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Simple Perfect Squared Squares and 2×1 Squared Rectangles of Orders 21 to 24

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In this note tables of all simple perfect squared squares and 2×1 squared rectangles of orders 21, 22, 23, and 24 are presented. © 1993 Academic Press, Inc.

INTRODUCTION

For describing the problem of the dissection of squares and rectangles into unequal squares in a non-trivial way I use the terminology of Brooks, Smith, Stone, and Tutte [1] and Bouwkamp [2]:

A dissection of a rectangle into a finite number $N > 1$ of non-overlapping squares is called a *squared rectangle* or a *squaring of order N*.

The N squares are called the *elements* of the dissection. The term "elements" is also used for the (length of the) sides of the elements.

If the elements are all unequal, the squaring is called *perfect* and the rectangle is called a *perfect rectangle*; otherwise the squaring is *imperfect*.

A squaring that contains a smaller rectangle or square, dissected into squares is called *compound*. All other squared rectangles or squares are *simple*.

The simple perfect squared square lowest order possible is 21 was found in March 1978 and is of order 21 [3]. The order-22 simple perfect 2×1 squared rectangle found in August 1978 appears to be of lowest order [4]. Two simple perfect squared squares of order 22 with side 110 and 192, respectively, were found in 1978 [5]. This result was communicated to Bouwkamp, Tutte, and Federico and it was sent by Federico to Willcocks. Willcocks was able to construct another squared square of order 22 with side 110 out of my solution of order 22 with the same side [6].

So far no solutions were known at all of orders 23 and 24. Of order 25, five simple perfect squared squares are known, due to Wilson [14].

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TABLE I

Simple Perfect Squared Square of Order 21

length= 112 width= 112
(50, 35, 27)(8, 19)(15, 17, 11)(6, 24)(29, 25, 9, 2)(7, 18)(16)(42)(4, 37)(33)

Federico found another three simple perfect squared squares of order 25 [15].

Because of a question by Dr. King of Georgia University (USA), on a subset of the three-connected planar graphs, I wanted to reuse the information stored on magnetic tape more than 12 years ago. It appeared that the information could not always be read from the tape in a reliable way after such a long time. Therefore I decided to redo the work done in 1977 and 1978 by means of the new facilities at our university. It took 50 h of computing time to verify that the order-21 simple perfect squared square solution is unique. Because of the success I decided to calculate the order-22 simple perfect squared squares and 2×1 squared rectangles. On September 5, 1990 this work was completed after 5 days. I found eight simple perfect squared squares of order 22. See Table II. I found only one simple perfect 2×1 squared rectangle of order 22. Its size is 272×136. See Table V.

TABLE II

List of Perfect Squared Squares of Order 22

length= 110 width= 110
(60, 50)(23, 27)(24, 22,14)(7, 16)(8, 6)(12, 15)(13)(2, 28)(26)(4, 21, 3)(18)(17)

length= 110 width= 110
(60, 50)(27, 23)(24, 22, 14)(4, 19)(8, 6)(3, 12, 16)(9)(2, 28)(26)(21)(1, 18)(17)

length= 139 width= 139
(80, 59)(21, 38)(29, 28, 17, 27)(7, 10)(18, 20)(4, 3)(32, 8)(1, 31)(30)(24, 2)(22)

length= 147 width= 147
(55, 44, 48)(40, 4)(52)(26, 29)(23, 3)(20, 31, 21)(5, 47)(43)(9, 17)(1, 8)(32)(25)

length= 147 width= 147
(59, 43, 45)(41, 2)(47)(34, 25)(21, 37, 8)(55)(22, 12)(10, 23)(32)(11, 26)(19, 4)(15)

length= 154 width= 154
(61, 52, 41)(11, 30)(9, 35, 19)(46, 24)(16, 33)(22, 2)(36, 17)(50)(47, 21)(5, 31)(26)

length= 172 width= 172
(97, 75)(22, 53)(39, 42, 38)(9, 44)(4, 19, 13, 2)(36, 3)(11)(33, 16)(24)(1, 18)(17)

length= 192 width= 192
(86, 49, 57)(41, 8)(28, 37)(19, 9)(47, 35, 4)(31, 14)(10, 36)(17, 26)(12, 71)(62)(59)

TABLE III

List of Simple Perfect Squared Squares of Order 23

```
length=  110 width=  110
( 44, 29, 37)( 21,  8)( 13, 32)( 28, 16)( 15, 19)( 12,  4)(  3,  1)(  2, 14)(  5)( 10, 41)( 38,  7)( 31)

length=  139 width=  139
( 80, 59)( 21, 38)( 26, 19, 15, 14, 27)(  1, 13)( 16)( 18, 20)(  7, 12)( 33)( 32,  8)( 28)( 24,  2)( 22)

length=  140 width=  140
( 54, 38, 48)( 28, 10)( 31, 27)( 33, 13,  8)(  5,  3)(  2,  1)( 29)( 20)(  4, 23)( 16, 19)( 53)( 45)( 42)

length=  140 width=  140
( 60, 44, 36)(  8, 28)( 30, 22)( 33, 15, 12)(  2, 26)( 24)(  3,  9)(  7, 23)( 18)( 16)( 50)( 47,  4)( 43)

length=  145 width=  145
( 84, 61)( 27, 34)( 30, 29, 25)( 12, 15)(  8, 26)(  9,  3)(  6, 20)(  4, 36)(  1, 32)( 31)(  2, 24)( 22)

length=  180 width=  180
( 92, 88)( 34, 22, 32)( 45, 47)( 12, 10)( 23, 19)( 25, 21)( 43,  2)( 41,  8)(  4, 15)( 37, 11)( 33)( 26)

length=  188 width=  188
( 96, 92)( 45, 47)( 26, 37, 33)( 15, 11)( 25,  8)(  4, 23, 21)( 19)( 51,  2)( 49)( 12, 34)( 32, 10)( 22)

length=  208 width=  208
( 71, 75, 62)( 22, 40)( 67,  4)( 60, 10,  9)(  1, 12, 18)( 11)( 23)( 17, 41)( 16, 24)(  3, 73)( 70)( 65)

length=  215 width=  215
( 79, 66, 70)( 22, 40,  4)( 74)( 60, 10,  9)(  1, 12, 18)( 11)( 23)( 17, 41)( 16, 24)( 76)(  3, 71)( 68)

length=  228 width=  228
( 99, 73, 56)( 17, 39)( 68, 22)( 36, 25)( 57, 42)(  9, 16)(  2,  7)( 10, 28)( 23)( 15, 87, 18)( 72)( 69)

length=  257 width=  257
(134,123)( 11, 24, 33, 55)( 60, 57, 28)( 15,  9)( 20, 22)( 29, 14)( 32,  2)( 79)(  3, 66, 17)( 63)( 49)

length=  332 width=  332
(123, 89,120)( 58, 31)( 83, 68)( 49, 50, 24)( 26, 56)( 48,  1)( 47, 30)( 15, 53)( 17,129, 38)(112)( 91)
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I continued the work for the search of the order 23. I started the computer on September 5, 1990 and the machine completed its work on November 22, 1990. Since I use a completely exhaustive search I was certain to find all solutions. In the meantime I communicated my results of order 22 to J. D. Skinner II, Lincoln, Nebraska. From my results he was able to construct two solutions of order 23 with side 180 and 188, respectively [7]. I found 12 simple perfect squared squares and eight simple perfect 2×1 squared rectangles of order 23. See Tables III and VI. Between January 4, 1991 and March 15, 1991 I calculated the order-24 solutions. I found 26 simple perfect squared squares and 15 simple perfect 2×1 squared rectangles of order 24. See Tables IV and VII. In 1970 Federico found several 2×1 squared rectangles of orders 23, 24, and 25 [13].

TABLE IV

List of Simple Perfect Squared Squares of Order 24

length = 120 width = 120
(47,32,41)(15,17)(8,33)(19,20,23)(25)(14,5)(4,13,3)(10,16)(9)(12,46)(40,6)(34)

length = 186 width = 186
(96,90)(36,28,26)(43,23,30)(2,24)(8,22)(16,7)(9,12,60)(4,18,3)(15)(47)(46)(33)

length = 194 width = 194
(72,54,68)(37,17)(3,65)(20)(53,19)(34,42)(69,18)(10,36,61)(28)(2,9,25)(23,7)(16)

length = 195 width = 195
(80,63,52)(11,41)(17,27,30)(61,26,10)(16,21)(18,53)(42)(39)(54,7)(4,49)(47,2)(45)

length = 196 width = 196
(105,91)(24,36,31)(51,44,10)(34)(5,26)(20,21)(2,17,1)(48)(18,47,15)(40,11)(32)(29)

length = 201 width = 201
(103,98)(22,26,50)(46,40,17)(23,12,4)(9,21)(11,1)(10)(6,58,20)(18,3)(53)(52)(38)

length = 201 width = 201
(108,93)(15,26,52)(44,39,19,10,11)(9,1)(16,22)(20,8)(24)(18,4)(56)(5,54)(49)(42)

length = 203 width = 203
(105,98)(18,32,48)(54,40,11)(29)(15,17)(24,58,2)(1,21,26)(20)(44,10)(36,5)(34)(31)

length = 247 width = 247
(136,111)(25,46,40)(32,36,48,45)(6,34)(24,28)(23,9)(5,19,12)(14)(3,66)(63)(62)(56)

length = 253 width = 253
(104,77,72)(31,18,23)(27,50)(13,5)(28)(24,20)(70,61)(4,44)(38,40)(9,88,2)(86)(79)

length = 255 width = 255
(137,118)(44,74)(59,53,25)(28,41)(16,65)(11,63)(27,22,10)(52)(26)(5,17)(32)(3,23)(20)

length = 288 width = 288
(159,129)(57,72)(56,39,25,16,23)(9,7)(2,28)(36)(42,15)(17,22)(87)(10,18)(73)(68)(60)

length = 288 width = 288
(165,123)(51,72)(64,69,32)(30,21)(93)(17,8,7)(1,36)(9)(26)(59,5)(54,20)(14,48)(34)

length = 290 width = 290
(166,124)(42,82)(61,59,48,40)(8,30,84)(11,14,31)(2,65,3)(63)(17)(9,21)(45,12)(33)

Table continued

TABLE IV—*Continued*

length= 292 width= 292
(175,117)(44, 73)(15, 29)(57, 54, 22, 24, 17, 1)(16)(102)(33)(20, 2)(26)(12, 8)(67)(3, 63)(60)

length= 304 width= 304
(132, 76, 96)(56, 20)(116)(72, 47, 69)(25, 22)(3, 48, 29, 11)(100)(35, 92)(12, 17)(7, 5)(57)(55)

length= 304 width= 304
(175,129)(53, 76)(72, 28, 24, 44, 7)(37, 23)(4, 20)(32)(99)(16, 85)(12, 36)(57, 27)(3, 33)(30)

length= 314 width= 314
(139, 79, 96)(62, 17)(113)(78, 59, 2)(64)(19, 40)(44, 16, 4)(12,105)(97)(28)(29, 11)(18, 65)(47)

length= 316 width= 316
(172,144)(65, 79)(78, 57, 37)(36, 52, 14)(93)(33, 24)(23, 13)(66, 12)(9, 15)(10, 3)(55)(54)(48)

length= 326 width= 326
(142, 83,101)(65, 18)(119)(80, 56, 6)(71)(24, 32)(11, 44, 15, 1)(14,106)(104)(43)(29)(10, 63)(53)

length= 423 width= 423
(186,114,123)(105, 9)(132)(62, 60, 31, 33)(29, 2)(27, 77, 36)(17, 49, 50)(47, 15)(168)(32)(128)(127)

length= 435 width= 435
(192,112,131)(93, 19)(74, 76)(100, 56, 23, 13)(10, 88, 80, 2)(78)(33)(44, 45)(8,150)(143, 1)(142)

length= 435 width= 435
(200,112,123)(101, 11)(134)(95, 50, 55)(30, 27, 44)(3, 24)(12, 21)(45, 5)(63, 9)(178)(54)(140)(117)

length= 459 width= 459
(198,126,135)(117, 9)(144)(98, 55, 45)(10,116, 36)(65)(180)(68, 30)(8, 57)(38)(16,100)(95, 11)(84)

length= 459 width= 459
(198,126,135)(117, 9)(144)(115, 38, 45)(21, 17)(10, 33, 83, 36)(4, 23)(25)(6, 50)(180)(31)(146)(133)

length= 479 width= 479
(175,140,164)(35, 29, 52, 24)(28,160)(6, 23)(130, 86)(43, 60)(26, 17)(77)(44, 68)(174)(5,155)(150)

For squarings of order up to 24 we need the c-nets of order up to 25. Therefore I generated and identified c-nets of orders 23 and 24. Those of order 25 were only generated and searched for the existence of perfect squared squares of order 24. I communicated all my results to Bouwkamp. Bouwkamp constructed the order-24 squared square with side 186 on December 4, 1990 and one of the solutions of order-24 with side 288 on November 17, 1990, from my results of order 23 by using transformation

TABLE V

Simple Perfect 2 × 1 Squared Rectangle of Order 22

length= 272 width= 136
(83, 49, 71, 69)(27, 22)(2, 67)(30, 65)(7, 20)(53, 24, 13)(11, 17, 5)(35)(29, 6)(23)

techniques [16]. Moreover, he was able to construct 12 new solutions of order 25 out of my squared squares of order 24, also using certain transformation techniques [16].

For an historical overview of the squared rectangle and squared square problem see Federico [12].

Mathematical Theory and Computer Procedures

Squared rectangles can be obtained from so called c-nets [1]. A *c-net* is a three-connected planar graph. The order of a c-net is its number of edges. The *dual* of a c-net is also a c-net. The c-nets are constructed using Tutte's theorem, known since 1947 and published in 1961 [9]. Let C_n be the set of c-nets of order n.

TABLE VI

List of Simple Perfect 2 × 1 Squared Rectangles of Order 23

length= 226 width= 113
(65, 68, 52, 41)(15, 26)(25, 23, 4)(19)(48, 17)(8, 18)(14, 45, 9)(2, 38, 10)(36)(31)(28)
length= 270 width= 135
(70, 48, 67, 38, 47)(29, 9)(20, 36)(22, 26)(7, 25, 68, 16)(65, 23, 4)(19, 18)(52)(43)(42)
length= 270 width= 135
(72, 81, 75, 42)(19, 23)(15, 4)(27)(63, 9)(14, 60, 1)(16)(54, 28, 8)(22)(43)(26, 2)(24)
length= 280 width= 140
(82, 42, 30, 26, 31, 69)(21, 5)(12, 18)(36)(54)(1, 20)(19)(73, 2)(71)(58, 24)(10, 44)(34)
length= 284 width= 142
(79, 64, 29, 19, 28, 65)(10, 9)(37)(39)(31, 33)(25, 77)(17, 22)(63, 16)(12, 5)(52)(47)(45)
length= 288 width= 144
(87, 46, 39, 51, 65)(7, 32)(53)(37, 14)(79)(15, 17)(13, 2)(57, 30)(56)(21, 45)(27, 3)(24)
length= 336 width= 168
(93, 72, 82, 89)(39, 33)(23, 15, 13, 24, 7)(17, 79)(75, 18)(2, 11)(8, 9)(63, 1)(62)(57)
length= 350 width= 175
(88, 50, 44, 71, 42, 55)(29, 13)(17, 27)(39, 11)(21, 47)(28)(18, 104, 5)(26)(87, 1)(86)(73)

TABLE VII

List of Simple Perfect 2 × 1 Squared Rectangles of Order 24

length= 220 width= 110
(66, 45, 48, 61)(19, 26)(23, 25)(12, 49)(2, 17)(44, 24)(10, 39)(37)(13, 14)(20, 4)(16, 1)(15)

length= 352 width= 176
(92, 71, 93, 53, 43)(10, 33)(40, 23)(21, 50)(17, 39)(84, 29)(45, 83, 22)(61)(55, 24)(7, 38)(31)

length= 360 width= 180
(101, 88, 97, 74)(23, 51)(35, 53)(47, 45, 28)(79, 22)(57)(24, 55)(39, 14)(38, 7)(11, 36)(31)(25)

length= 370 width= 185
(105,100, 43, 29, 93)(14, 15)(39, 17, 1)(16)(33)(18, 21)(34, 92)(30, 85, 3)(24)(80, 25)(58)(55)

length= 386 width= 193
(107, 84, 54, 52, 89)(15, 37)(41, 13)(28)(44, 40)(6, 16,104)(29, 36, 10)(26)(86, 21)(69)(65)(62)

length= 388 width= 194
(111,112, 37, 35, 93)(2, 33)(39)(8, 25)(30, 17)(34,101)(9, 21)(83, 28)(27, 82, 3)(12)(67)(55)

length= 392 width= 196
(109, 88, 51, 52, 92)(38, 13)(12, 40)(25)(43, 45)(26, 34, 3)(31,104)(87, 22)(18, 8)(73)(65)(63)

length= 396 width= 198
(109, 96, 84,107)(12, 49, 23)(26, 45, 37)(39, 91)(89, 20)(7, 19)(27)(8, 65, 13)(15, 57)(52)(42)

length= 400 width= 200
(104, 79, 82, 73, 62)(11, 51)(44, 40)(25, 28, 26)(47, 35)(96, 33)(2, 24)(30)(4, 87)(83)(71)(63)

length= 446 width= 223
(127,124, 81,114)(43, 38)(5,109)(16, 27)(34, 99, 22, 12)(96, 31)(1, 4, 11)(10, 3)(7)(77)(65)

length= 462 width= 231
(140,104, 94,124)(64, 30)(50, 54)(47,107)(91, 35, 14)(21, 39, 4)(36, 73, 13)(60)(56)(38, 1)(37)

length= 464 width= 232
(133,136, 67, 39, 38, 51)(1, 24, 13)(17, 23)(64)(11, 6)(53)(78)(117)(99, 34)(31, 96, 9)(87)(65)

length= 48 8width= 244
(133, 88, 62, 77,128)(47, 15)(41, 51)(67, 21)(59, 9)(50)(63,116)(111, 22)(89)(76, 33)(10, 53)(43)

length= 600 width= 300
(179, 85,124, 84,128)(40, 44)(33, 25, 27)(8, 17)(15, 12)(41)(176)(32)(172)(73)(121, 58)(5, 68)(63)

length= 638 width= 319
(164,173,135, 75, 91)(59, 16)(40, 67)(3, 37)(62)(74, 61)(10, 57)(155, 9)(47)(146, 36)(123)(110)(104)

If $s \in C_n$ is not a wheel, then at least one of the nets s and its dual s' can be constructed from an element $\sigma \in C_{n-1}$ by addition of an edge joining two vertices.

A wheel is a c-net with an even number of edges E, with one edge of degree $E/2$, and $E/2$ vertices of degree 3. The *degree* of a vertex is the number of edges joining the vertex.

Generation of c-nets of order $n+1$ out of c-nets of order n gives rise to many duplicate c-nets. These can be removed using an identification method described in 1962 [10] and improved in 1978 [11].

Squarings can be obtained from c-nets by considering these graphs as electrical networks of unit resistances (see [1]). Basically starting from c-nets of order n, those of order $n+1$ are generated and identified using electronic computer equipment. Duplicates are removed, currents are calculated, and simple perfect squared squares and 2×1 rectangles are filtered. For detailed discussions of the procedure see [10].

Results

Results are presented in tables. Tables I through IV show the Bouwkamp codes of simple perfect squared squares of orders 21, 22, 23, and 24, respectively. Tables V through VII show the Bouwkamp codes of simple perfect squared 2×1 rectangles of orders 22, 23, and 24, respectively.

There are eight doublets with side 139 (order 22 and 23), 140 (both order 23), 147 (both order 22), 201 (both order 24), 288 (both order 24), 304 (both order 24), 435 (both order 24), and 459 (both order 24).

There exists one triplet with side 110 (two of order 22 and one of order 23). And there is one doublet of squared 2×1 rectangles with size 270×135, both of order 23.

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