v} \). In practice the Fourier coefficients are computed by truncating the series after an odd harmonic, as explained by De Swart and Zeman (1993), and solving the resulting set of linear algebraic equations. From the components \( \tilde{S}_n \) and \( \tilde{S}_n \) of the transformed sediment flux can be computed. Averaging these results over a tidal period and substituting the expressions in the bottom evolution equation finally yields

\[
\frac{\partial \tilde{h}}{\partial T} = \mu \omega \tilde{h}.
\]

Thus from equations (14) and (17) it follows that the bottom topography can be written as \( h' \sim \exp(-i \tilde{x} \cdot \bar{\mathbf{x}} + \mu \omega T) \). In general \( \omega \) is a complex frequency where its real part denotes the growth rate and its imaginary part the frequency of the perturbation. However, in the present model \( \omega \) is always real, thus the crests of the bed forms do not move. This is because the equations of motion are invariant to arbitrary rotation of the coordinate axes. Consequently, there is no preferred direction in the model which implies that bed forms cannot propagate. Thus, in eq. (17) \( \omega \) is the growth rate, which depends on the model parameters, the components of the wavevector \( \bar{\mathbf{k}} \) of the bottom perturbations and the even Fourier components of the transformed vorticity field defined in eq. (16).

4 Two-dimensional models

In this section we briefly discuss the results for a two-dimensional model where we choose \( R_1 = r = \text{constant}, \ R_2 = \phi_1 = \phi_2 = 0, \mu = r^{-k} \). As can be traced back from equations (4) and (5) this implies that the bed shear stress coincides with the de-averaged velocity and that there are no phase lags. In figure 4a a contour plot of a growth rate \( \omega \) is shown in the wavenumber space for the reference case \( f = 0.8, 0.6, \beta = 0, \lambda = 0.01, b = 3, \tau_c = 0 \) and \( A = 0 \). This corresponds to a unidirectional M2 tide with a velocity amplitude of 1 ms\(^{-1}\) in a coastal sea at mid-latitudes with undisturbed water depth of 30 m. In this case a dimensionless wavenumber \( (k^2 + f) \) corresponds to a wave-length of approximately 40 km. The figure shows that the a wide range of bed forms with positive growth rates. The maximum value is attained for a perturbation with a wave-length of approximately 8 km and its crests are rotating cyclonically (counterclockwise on the Northern Hemisphere), by \( \sim 30^\circ \), with respect to the principal tidal current direction. These length scales and preferred orientations can also be observed for most offshore sand ridges, see figure 1.

As was already explained in the introduction the physical mechanism is related to the presence of horizontal residual circulations around the banks, see also figure...
These circulations cause a net convergence of sediment transport at the crests of the banks which is most effective for bed forms which have their crests turned cyclonically with respect to the direction of principal tidal currents. This explains the observed asymmetry in the growth rate curves of figure 1 and it can be ascribed to earth rotation effects. The reason that small-scale perturbations (large wavenumbers) are damped is due to the down-slope correction term in the sediment transport (4). When substituted in the bottom evolution equation (3) it can be seen that this mechanism causes diffusion of bed forms which becomes very effective if gradients are large. The figure is qualitatively similar to results presented in earlier studies. The important difference is here that it has been verified that the present solution differs less than 1% from the exact solution. This requires a truncation of the Fourier series in eq. (16) after the third overtide and results in an increase of growth rates by 30%.

We have carried out many sensitivity studies with this model. The results can be summarized as follows:

*Both the incorporation of lateral mixing terms in the vorticity equation (i.e., nonzero A) and a critical shear stress in the sediment flux (i.e., nonzero τc) cause a significant reduction of the wavenumber of the preferred bed form and the corresponding maximum growth rate. However, the preferred orientation of the banks is only slightly affected.

*Maximum growth rates increase with increasing values of the parameters f, r and b. This is obvious since increasing Coriolis and bottom frictional forces causes more intense residual circulations around the banks whereas parameter b controls the amount of sediment transport.

*If the ellipticity of the tide is increased the distribution of positive growth rates in wavenumber space becomes more symmetric whereas the preferred wave-length and orientation of the banks are hardly affected, see figure 4b.
There is no combination of physically realistic model parameters for which a nontrivial selective bed form is obtained. It is a direct consequence of the flow-topography interaction mechanism and can be understood as follows. Consider a bottom perturbation with a fixed orientation \( \alpha \) of its crests with respect to the direction of the principal tidal currents and assume a noncircular tide (\(|\beta| < 1\) in eq. (15). It appears that the growth rate \( \omega \) of this bed form is an even function of the wavenumber \( |\kappa| \equiv \kappa \) and furthermore

\[
\omega(\kappa^2) = 0, \quad \frac{\partial \omega}{\partial \kappa^2} \leq \frac{\partial \omega}{\partial \kappa^2}_{|\kappa^2=0}.
\]

These results show that mode \( \kappa = 0 \) is neutrally stable and that the spectrum of perturbations with positive growth rates always include the ultralong waves. This is illustrated in figure 5 which shows the maximum growth rate \( \omega_m \) and corresponding wavenumber \( \kappa_m \) as a function of the bottom friction parameter \( r \) for three different values of the Coriolis number \( f \). As can be seen the wavenumber \( \kappa_m \) decreases with decreasing \( r \). For \( f = 0 \) and \( r < 0.06 \) all perturbations decay and the ultralong waves are the last to become stable.

Figure 5: Maximum growth rate \( \omega_m \) (a) and corresponding wavenumber \( \kappa_m \) (b) as a function of the bottom friction parameter \( r \) for \( f = 0 \) (solid curve), \( f = 0.4 \) (dashed curve) and \( f = 0.8 \) (dotted curve).

The only exception occurs for \(|\beta| = 1\) which corresponds to a circular tide. There is no longer a principal direction of tidal currents in this degenerate case. The formation of sand ridges is now fully due to asymmetry in the velocity field caused by the basic tide and the first overtide \( (M_4) \), thus residual current effects are unimportant. Figure 6 shows the neutral curves \( \omega = 0 \) in dependence of the wavenumber \( \kappa \) and friction parameter \( r \) for \( \beta = -1 \) and \( \beta = 1 \). Both curves reach a minimum at nonzero \( r \) and \( \kappa \), which indicates that there are selective modes. Note that the minimum for \( \beta = -1 \) (cyclonically rotating tidal current vector, on the Northern Hemisphere counterclockwise) is reached for smaller friction parameter than the situation where
\( \beta = 1 \). This is because a cyclonically polarized tidal current reinforces Coriolis torques thereby causing larger tidal asymmetries and larger growth rates.

Figure 6: Neutral curves \( \omega = 0 \) as a function of the friction parameter \( r \) and wavenumber \( \kappa \) for a cyclonically (a) and anticyclonically (b) rotating circular tide.

5 An extended two-dimensional model

In this section we investigate the effect of allowing for bed shear stresses which have a direction and phase which differs from those of the depth-averaged current. Thus we consider the case that the coefficients \( R_1, R_2, \phi_1 \) and \( \phi_2 \) are nonzero. Realistic values for these parameters are obtained by identifying the bed shear stress defined in eq. (5) with the bed shear stress computed from the basic state equations

\[
\frac{\partial \vec{v}_0}{\partial t} + f \hat{e}_z \times \vec{v}_0 = -\nabla \zeta_0 + \frac{1}{2} E \frac{\partial^2 \vec{v}_0}{\partial z^2},
\]

\[
\vec{v}_0 \equiv \int_{-1}^{0} \vec{v}_0 dz = \sin t \hat{e}_z + \beta \cos t \hat{e}_y,
\]

\[
\frac{\partial \vec{v}_0}{\partial z} = 0 \quad \text{at} \quad z = 0,
\]

\[
\frac{1}{2} E \frac{\partial \vec{v}_0}{\partial z} = s \vec{v}_0 \quad \text{at} \quad z = -1.
\]

They describe a basic tidal current over a flat bottom in a three-dimensional shallow water model and they can be solved by straightforward methods, see Prandle (1982). Here \( \vec{v} \) is a horizontal velocity field which depends on the vertical coordinate \( z \) and \( \vec{v}_0 \) is its depth-averaged value. Furthermore parameter \( E \) is the Stokes number, which measures the effect of internal friction, and \( s \) is the slip parameter which describes the relation between velocity and shear stress at the bottom. The bed shear stress is defined as

\[
\tau = \frac{1}{2} E \frac{\partial \vec{v}_0}{\partial z} \bigg|_{z=-1}
\]

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In the experiments we have fixed \( b = 3, A = 0 \) and \( \beta = 0 \) (unidirectional tides) an\( \lambda = 0.02, \tau_c = 0.1 \). Furthermore, \( \mu = 1 \), so the results are not rescaled. In figure 7th parameters \( R_1, R_2, \phi_1 \) and \( \phi_2 \) are shown as a function of the Stokes number \( E \) for a slippage parameter \( s = 5 \). The latter is a realistic value for tidal flow in a shallow sea. Note that \( \phi_1 \) is always negative, so the depth-averaged flow lags behind the bed shear stress, as to be expected because of internal friction effects. In this model earth rotation effec

![Graph](image)

Figure 7: a. Dependence of the parameters \( R_1 \) (solid curve) and \( R_2 \) (dashed curve) on the Stokes number \( E \) for \( s = 5, \beta = 0 \) and \( f = 0.8 \). b. As a. but for the parameters \( \phi_1 \) and \( \phi_2 \) (in degrees), respectively.

cause the end points of the bed shear stress vector of the basic state to follow an ellipse during a tidal cycle in a cyclonic direction. Furthermore, the principal axis of the ellipse is rotated anticyclonically (angle \( \psi \)) with respect to the direction of the depth-averaged current. If this orientation is large enough (as occurs for small Stokes numbers and large slip parameters) then the preferred bed forms will have their crests more or less parallel to the principal direction of the bed shear stress vector. This can be seen from figure 8 which shows the growth rates for the extended two-dimensional model for \( E = 0.25, s = 5 \). In this case \( R_1 = 0.42, R_2 = 0.12, \phi_1 = -19.27^\circ, \phi_2 = 25.3^\circ \) and \( \psi = -11.52^\circ \). The preferred wave-length and crest orientation are 5.0 km and 15.0\(^\circ\), and thus the crests are rotated anticyclonically with respect to the principal direction of the depth-averaged current. This is fundamentally different from the two-dimensional model discussed in the previous section. The reason for this behaviour is that the preferred bed forms cause large bottom frictional torques, see the second term on the right-hand side of vorticity equation (12). This tidal vorticity is subsequently advected by the depth-averaged current of which the direction in the present model does not coincide with that of the crests. Consequently an intense residual current can develop if the angle \( |\psi| \) has large values.

It appears that growth rates are quite small for the present values of the Stokes number and slip parameter. Experiments have demonstrated that they rapidly increase if \( E \) and \( s \) are increased. Since in that case also the angle \( \psi \) between principal
Figure 8: Contour plot of the growth rates in the wavenumber space for the extended two-dimensional model. For parameters see text.

Shear stress direction and depth-averaged flow direction decreases the preferred bed forms will have the same characteristics as in the two-dimensional model.

6 A parameterized three-dimensional model

So far we have studied the sediment transport and morphodynamics related to depth-averaged currents. This resulted in preferred bed forms with characteristics similar to observed sand ridges. For the occurrence of smaller scale bottom patterns secondary circulations, which have their main orientation in the vertical plane, are important. Evidence for this can be found in Blondeaux (1990), Vittori and Blondeaux (1990), Fredsoe and Deigaard (1992) and references therein. This knowledge was used in Hulscher et al. (1993) to propose a simple parameterization in a depth-averaged model to account for the effect of secondary currents on the sediment transport. As a result the formation of both sand ridges and sand waves was simulated, where the latter features have wave-lengths of a few hundred meters and their crests are perpendicular to the principal tidal current direction. In this section an extended and more explicit parameterization of the contribution of secondary currents to the sediment transport will be presented. This is motivated by the results of Hulscher (1995) with a three-dimensional shallow water model. The advantage of a parameterized model is that it yields insight in the basic mechanism of sand wave growth whereas also the growth rates can be computed more easily. It is derived from an analysis of the perturbed three-dimensional shallow water equations which read

$$\frac{\partial \phi'}{\partial t} + (\mathbf{v}_0 \cdot \nabla) \phi' + \nu \frac{\partial \phi'}{\partial z} + f \mathbf{e}_z \times \phi' = -\nabla' \zeta' + \frac{1}{2} E \frac{\partial^2 \phi'}{\partial z^2},$$

(23)
\vec{\nabla} \cdot \vec{\sigma} + \frac{\partial w'}{\partial z} = 0, \tag{24}

with boundary conditions
\[ \frac{\partial \vec{v}'}{\partial z} = 0, \quad w' = 0 \quad \text{at} \quad z = 0, \tag{25} \]
\[ \frac{\partial \vec{v}'}{\partial z} = \frac{2s}{E} \vec{v}' + h' \left[ -\frac{\partial^2 \vec{v}_0}{\partial z^2} + \frac{2s}{E} \frac{\partial \vec{v}_0}{\partial z} \right], \quad \vec{w}' = \left( \vec{u}_0 \cdot \vec{\nabla} \right) h' \quad \text{at} \quad z = -1. \tag{26} \]

Here the basic state horizontal velocity field \( \vec{v}_0 \) and the parameters \( E \) (Stokes number) and \( s \) (slip parameter) are defined in eq. (19)-(21). Furthermore \( \vec{v}'(x, z, t) \) is the perturbed horizontal flow field due to the presence of small bed forms whereas \( w' \) and \( \zeta' \) are the perturbed vertical velocity component and free surface elevation, respectively.

The state variables are decomposed into depth-averaged parts and fluctuations (denoted by subscripts \( s \)) which describe the secondary flow. We assume that the depth-averaged equations yield results which are already governed by the extended two-dimensional model of the previous section. We now focus on the secondary flow mechanism. This is only significant if advection terms are large, i.e., if the wavenumbers of the bed form perturbations are large. The momentum equations for the secondary flow can thus be approximated by
\[ \left( \vec{u}_0 \cdot \vec{\nabla} \right) \vec{v}_s + w_s \frac{\partial \vec{v}_0}{\partial z} = -\vec{\nabla} \zeta' + \frac{1}{2} \frac{E}{|\zeta'|} \frac{\partial^2 \vec{v}_s}{\partial z^2}, \tag{27} \]

which is a balance between advection, pressure and friction terms. These equations are supplemented by a continuity equation and boundary conditions which follow from (24)-(26) by adding subscripts \( s \) to the variables \( \vec{v}', w', \zeta' \). Here it is assumed that in the regime where eq. (27) is valid the secondary flow dominates over the depth-averaged flow. These equations include the generation of secondary flow discussed by Zimmerman (1981) and Hulscher et al. (1993). Basically, the internal friction terms cause the generation of tidal vorticity which is subsequently advected by the basic tidal currents. As a result residual secondary circulation cells as sketched in figure 3b will be generated. However, in the present model also vertical advection of basic momentum is incorporated which cannot be neglected using scaling arguments.

This system of equations is analyzed as follows. First the variables are Fourier transformed in the horizontal domain, similar to what was done in section 3, eq. (14). It follows that the intensity of the advection terms in the momentum equations is measured by the wavenumber \( \kappa \equiv |\vec{\kappa}| \), which is much larger than the Stokes number \( E \) which multiplies the highest-order derivatives. Hence the problem can be analyzed by singular perturbation methods. Here we have only analyzed the case for finite values of the slip parameter, such that the basic velocity field \( \vec{v}_0 \) at the bottom \( z = -1 \) is nonzero. In this case the thickness of the boundary layer is \( \delta_i = [E/(2\kappa)]^{1/2} \).

The analysis of the boundary layer equations is straightforward because the equations are linear differential equations with constant coefficients in the vertical boundary
layer coordinate. In particular the vertical velocity component is constant in this layer and determined by the boundary condition at the bottom. Approximate solutions can then be found by harmonic truncation from which the shear stress $\tau$ can be computed. This is subsequently substituted in eq. (11) for the perturbed sediment transport and thus yields a secondary flow correction on the growth rates of bed forms.

An example of how secondary flow corrections affect the growth rates of bed forms is shown in figure 9 a which is a contour plot of the growth rates in wavenumber space for $E = 1, s = 5$ and other parameters have the same values as in the previous section. In this case the maximum growth rate is $\omega_m = 141.0$ which occurs for $k_m = 131.8, l_m = 12.5$. This describes a preferred bed form with a wave-length of 300 m of which the crests are almost perpendicular to the principal direction of the bed shear stress (crest orientation 84.5° with respect to depth-averaged tidal current direction).

![Contour plot of the growth rates in the parameterized three-dimensional model which includes effects of secondary currents, parameters $E = 1, s = 5$.](image)

Note that the growth rates are much larger than those obtained in the models of the previous sections where only sand ridges were obtained. Thus for this choice of the model parameters sand waves dominate over sand ridges. However, if smaller values of $s$ are considered the model predicts preferred bed forms which again resemble sand ridges. For an extensive study, based on a three-dimensional model, see Hulscher (1995).

For the analysis of the nonlinear dynamics it would be helpful if parameters could be selected for which only a narrow spectrum of bed forms with positive growth rates is found. Therefore it was investigated how the maximum growth rate $\omega_m$ and the corresponding wavenumbers $k_m, l_m$ depend on the slip parameter $s$. It appears that they decrease monotonically with decreasing $s$ and for $s < s_c = 0.14$ the basic state (a flat bottom) is stable. At critical conditions again the perturbation with wavenumber $k = 0$ is the first to become unstable. Hence this is not a suitable starting point for a weakly nonlinear analysis.

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7 Conclusions

In this paper the formation of offshore tidal sand ridges and sand waves has been investigated. It appears that a positive feedback between undulations in the bottom and horizontal residual circulations results in the occurrence of sand ridges, as described by Zimmerman (1981) and Huthnance (1982). From a sensitivity study of the two-dimensional depth-integrated model it is concluded that the crests of these bed forms are always rotated counterclockwise (on the Northern Hemisphere) with respect the principal current direction.

A serious limitation of these two-dimensional models is that the sediment transport is in the direction of the depth-averaged flow. Generally, due to earth rotation effects these directions will differ. This effect has been included in an extended two-dimensional model. Experiments have demonstrated that the most preferred bed forms still resemble sand ridges but for certain parameter values the crests can be rotated clockwise. This result may be helpful for a better understanding of shore-face connected ridges which have this property, see figure 1.

Finally a parameterized three-dimensional model is discussed. It incorporates the effects of secondary circulations, which have their main orientation in the vertical plane, on the sediment transport and morphodynamics. The parameterization is motivated by the results of Hulscher (1995) and extends the one discussed by Hulscher et al. (1993) by including also vertical advection of basic momentum. As a result both the formation of sand ridges and sand waves is simulated. The mechanism of sand wave generation is due to residual secondary circulations and agrees with the description of Zimmerman (1981) and the results of Blondeaux (1990) for a sea ripple model.

All the models investigated in this paper suffer from the problem that no combination of parameter values can be found for which a narrow spectrum of bed forms with finite wavelengths exists. It appears that the ultralong waves are always the first modes to be selected if a parameter such as the tidal current amplitude is increased. This is due to the behaviour of the horizontal residual vorticity response. Consequently for realistic parameter values a strong nonlinear analysis is required to yield information on the amplitude behaviour of the ridges. The only exception is for circular tides, such that the bed forms are not generated by residual circulations but rather by tidal asymmetries. However, this turns out to be a degenerate case.

Finally it is important to realize that there are inconsistencies between depth-averaged and three-dimensional shallow water models. It appears that the depth-averaged results obtained from a stability analysis of the three-dimensional model differ from those which follow from the stability analysis of the two-dimensional model, see also the discussion in Zimmerman (1986). Hence for a full understanding of the dynamics of sand ridges and sand waves it is recommended to use the three-dimensional shallow water model, as was done by Hulscher (1995).

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References


Van der Meene, J. (1994) The shoreface connected ridges along the central Dutch coast, Neth. Geogr. Studies 174, KNAG, Utrecht, 256 pp..


